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THE BOUNDARY VALUE PROBLEM FOR ONE CLASS OF HIGHER-ORDER SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The boundary value problem for a class of higher-order semilinear partial differential equations is studied. The theorems on existence, uniqueness and nonexistence of solutions of this problem are proved.

1. Statement of the problem

Consider in the Euclidean space $\mathbb{R}^{n+1}$ with the variables $\mathbf{x} = (x_1, x_2, ..., x_n)$ and t the following higher-order semilinear partial differential equation

$$L_f u := \frac{\partial^{2(2k+1)} u}{\partial t^{2(2k+1)}} - \Delta^2 u + f(u, \nabla u) = F(x,t), \tag{1.1}$$

where f and F are given, and u is an unknown real functions,
$\nabla := (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t})$, $\Delta := \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n \geq 2$, and $k \geq 0$ is an integer number.

For the equation (1.1) we consider the boundary value problem: find in the cylindrical domain $D_T := \Omega \times (0,T)$, where Ω is an open Lipschitz domain in $\mathbb{R}^n$, a solution $u = u(x,t)$ of that equation according to the boundary conditions

$$\left.\frac{\partial^i u}{\partial t^i}\right|_{\Omega_0 \cup \Omega_T} = 0, \ i = 0, ..., 2k, \tag{1.2}$$

$$u|_\Gamma = 0, \quad \left.\left(\Delta u - a\frac{\partial u}{\partial \boldsymbol{\nu}}\right)\right|_\Gamma = 0, \tag{1.3}$$

where $\Gamma : \partial\Omega \times (0,T)$ is the lateral face of the cylinder D_T, $\Omega_0 : \mathbf{x} \in \Omega$, $t = 0$ and $\Omega_T : \mathbf{x} \in \Omega$, $t = T$ are the lower and upper bases of this cylinder, respectively, $a : \overline{\Gamma} \to \mathbb{R}$ is a given continuous function, while $\frac{\partial}{\partial \boldsymbol{\nu}}$ is a derivative with respect to the outer normal to the boundary ∂D_T of the domain D_T. Here $\boldsymbol{\nu} = (\nu_1, ..., \nu_n, \nu_{n+1})$ is a unit vector of the outer normal to ∂D_T and, obviously, $\nu_{n+1}|_\Gamma = 0$.

In the theory of partial differential equations along with second order equations belonging to one of the standard types, e.g. elliptic, hyperbolic, parabolic, mixed, etc., for which certain problems are set and investigated for correctness, high

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order partial differential equations, which, generally speaking, are not subject to standard classification by type, are also considered. For example, the linear part of the operator on the left-hand side of equation (1.1) is hypoelliptic in the terminology of L. Hörmander [6]. The study of partial differential equations and systems of non-standard structure from the point of view of existence or absence of their solutions, the statement of correct local, nonlocal and other problems is certainly of scientific interest in the theory of partial differential equations.

In work [9], for equation (1.1) in a cylindrical domain D_T a boundary value problem with conditions (1.2) and homogeneous boundary conditions

$$u|_\Gamma = 0, \quad \frac{\partial u}{\partial \boldsymbol{\nu}}\Big|_\Gamma = 0$$

are considered instead of (1.3). To the study of initial, boundary and mixed problems for higher-order nonlinear partial differential equations with a structure different from (1.1) has been devoted numerous literature (see, for example, papers [1],[2],[4],[5],[7],[8],[10]-[12],[15],[16],[18] and the literature cited there). For example, paper [18] proposes a general approach to a priori estimates of solutions of nonlinear partial differential equations and systems, in particular of higher-order, allowing to investigate the nonexistence of their solutions. In the book [4] a unified approach to the consideration of blow-up of solutions of four types of nonlinear evolution equations in partial derivatives of higher-order: parabolic, hyperbolic, dispersive and Schrödinger is proposed. The traditional issues of existence, non-existence, uniqueness, non-uniqueness, and global asymptotics of solutions are also considered. In work [1], the Cauchy problem for a higher-order pseudohyperbolic equation is investigated. Using estimates of type $L_p \to L_q$ for the corresponding linear problem, the authors established criteria for existence and non-existence of global solutions. The questions of the existence and uniqueness of smooth global solutions are also considered. The behavior of solutions and their derivatives as $t \to +\infty$ is also established. Let us also note papers [2],[5],[15],[16] in which initial boundary value problems for nonlinear partial differential equations of higher-order are investigated. The existence, non-existence, uniqueness, and asymptotic behavior of their solutions are considered in these papers. In works [7], [12], the boundary value problems of nonlinear equations with an iterated multidimensional wave operator in the main part were studied when the entire data carrier of the considered problems is a conical characteristic manifold, and in paper [10], the boundary value problem in a cylindrical domain when the Dirichlet and Neumann conditions are set on the whole boundary of the domain was studied for a class of nonlinear systems with an iterated multidimensional wave operator in the main part. Note also papers [8] and [11], where boundary value problems in the cylindrical domain were studied for some classes of nonlinear partial differential equations and systems of higher-order. In these works, depending on the conditions imposed on the nonlinear terms included in the equations under consideration, uniqueness, existence and absence of their solutions were established. Note that equation (1.1) is not included in the classes of equations considered in works [1],[2],[4],[5],[7],[8],[10]-[12],[15],[16],[18].

Denote by $C^{4,4k+2}(\overline{D}_T)$ the space of functions u continuous in $\overline{D}_T$ and having there continuous partial derivatives $\partial_x^\beta u, \frac{\partial^l u}{\partial t^l}$, where $\partial_x^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} ... \partial x_n^{\beta_n}}$, $\boldsymbol{\beta} = (\beta_1, ..., \beta_n)$, $|\beta| = \sum_{i=1}^n \beta_i \leq 4$; $l = 1, ..., 4k+2$.

Let

$$C_0^{4,4k+2}(\bar{D}_T) := \Big\{ u \in C^{4,4k+2}(\overline{D}_T) : u|_\Gamma = 0, \\ \frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0 \cup \Omega_T} = 0, \, i = 0, ..., 2k \Big\}. \tag{1.4}$$

Introduce the Hilbert space $W_0^{2,2k+1}(D_T)$ as a completion of the classical space $C_0^{4,4k+2}(\overline{D}_T)$ with respect to the norm

$$||u||^2_{W_0^{2,2k+1}(D_T)} = \\ = \int_{D_T} \Big[u^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2 + \sum_{i,j=1}^n \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2 + \sum_{i=1}^{2k+1} \Big(\frac{\partial^i u}{\partial t^i}\Big)^2 \Big] dxdt. \tag{1.5}$$

Remark 1.1. It follows from (1.5) that if $u \in W_0^{2,2k+1}(D_T)$, then $u \in \overset{o}{W}{}_2^1(D_T)$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial^l u}{\partial t^l} \in L_2(D_T)$; $i, j = 1, ..., n$; $l = 1, ..., 2k+1$. Here $W_2^m(D_T)$ is the well-known Sobolev space consisting of the elements of $L_2(D_T)$, having up to the m-th order generalized derivatives from $L_2(D_T)$, and $\overset{o}{W}{}_2^1(D_T) = \Big\{ u \in W_2^1(D_T) : \ u|_{\partial D_T} = 0 \Big\}$, where the equality $u|_{\partial D_T} = 0$ is understood in the sense of the trace theory [14]. Moreover, in the case when the domain Ω is convex, implying that D_T is also convex, since the following estimate is valid [14]

$$\int_{D_T} \Big[\sum_{i,j=1}^n \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial^2 u}{\partial x_i \partial t}\Big)^2 + \Big(\frac{\partial^2 u}{\partial t^2}\Big)^2 \Big] dxdt \leq \\ c \int_{D_T} \Big[\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial t^2} \Big]^2 dxdt \\ \forall u \in \overset{o}{C}{}^2(\overline{D}_T) := \{u \in C^2(\overline{D}_T) : u|_{\partial D_T} = 0\} \tag{1.6}$$

with positive constant c, not dependent on u and domain D_T, then from (1.5) and (1.6) we obtain continuous embedment of the spaces

$$W_0^{2,2k+1}(D_T) \subset W_2^2(D_T). \tag{1.7}$$

Below, we assume that domain Ω is convex.

Before introducing a notion of a weak generalized solution of the problem (1.1), (1.2), (1.3) from the space $W_0^{2,2k+1}(D_T)$, let us suppose that $u \in C_0^{4,4k+2}(\overline{D}_T)$ is a classical solution of this problem. Multiplying both parts of the equation (1.1) by an arbitrary function $\varphi \in C_0^{4,4k+2}(\bar{D}_T)$ and integrating the obtained equation by parts over the domain D_T, due to (1.2) we obtain

$$-\int\limits_{D_T} \frac{\partial^{2k+1} u}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1} \varphi}{\partial t^{2k+1}} dxdt - \int\limits_{\partial D_T} \frac{\partial \varphi}{\partial \boldsymbol{\nu}} \Delta u ds + \int\limits_{\partial D_T} \varphi \frac{\partial}{\partial \boldsymbol{\nu}} \Delta u ds -$$

$$\int_{D_T} \Delta u \cdot \Delta\varphi dxdt + \int_{D_T} f(u,\nabla u)\varphi dxdt = \int_{D_T} F\varphi dxdt. \tag{1.8}$$

Taking into account the second boundary condition from (1.3) and the valid conditions $\varphi|_{\partial D} = 0$, $\frac{\partial\varphi}{\partial\boldsymbol{\nu}}\big|_{\Omega_0\cup\Omega_T} = 0$, then since $\varphi \in C_0^{4,4k+2}(\bar{D}_T)$ from (1.8) we get

$$\int_{D_T} \Big[\frac{\partial^{2k+1}u}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1}\varphi}{\partial t^{2k+1}} + \Delta u \cdot \Delta\varphi\Big] dxdt + \int_{\Gamma} a\frac{\partial u}{\partial\boldsymbol{\nu}} \cdot \frac{\partial\varphi}{\partial\boldsymbol{\nu}} ds -$$

$$\int_{D_T} f(u,\nabla u)\varphi dxdt = -\int_{D_T} F\varphi dxdt \;\; \forall\varphi \in C_0^{4,4k+2}(\overline{D}_T). \tag{1.9}$$

In certain sense it is true also the reverse proposition, i.e. if $u \in C^{4,4k+2}(\overline{D}_T)$ satisfies conditions (1.2), the first boundary value condition from (1.3) and integral identity (1.9) for any $\varphi \in C_0^{4,4k+2}(\overline{D}_T)$, then by standard reasoning it follows that u represents a solution to equation (1.1) in the domain D_T and satisfies the second boundary value condition from (1.3) in the classical sense, i.e. u is a classical solution to the problem (1.1), (1.2), (1.3).

We intend to take the equality (1.9) as a basis for definition of a weak generalized solution u of the problem (1.1), (1.2), (1.3) in the space $\overset{\circ}{W}{}_0^{2,2k+1}(D_T)$ but first, we need to impose such conditions on the power of nonlinearity of function $f(u,\nabla u)$ with respect to independent variables of this function that provide the existence of the following integral

$$\int_{D_T} f(u,\nabla u)\varphi dxdt$$

from the left hand side of equation (1.9) for any $\varphi \in W_0^{2,2k+1}(D_T)$. Below, for function $f = f(s_0, s_1, ..., s_{n+1}), (s_0, s_1, ..., s_{n+1}) \in \mathbb{R}^{n+2}$ we assume that

$$f \in C(\mathbb{R}^{n+2}) \tag{1.10}$$

and

$$|f(s_0, s_1, ..., s_{n+1})| \le M + \sum_{i=0}^{n+1} M_i|s_i|^{\alpha_i} \;\; \forall\boldsymbol{s} = (s_0, s_1, ..., s_{n+1}) \in \mathbb{R}^{n+2}, \tag{1.11}$$

where $M, M_i, \alpha_i = const > 0,\ i = 0, 1, ..., n+1$, and

$$1 < \alpha_0 < \frac{n+1}{n-3} \text{ for } n > 3;\;\; \alpha_0 > 1 \text{ for } n = 2, 3, \tag{1.12}$$

$$1 < \alpha_i < \frac{n+1}{n-1},\;\; i = 1, ..., n+1,\;\; n \ge 2. \tag{1.13}$$

Remark 1.2. As it is known the space $W_2^2(D_T)$ is continuously and compactly embedded into $L_p(D_T)$ for $p < \frac{2(n+1)}{n-3}$ when $n > 3$ and for any $p \ge 1$ when $n = 2, 3$; analogously, the space $W_2^1(D_T)$ is continuously and compactly embedded into $L_q(D_T)$ for $q < \frac{2(n+1)}{n-1}$ [14]. Therefore, taking into account continuous embedment of spaces (1.7), the inequality (1.11) with powers of nonlinearities α_i satisfy conditions (1.12), (1.13), and due to the properties of the Nemitskii operators N_i, $i=0,1,...,n+1$, acting by formulas $N_i v=|v|^{\alpha_i}$, we obtain that the nonlinear operator

$$N : \overset{\circ}{W}{}^{2,2k+1}_0(D_T) \to L_2(D_T) \tag{1.14}$$

acting by formula

$$Nu = f(u, \nabla u) \tag{1.15}$$

is continuous and compact [13]. Hence, in particular, it follows that if $u \in \overset{\circ}{W}{}^{2,2k+1}_0(D_T)$ then $f(u,\nabla u) \in L_2(D_T)$ and integral $\int\limits_{D_T} f(u,\nabla u)\varphi dxdt$ from the left hand side of equation (1.9) exists and for $u_m \to u$ in the space $W^{2,2k+1}_0(D_T)$ we have $f(u_m,\nabla u_m) \to f(u,\nabla u)$ in the space $L_2(D_T)$.

Definition 1.1. *Let Lipschitz domain Ω be convex and function f satisfy the conditions (1.10), (1.11), (1.12), (1.13); $a \in C(\overline{\Gamma}), F \in L_2(D_T)$. The function $u \in W^{2,2k+1}_0(D_T)$ is said to be a weak generalized solution of the problem (1.1), (1.2), (1.3), if the integral equality (1.9) is valid for any function $\varphi \in W^{2,2k+1}_0(D_T)$, i.e.,*

$$\int\limits_{D_T} \Big[\frac{\partial^{2k+1} u}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1} \varphi}{\partial t^{2k+1}} + \Delta u \cdot \Delta\varphi\Big] dxdt + \int\limits_{\Gamma} a \frac{\partial u}{\partial \boldsymbol{\nu}} \cdot \frac{\partial \varphi}{\partial \boldsymbol{\nu}} ds -$$

$$\int\limits_{D_T} f(u,\nabla u)\varphi dxdt = -\int\limits_{D_T} F\varphi dxdt \;\; \forall \varphi \in W^{2,2k+1}_0(D_T). \tag{1.16}$$

2. Equivalent norms in the space $W^{2,2k+1}_0(D_T)$

Above we introduced Hilbert space $W^{2,2k+1}_0(D_T)$ with norm $||u||_0 = ||u||_{W^{2,2k+1}_0(D_T)}$ defined by the right hand side of equality (1.5), which is generated by scalar product

$$(u,v)_0 = \int_{D_T} \Bigg[u\cdot v + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i} + \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}\cdot \frac{\partial^2 v}{\partial x_i \partial x_j} + $$

$$+ \sum_{i=1}^{2k+1} \frac{\partial^i u}{\partial t^i}\cdot\frac{\partial^i v}{\partial t^i}\Bigg] dxdt. \tag{2.1}$$

Below we show that if

$$a \in C(\overline{\Gamma}), \;\; a|_\Gamma \ge 0, \tag{2.2}$$

then the bilinear form

$$(u,v)_1 = \int_{D_T} \left[\frac{\partial^{2k+1} u}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1} v}{\partial t^{2k+1}} + \Delta u \cdot \Delta v \right] dxdt + \int_\Gamma a\frac{\partial u}{\partial \boldsymbol{\nu}}\cdot\frac{\partial v}{\partial \boldsymbol{\nu}} ds \tag{2.3}$$

also defines a scalar product in the Hilbert space $W^{1,2k+1}_0(D_T)$ with the norm

$$||u||_1^2 = \int_{D_T} \left[\Big(\frac{\partial^{2k+1} u}{\partial t^{2k+1}}\Big)^2 + (\Delta u)^2 \right] dxdt + \int_\Gamma a\Big(\frac{\partial u}{\partial \boldsymbol{\nu}}\Big)^2 ds. \tag{2.4}$$

Lemma 2.1. *Let condition (2.2) be fulfilled and* Ω *be a convex domain with boundary* $\partial\Omega$ *of the class* C^2*. The following inequalities*

$$c_1||u||_0 \leq ||u||_1 \leq c_2||u||_0 \ \ \forall u \in C_0^{4,4k+2}(\overline{D}_T) \tag{2.5}$$

hold, where the positive constants c_1 *and* c_2 *do not depend on* u*.*

Proof. First let us estimate the values $||\frac{\partial^i u}{\partial t^i}||^2_{L_2(D_T)}$, $i = 0, 1, ..., 2k$, by the value $||\frac{\partial^{2k+1} u}{\partial t^{2k+1}}||^2_{L_2(D_T)}$. Since $u \in C_0^{4,4k+2}(\overline{D}_T)$ satisfy the equalities (1.2) then we have

$$\frac{\partial^i u(\cdot,t)}{\partial t^i} = \frac{1}{(2k-i)!}\int_0^t (t-\tau)^{2k-i}\frac{\partial^{2k+1}u(\cdot,\tau)}{\partial t^{2k+1}}d\tau, \ \ i = 0, 1, ..., 2k. \tag{2.6}$$

Using the Cauchy's inequality from (2.6) we have

$$\Big(\frac{\partial^i u(\cdot,t)}{\partial t^i}\Big)^2 \leq \frac{1}{((2k-i)!)^2}\int_0^t (t-\tau)^{2(2k-i)}d\tau \int_0^t \Big(\frac{\partial^{2k+1}u(\cdot,\tau)}{\partial t^{2k+1}}\Big)^2 d\tau \leq$$

$$\leq T^{4k-2i+1}\int_0^T \Big(\frac{\partial^{2k+1}u(\cdot,\tau)}{\partial t^{2k+1}}\Big)^2 d\tau,$$

whence we obtain

$$\int_0^T \Big(\frac{\partial^i u(\cdot,t)}{\partial t^i}\Big)^2 dt \leq T^{4k-2i+2}\int_0^T \Big(\frac{\partial^{2k+1}u(\cdot,\tau)}{\partial t^{2k+1}}\Big)^2 d\tau, \ \ i = 0, 1, ..., 2k. \tag{2.7}$$

Integrating both parts of the inequality (2.7) over domain Ω we get

$$\int_{D_T} \Big(\frac{\partial^i u}{\partial t^i}\Big)^2 dxdt \leq T^{4k-2i+2}\int_{D_T} \Big(\frac{\partial^{2k+1}u}{\partial t^{2k+1}}\Big)^2 dxdt, \ \ i = 0, ..., 2k. \tag{2.8}$$

If $u \in C_0^{4,4k+2}(\overline{D}_T)$ then $u(\cdot,t) \in C^2(\overline{\Omega})$, $u(\cdot,t)|_{\partial\Omega} = 0$ for fixed $t \in [0,T]$ and according to the known inequality [14]

$$\int_\Omega \Big[u^2(\cdot,t) + \sum_{i=1}^n \Big(\frac{\partial u(\cdot,t)}{\partial x_i}\Big)^2\Big]dx \leq c_0\int_\Omega \big(\Delta u(\cdot,t)\big)^2 dx, \tag{2.9}$$

where the positive constant $c_0 = c_0(\Omega)$ does not depend on $t \in [0,T]$ and u. Integrating the inequality (2.9) with respect to t we obtain

$$\int_{D_T} \Big[u^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2\Big]dxdt \leq$$

$$c_0\int_{D_T} (\Delta u)^2 dxdt \ \ \forall u \in C_0^{4,4k+2}(\overline{D}_T). \tag{2.10}$$

If $u \in C_0^{4,4k+2}(\overline{D}_T)$, and, therefore, $u(\cdot,t) \in C^2(\overline{\Omega})$, $t \in [0,T]$, then since the boundary $\partial\Omega$ of the domain Ω according supposition is of the class C^2, then for $\frac{\partial u(\cdot,t)}{\partial \boldsymbol{\nu}}|_{\partial\Omega}$ there is valid the following estimate [17]

$$\int_{\partial\Omega} \Big(\frac{\partial u(\cdot,t)}{\partial \boldsymbol{\nu}}\Big)^2 ds = \Big|\Big|\frac{\partial u(\cdot,t)}{\partial \boldsymbol{\nu}}\Big|\Big|^2_{L_2(\partial\Omega)} \leq \tilde{c}_0||u(\cdot,t)||^2_{W_2^2(\Omega)} =$$

$$= \tilde{c}_0\int_\Omega \Big[u^2(\cdot,t) + \sum_{i=1}^n \Big(\frac{\partial u(\cdot,t)}{\partial x_i}\Big)^2 + \sum_{i,j=1}^n \Big(\frac{\partial^2 u(\cdot,t)}{\partial x_i \partial x_j}\Big)^2\Big]dx, \tag{2.11}$$

where positive constant $\tilde{c}_0 = \tilde{c}_0(\Omega)$ does not depend on t and u.

Taking into account (2.2) let $a_0 = \max\limits_{(x,t)\in\overline{\Gamma}} a(x,t) \geq 0$. From (2.11) follows that

$$\int_{\Gamma} a\Big(\frac{\partial u}{\partial \boldsymbol{\nu}}\Big)^2 ds \leq a_0 \int_{\Gamma} \Big(\frac{\partial u}{\partial \boldsymbol{\nu}}\Big)^2 ds =$$

$$= a_0 \int_0^T \Big[\int_{\partial\Omega} \Big(\frac{\partial u(\cdot,t)}{\partial \boldsymbol{\nu}}\Big)^2 ds\Big] dt \leq a_0 \tilde{c}_0 \int_0^T ||u(\cdot,t)||^2_{W_2^2(\Omega)} dt =$$

$$= a_0 \tilde{c}_0 \int_{D_T} \Big[u^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2 + \sum_{i,j=1}^n \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2\Big] dxdt. \qquad (2.12)$$

Finally, since the domain Ω is convex and thereafter the inequality (1.6) is valid, then from (1.5), (2.1)-(2.4), (2.8), (2.12) it is clear that (2.5) is valid. The Lemma 2.1 is proved. □

Remark 2.1. In view of the Lemma 2.1 by completion of the space $C_0^{4,4k+2}(\overline{D}_T)$ by the norm (2.4) we obtain the same Hilbert space $W_0^{2,2k+1}(D_T)$ with equivalent scalar products (2.1) and (2.3).

3. Reduction of the problem (1.1), (1.2), (1.3) to the nonlinear functional equation $u = Au$ in the space $W_0^{2,2k+1}(D_T)$ and the estimate of $||Au||_1$

First let us consider the linear problem corresponded to (1.1), (1.2), (1.3), i.e. when $f = 0$. In this case for $F \in L_2(D_T)$ we introduce analogously a notion of a weak generalized solution $u \in W_0^{2,2k+1}(D_T)$ of this problem for which due to (1.16) and (2.3) it is valid the following integral equality

$$(u,\varphi)_1 = \int_{D_T} \left[\frac{\partial^{2k+1} u}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1} \varphi}{\partial t^{2k+1}} + \Delta u \cdot \Delta \varphi\right] dxdt + \int_{\Gamma} a \frac{\partial u}{\partial \boldsymbol{\nu}} \cdot \frac{\partial \varphi}{\partial \boldsymbol{\nu}} ds =$$

$$= - \int_{D_T} F\varphi dxdt \ \ \forall \varphi \in W_0^{2,2k+1}(D_T). \qquad (3.1)$$

Taking into account (2.5) we have

$$\left|\int_{D_T} F\varphi dxdt\right| \leq ||F||_{L_2(D_T)} ||\varphi||_{L_2(D_T)} \leq$$

$$\leq ||F||_{L_2(D_T)} ||\varphi||_0 \leq c_1^{-1} ||F||_{L_2(D_T)} ||\varphi||_1. \qquad (3.2)$$

According to the Remark 2.1, (3.1) and (3.2) from the Riesz theorem it follows that there exists a unique function $u \in W_0^{2,2k+1}(D_T)$ which satisfies the equality (3.1) for any $\varphi \in W_0^{2,2k+1}(D_T)$ and for its norm the following estimate

$$||u||_1 \leq c_1^{-1} ||F||_{L_2(D_T)} \qquad (3.3)$$

is valid. Thus, introducing the notation $u = L_0^{-1} F$, we find that to the linear problem corresponding to (1.1), (1.2), (1.3), i.e. for $f = 0$, there corresponds the linear bounded operator

$$L_0^{-1} : L_2(D_T) \to W_0^{2,2k+1}(D_T)$$

and for its norm the estimate

$$\|L_0^{-1}\|_{L_2(D_T)\to W_0^{2,2k+1}(D_T)} \le c_1^{-1} \tag{3.4}$$

holds by virtue of (3.3).

Remark 3.1. According to the Definition 1.1 of a weak generalized solution to the problem (1.1), (1.2), (1.3) and above introduced operator L_0^{-1} the integral identity (1.16), which is equivalent to this problem, can be written in the form of nonlinear functional equation

$$u = L_0^{-1}\left[f(u,\nabla u) - F\right] \tag{3.5}$$

in the Hilbert space $W_0^{2,2k+1}(D_T)$. Due to (1.15) the equation (3.5) can be rewritten in the form

$$u = Au := L_0^{-1}(Nu - F), \tag{3.6}$$

where in view of (3.4) and the Remark 1.2, if nonlinear function f satisfies conditions (1.10) - (1.13), then the operator $A : W_0^{2,2k+1}(D_T) \to W_0^{2,2k+1}(D_T)$ from (3.6) will be continuous and compact.

If $v \in \overset{o}{W_2^1}(D_T, \Omega_0 \cup \Omega_T) := \{v \in W_2^1(D_T) : v|_{\Omega_0\cup\Omega_T} = 0\}$, then due to the structure of cylindrical domain $D_T = \Omega \times (0,T)$ we have multiplicative inequality [14]

$$||v||_{Lp(D_T)} \le \beta||\nabla v||^{\tilde{\alpha}}_{Lm(D_T)}||v||^{1-\tilde{\alpha}}_{Lr(D_T)} \ \ \forall v \in \overset{0}{W_2^1}(D_T, \Omega_0 \cup \Omega_T), \tag{3.7}$$

$$\tilde{\alpha} = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{1}{r} - \frac{1}{\tilde{m}}\right)^{-1}, \ \tilde{m} = \frac{(n+1)m}{n+1-m}, \ \ r \le p,$$

with positive constant $\beta = \beta(\Omega)$ not dependent on v and T, where for $r = 1$ and $m = 2$ the parameter $p \in \left[1, \frac{2(n+1)}{n-1}\right]$.

Due to inequality [20]

$$\int_{D_T} |v|dxdt \le (\text{mes}\, D_T)^{1-\frac{1}{p}}||v||_{Lp(D_T)}\,, \ p \ge 1,$$

from (3.7) follows

$$||v||_{L_p(D_T)} \le \beta_0(\text{mes}\, D_T)^{\frac{1}{p}+\frac{1}{n+1}-\frac{1}{2}}||v||_{W_2^1(D_T)} \ \ \forall v \in \overset{0}{W_2^1}(D_T, \Omega_0 \cup \Omega_T), \tag{3.8}$$

with positive constant $\beta_0 = \beta_0(\Omega)$ not dependent on v and T.

Since $\text{mes} D_T = T\text{mes}\Omega$ then from (3.8) we have

$$||v||_{L_p(D_T)} \le \beta_1 T^{\frac{1}{p}+\frac{1}{n+1}-\frac{1}{2}}||v||_{W_2^1(D_T)} \ \ \forall v \in \overset{0}{W_2^1}(D_T, \Omega_0 \cup \Omega_T), \tag{3.9}$$

where $\beta_1 = \beta_0(\text{mes}\Omega)^{\frac{1}{p}+\frac{1}{n+1}-\frac{1}{2}}$, besides, it is easy to see that the condition $\frac{1}{p} + \frac{1}{n+1} - \frac{1}{2} > 0$ is equivalent to the condition $p < \frac{2(n+1)}{n-1}$.

Due to (1.4) and definition of space $W_0^{2,2k+1}(D_T)$, as a result of the completion of classical space $C_0^{4,4k+2}(\overline{D}_T)$ with respect to norm (1.5), according to the existence of traces of the elements of space $W_2^2(D_T)$ on $(\Omega_0 \cup \Omega_1) \subset \partial D_T$, and taking into account the embedding of spaces (1.7) we obtain

$$u,\ u_t,\ u_{x_i} \in \overset{0}{W_2^1}(D_T, \Omega_0 \cup \Omega_T),\ i = 1, ..., n. \tag{3.10}$$

Thus, if the power exponents $\alpha_i,\ i = 1, ..., n+1$, satisfy inequalities (1.13), then in view of (1.5), (1.7), (3.9) and (3.10) we have

$$\Big[\int_{D_T} |u_{x_i}|^{2\alpha_i} dxdt\Big]^{\frac{1}{2}} = ||u_{x_i}||^{\alpha_i}_{L_{2\alpha_i}(D_T)} \le \beta_1^{\alpha_i} T^{\alpha_i(\frac{1}{2\alpha_i}+\frac{1}{n+1}-\frac{1}{2})} ||u_{x_i}||^{\alpha_i}_{W_2^1(D_T)} \le$$
$$\le \beta_1^{\alpha_i} T^{\alpha_i(\frac{1}{2\alpha_i}+\frac{1}{n+1}-\frac{1}{2})} ||u||^{\alpha_i}_{W_0^{2,2k+1}(D_T)} =$$
$$= \beta_1^{\alpha_i} T^{\alpha_i(\frac{1}{2\alpha_i}+\frac{1}{n+1}-\frac{1}{2})} ||u||^{\alpha_i}_0 \ \ \forall u \in W_0^{2,2k+1}(D_T),\ \ i = 1, ..., n, \tag{3.11}$$

analogously,

$$\Big[\int_{D_T} |u_t|^{2\alpha_{n+1}} dxdt\Big]^{\frac{1}{2}} \le$$
$$\le \beta_1^{\alpha_{n+1}} T^{\alpha_{n+1}(\frac{1}{2\alpha_{n+1}}+\frac{1}{n+1}-\frac{1}{2})} ||u||^{\alpha_{n+1}}_0 \ \ \forall u \in W_0^{2,2k+1}(D_T). \tag{3.12}$$

Below, for simplicity, instead of condition (1.12) imposed on exponent α_0 we require fulfilment of the following condition

$$1 < \alpha_0 < \frac{n+1}{n-1} \tag{3.13}$$

analogously to conditions (1.13) imposed on the rest exponents $\alpha_i,\ i = 1, ..., n+1$.

In view of (3.13), analogously to estimates (3.11) and (3.12) we have

$$\Big[\int_{D_T} |u|^{2\alpha_0} dxdt\Big]^{\frac{1}{2}} \le \beta_1^{\alpha_0} T^{\alpha_0\left(\frac{1}{2\alpha_0}+\frac{1}{n+1}-\frac{1}{2}\right)} ||u||^{\alpha_0}_0 \ \ \forall u \in W_0^{2,2k+1}(D_T). \tag{3.14}$$

Note that due to (1.13) and (3.13) we have

$$\gamma_i = \alpha_i\Big(\frac{1}{2\alpha_i} + \frac{1}{n+1} - \frac{1}{2}\Big) > 0,\ i = 0, ..., n+1. \tag{3.15}$$

Below we consider another analog for the first inequality from (2.5) which we need further. Due to (1.5), (1.6), (2.3), (2.8) and (2.10) we have

$$||u||_0^2 \le \int_{D_T} \Big[c_0(\Delta u)^2 + 2c\Big((\Delta u)^2 + \Big(\frac{\partial^2 u}{\partial t^2}\Big)^2\Big) + \sum_{i=1}^{2k+1} \Big(\frac{\partial^i u}{\partial t^i}\Big)^2\Big] dxdt \le$$
$$\le \int_{D_T} \Big[(c_0 + 2c)(\Delta u)^2 + 2cT^{4k-2}\Big(\frac{\partial^{2k+1} u}{\partial t^{2k+1}}\Big)^2 +$$
$$+ \Big\{\sum_{i=1}^{2k+1} T^{4k-2i+2}\Big\}\Big(\frac{\partial^{2k+1} u}{\partial t^{2k+1}}\Big)^2\Big] dxdt \le$$
$$\le \lambda^2(T) \int_{D_T} \Big[\Big(\frac{\partial^{2k+1} u}{\partial t^{2k+1}}\Big)^2 + (\Delta u)^2\Big] dxdt =$$
$$= \lambda^2(T)||u||_1^2 \ \forall u \in W_0^{2,2k+1}(D_T), \tag{3.16}$$

where

$$\lambda(T)=\begin{cases}(c_0+4c+2k+1)^{\frac{1}{2}}, & T\le 1,\\ (c_0+4c+2k+1)^{\frac{1}{2}}T^{2k}, & T>1.\end{cases} \tag{3.17}$$

Now, taking into account (1.11), (3.3), (3.4), (3.11)-(3.17) we estimate $||Au||_{W_0^{2,2k+1}(D_T)}=||Au||_1$ from (3.6)

$$||Au||_{W_0^{2,2k+1}(D_T)}=||Au||_1\le ||L_0^{-1}||_{L_2(D_T)\to W_0^{2,2k+1}(D_T)}||Nu-F||_{L_2(D_T)}\le$$
$$\le c_1^{-1}||Nu||_{L_2(D_T)}+c_1^{-1}||F||_{L_2(D_T)}\le c_1^{-1}\Big[\int_{D_T}\Big(M+M_0|u|^{\alpha_0}+\sum_{i=1}^{n}M_i|u_{x_i}|^{\alpha_i}+$$
$$+M_{n+1}|u_t|^{\alpha_{n+1}}\Big)^2dxdt\Big]^{\frac{1}{2}}+c_1^{-1}||F||_{L_2(D_T)}\le c_1^{-1}\Big[\int_{D_T}(n+3)\big(M^2+M_0^2|u|^{2\alpha_0}+$$
$$+\sum_{i=1}^{n}M_i^2|u_{x_i}|^{2\alpha_i}+M_{n+1}^2|u_t|^{2\alpha_{n+1}}\Big)dxdt\Big]^{\frac{1}{2}}+c_1^{-1}||F||_{L_2(D_T)}\le$$
$$\le c_1^{-1}(n+3)^{\frac{1}{2}}\Big[\Big(\int_{D_T}M^2dxdt\Big)^{\frac{1}{2}}+\Big(\int_{D_T}M_0^2|u|^{2\alpha_0}dxdt\Big)^{\frac{1}{2}}+$$
$$+\sum_{i=1}^{n}\Big(\int_{D_T}M_i^2|u_{x_i}|^{2\alpha_i}dxdt\Big)^{\frac{1}{2}}+\Big(\int_{D_T}M_{n+1}^2|u_t|^{2\alpha_{n+1}}dxdt\Big)^{\frac{1}{2}}\Big]+c_1^{-1}||F||_{L_2(D_T)}\le$$
$$\le c_1^{-1}(n+3)^{\frac{1}{2}}\Big[\Big(M^2\mathrm{mes}D_T\Big)^{\frac{1}{2}}+\sum_{i=0}^{n+1}M_i\beta_1^{\alpha_i}T^{\gamma_i}||u||_0^{\alpha_i}\Big]+$$
$$+c_1^{-1}||F||_{L_2(D_T)}\le c_1^{-1}(n+3)^{\frac{1}{2}}\sum_{i=0}^{n+1}M_i\beta_1^{\alpha_i}T^{\gamma_i}\lambda^{\alpha_i}(T)||u||_1^{\alpha_i}+$$
$$+c_1^{-1}(n+3)^{\frac{1}{2}}(M^2\mathrm{mes}\,D_T)^{\frac{1}{2}}+c_1^{-1}||F||_{L_2(D_T)}=$$
$$=\sum_{i=0}^{n+1}\tilde a_i(T)||u||_1^{\alpha_1}+b(T)\ \ \forall u\in W_0^{2,2k+1}(D_T). \tag{3.18}$$

Here

$$\tilde a_i(T)=c_1^{-1}(n+3)^{\frac{1}{2}}M_iT^{\gamma_i}\lambda^{\alpha_i}(T),\ i=0,...,n+1, \tag{3.19}$$

$$b(T)=c_1^{-1}(n+3)^{\frac{1}{2}}(M^2\mathrm{mes}\,\Omega)^{\frac{1}{2}}T^{\frac{1}{2}}+c_1^{-1}||F||_{L_2(D_T)}, \tag{3.20}$$

besides, in derivation of the estimate (3.18) we used the following inequalities

$$\Big(\sum_{i=1}^{m}k_i\Big)^2\le m\sum_{i=1}^{m}k_i^2,\ \ \Big(\sum_{i=1}^{m}k_i^2\Big)^{\frac{1}{2}}\le\sum_{i=1}^{m}|k_i|.$$

Let us simplify the right hand part of the estimate (3.18). Since $\alpha_i>1,\ i=0,...,n+1$, then for $||u_1||\le 1$ we have $||u||_1^{\alpha_i}\le 1$, and for $||u||_1>1$ we have $||u||_1^{\alpha_i}\le||u||_1^{\alpha}$, where

$$\alpha=\max_{0\le i\le n+1}\alpha_i>1. \tag{3.21}$$

Therefore, in view of (3.19), (3.20), from (3.18) we obtain

$$||Au||_{W_0^{2,2k+1}(D_T)}\le a_1(T)||u||_1^{\alpha}+b_1(T)\ \ \forall u\in W_0^{2,2k+1}(D_T), \tag{3.22}$$

where

$$a_1(T) = \sum_{i=0}^{n+1} \tilde{a}_i(T), \ \ b_1(T) = \sum_{i=0}^{n+1} \tilde{a}_i(T) + b(T), \tag{3.23}$$

where $\tilde{a}_i(T), \ i = 0, 1, ..., n+1$, and $b(T)$ are defined by the equalities (3.19) and (3.20).

4. Some cases of existence and absence of solutions of the problem (1.1), (1.2), (1.3)

In this section, assuming that

$$F : D_\infty \to \mathbb{R}, \ \ F|_{D_T} \in L_2(D_T) \ \forall T > 0, \tag{4.1}$$

where $D_\infty := \Omega \times (0, \infty)$, under certain assumptions imposed on nonlinear function f we prove the existence of positive number T_0 such that for $0 < T < T_0$ the problem (1.1), (1.2), (1.3) has at least one generalized solution $u \in \overset{\circ}{W}{}^{2,2k+1}_0(D_T)$ in the sense of Definition 1.1, and for sufficiently large T this problem may not have a solution in the domain D_T. We also single out the class of nonlinear functions f, when for any F satisfying condition (4.1) the problem (1.1), (1.2), (1.3) is solvable in the domain D_T for any $T > 0$.

According to the estimate (3.22) consider the following algebraic equation

$$a_1 z^\alpha + b_1 = z \tag{4.2}$$

with respect to unknown $z > 0$, where $a_1 = a_1(T)$ and $b_1 = b_1(T)$, are defined by equalities (3.23).

For $T > 0$ due to (3.19), (3.20) and (3.23) it is clear that $a_1 > 0$ and $b_1 > 0$. Simple analysis, analogous to that given in [19], for $\alpha = 3$ shows that: 1) in the case $0 < b_1 < b_0$, where

$$b_0 = [\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}] a_1^{-\frac{1}{\alpha-1}}, \tag{4.3}$$

equation (4.2) has two positive roots z_1 and z_2. For $b_1 = b_0$ these roots coincide and we have only one positive root

$$z_1 = z_2 = z_0 = (\alpha a_1)^{-\frac{1}{\alpha-1}};$$

2) if $b_1 > b_0$, then equation (4.2) does not have non-negative roots.

Note that in the case $0 < b_1 < b_0$ there are valid inequalities

$$z_1 < z_0 = (\alpha a_1)^{-\frac{1}{\alpha-1}} < z_2.$$

In view of (3.17), (3.19), (3.20), (3.23) and (4.3) the condition $b_1 < b_0$ is equivalent to the condition

$$g(T) := a_1^{\frac{\alpha}{\alpha-1}}(T) + a_1^{\frac{1}{\alpha-1}}(T)\big[c_1^{-1}(n+3)^{\frac{1}{2}}(M^2 \mathrm{mes}\,\Omega)^{\frac{1}{2}} T^{\frac{1}{2}} +$$
$$+ c_1^{-1}\|F\|_{L_2(D_T)}\big] < \alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}. \tag{4.4}$$

Since the Lebesgue measure is absolutely continuous then due to (4.1) we have

$$\lim_{T\to 0} \|F\|_{L_2(D_T)} = 0,$$

hence, due to (3.15), (3.19), (3.23) and (4.4) from (4.3) we get

$$\lim_{T\to 0} g(T) = 0. \tag{4.5}$$

At the same time since $\alpha > 1$, then the right hand part of inequality (4.4) is positive. Therefore, due to (4.5) there exists a positive number $T_0 = T_0(F)$ such that $b_1 < b_0$ when the condition

$$0 < T < T_0(F) \tag{4.6}$$

is fulfilled. Thus, if T satisfies inequality (4.6), then operator

$$A : \overset{\circ}{W}{}^{2,2k+1}(D_T) \to \overset{\circ}{W}{}^{2,2k+1}(D_T),$$

acting by formula (3.6), maps the ball $B(0, z_2) := \{u \in \overset{\circ}{W}{}^{2,2k+1}(D_T) : ||u||_{\overset{\circ}{W}{}^{2,2k+1}(D_T)} \le z_2\}$ into itself, where $z_2 = z_2(T)$ is a maximal positive root of the equation (4.2). Indeed, if $u \in B(0, z_2)$, then due to (3.22) and (4.2) we have

$$||Au||_{\overset{\circ}{W}{}^{2,2k+1}(D_T)} \le a_1||u||_1^\alpha + b_1 \le a_1 z_2^\alpha + b_1 = z_2.$$

Therefore, taking into account that operator A is continuous and compact, and maps closed convex ball $B(0, z_2) \subset \overset{\circ}{W}{}^{2,2k+1}(D_T)$ into itself, then according to the Schauder's theorem [3] the equation (3.6) has at least one solution u from the space $\overset{\circ}{W}{}^{2,2k+1}(D_T)$ and at the same time it represents a weak generalized solution of the problem (1.1), (1.2), (1.3) in the sense of Definition 1.1.

Thus, due to suppositions made above on domain Ω, nonlinear function f and right hand side F of equation (1.1), and also reasonings given above, the following theorem is valid.

Theorem 4.1. *Let the domain Ω be bounded and convex in $\mathbb{R}^n$ with boundary $\partial\Omega$ of the class C^2; function $a = a(x,t)$, $(x,t) \in \Gamma$, satisfy condition (2.2); nonlinear function f satisfy conditions (1.10), (1.11), (1.13) and (3.13); function F satisfy condition (4.1). Then there exists a number $T_0 = T_0(F) > 0$ such that for $0 < T < T_0$ the problem (1.1), (1.2), (1.3) has at least one weak generalized solution $u \in \overset{\circ}{W}{}^{2,2k+1}(D_T)$ in the sense of the Definition 1.1.*

Now let us consider the case of nonlinear function f:

$$f(s_0, s_1, ..., s_{n+1}) \le -|s_0|^\alpha \ \ \forall (s_0, s_1, ..., s_{n+1}) \in \mathbb{R}^{n+2}, \tag{4.7}$$

where $\alpha = const > 1$, when the problem (1.1), (1.2), (1.3) may not have a solution.

Theorem 4.2. *Let $\Omega : |x| < 1$, function a satisfy condition (2.2), function f satisfy conditions (1.10), (1.11), (1.13), (3.13) and (4.7), function $F = \mu F_0$ with $\mu = const > 0$ and $F_0 > 0$ be defined in D_∞ and $F_0\big|_{D_T} \in L_2(D_T)$ $\forall T > 0$. Then for any fixed $T > 0$ there exists a number $\mu_0 = \mu_0(F_0, \alpha) > 0$ such that for $\mu > \mu_0$ the problem (1.1), (1.2), (1.3) cannot have a weak generalized solution in the space $\overset{\circ}{W}{}^{2,2k+1}(D_T)$ in the sense of Definition 1.1.*

Proof. Assume that the problem (1.1), (1.2), (1.3) has a solution $u \in \overset{\circ}{W}{}^{2,2k+1}(D_T)$ for $F = \mu F_0$ and any $\mu > 0$. According to Definition 1.1 function u satisfies equality (1.16) for any $\varphi \in \overset{\circ}{W}{}^{2,2k+1}(D_T)$. Below we use the method of test functions

[18]. As a test function we take a function φ satisfying the following conditions

$$\varphi\in C_0^{4,4k+2}(\overline{D}_T),\quad \varphi|_\Gamma=\frac{\partial\varphi}{\partial\boldsymbol{\nu}}\bigg|_\Gamma=0,\quad \varphi|_{D_T}>0, \tag{4.8}$$

where the space $C_0^{4,4k+2}(\overline{D}_T)$ is defined in (1.4).

By integration by parts in integral equality (1.16) and taking into account (4.8) we get

$$-\int\limits_{D_T} f(u,\nabla u)\varphi dxdt=\int\limits_{D_T} u\Big[\frac{\partial^{2(2k+1)}\varphi}{\partial t^{2(2k+1)}}-\Delta^2\varphi\Big]dxdt-$$

$$-\int\limits_{D_T} F\varphi dxdt=\int\limits_{D_T} uL_0\varphi dxdt-\mu\int\limits_{D_T} F_0\varphi dxdt, \tag{4.9}$$

where operator L_0 is defined in (1.1), when $f=0$ and $F=\mu F_0$.

Due to (4.7) and (4.8) from (4.9) follows that

$$\int_{D_T}|u|^\alpha\varphi dxdt\le\int_{D_T} uL_0\varphi dxdt-\mu\int\limits_{D_T} F_0\varphi dxdt. \tag{4.10}$$

If in Young's inequality with parameter $\varepsilon>0$

$$ab\le\frac{\varepsilon}{\alpha}a^\alpha+\frac{1}{\alpha'\varepsilon^{\alpha'-1}}b^{\alpha'};\ a,b\ge0,\ \alpha'=\frac{\alpha}{\alpha-1}$$

we take $a=|u|\varphi^{1/\alpha}$, $b=|L_0\varphi|/\varphi^{1/\alpha}$, then taking into account that $\alpha'/\alpha=\alpha'-1$ we have

$$|uL_0\varphi|=|u|\varphi^{1/\alpha}\frac{|L_0\varphi|}{\varphi^{1/\alpha}}\le\frac{\varepsilon}{\alpha}|u|^\alpha\varphi+\frac{1}{\alpha'\varepsilon^{\alpha'-1}}\frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}. \tag{4.11}$$

From (4.10), (4.11) we have the inequality

$$\Big(1-\frac{\varepsilon}{\alpha}\Big)\int\limits_{D_T}|u|^\alpha\varphi dxdt\le\frac{1}{\alpha'\varepsilon^{\alpha'-1}}\int\limits_{D_T}\frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}dxdt-\mu\int\limits_{D_T} F_0\varphi dxdt,$$

whence for $\varepsilon<\alpha$ we get

$$\int\limits_{D_T}|u|^\alpha\varphi dxdt\le\frac{\alpha}{(\alpha-\varepsilon)\,\alpha'\varepsilon^{\alpha'-1}}\int\limits_{D_T}\frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}dxdt-\frac{\alpha\mu}{\alpha-\varepsilon}\int\limits_{D_T} F_0\varphi dxdt. \tag{4.12}$$

Taking into account the equalities $\alpha'=\frac{\alpha}{\alpha-1}$, $\alpha=\frac{\alpha'}{\alpha'-1}$ and $\min\limits_{0<\varepsilon<\alpha}\frac{\alpha}{(\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}}=1$ which is achieved at $\varepsilon=1$, from (4.12) we find that

$$\int\limits_{D_T}|u|^\alpha\varphi dxdt\le\int\limits_{D_T}\frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}dxdt-\alpha'\mu\int\limits_{D_T} F_0\varphi dxdt. \tag{4.13}$$

It is not difficult to show the existence of a test function φ such that it satisfies conditions (4.8) and the following condition

$$\kappa_0=\int\limits_{D_T}\frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}dxdt<+\infty. \tag{4.14}$$

Indeed, as it can be easily verified, the function

$$\varphi(x,t) = [(1-|x|^2)t(T-t)]^m$$

for a sufficiently large positive m satisfies conditions (4.8) and (4.14).

Since $F_0 \in L_2(D_T)$, $F_0\big|_{D_T} > 0$, and $\varphi|_{D_T} > 0$ we have

$$0 < \kappa_1 = \int\limits_{D_T} F_0 \varphi dxdt < +\infty. \tag{4.15}$$

Denote by $g(\mu)$ the right-hand side of the inequality (4.13) which is a linear function with respect to μ. In view of (4.14) and (4.15) we have

$$g(\mu) < 0 \text{ for } \mu > \mu_0 \text{ and } g(\mu) > 0 \text{ for } \mu < \mu_0, \tag{4.16}$$

where

$$g(\mu) = \kappa_0 - \alpha' \mu \kappa_1, \ \ \mu_0 = \frac{\kappa_0}{\alpha' \kappa_1} > 0.$$

Due to (4.16) for $\mu > \mu_0$, the right-hand side of the inequality (4.13) is negative, whereas the left-hand side of that inequality is nonnegative. The obtained contradiction proves the theorem. □

Remark 4.1. Note that in the Theorem 4.2 for simplicity we assume $\Omega : |x| < 1$. However, this theorem is valid in more general case, when Ω represents a convex, sufficiently smooth boundary $\partial\Omega$. Our assumption was caused by the construction of a test function φ satisfying conditions (4.8) and (4.14) according to the formula

$$\varphi(x,t) = [(1-|x|^2)t(T-t)]^m \tag{4.17}$$

for a sufficiently large positive m. If the boundary of the convex domain Ω is given by the equation $\partial\Omega : \omega(x) = 0$, where $\nabla_x \omega|_{\partial\Omega} \neq 0$, $\omega|_{\Omega} > 0$ and $\omega \in C^4(R^n)$, then, instead of the test function defined by formula (4.17), we should take

$$\varphi(x,t) = [\omega(x)t(T-t)]^m,$$

where m is a sufficiently large positive number, and in this case the Theorem 4.2 remains valid.

Remark 4.2. From the Theorems 4.1 and 4.2 follows that for function F, satisfying condition (4.1) for sufficiently small $T > 0$ the problem (1.1), (1.2), (1.3) is always solvable, although for sufficiently large T it may not have a solution. Below we consider the class of nonlinear functions f when for any function F, satisfying condition (4.1) the problem (1.1), (1.2), (1.3) has at least one solution for any fixed $T > 0$.

Theorem 4.3. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of the class C^2, function $a = a(x,t)$, $(x,t) \in \Gamma$ satisfy condition (2.2), nonlinear function $f = f(s_0, s_1, ..., s_{n+1})$, satisfying conditions (1.10) - (1.13), be representable in the form of product $f = f_0(s_0) f_1(s_1, ..., s_{n+1})$, where $f_0 \in C(\mathbb{R})$, $f_1 \in C(\mathbb{R}^{n+1})$ and*

$$0 \le f_1(s_1, ..., s_{n+1}) \le const < +\infty, \ \ \lim_{|s_0| \to \infty} \sup \frac{f(s_0)}{s_0} \le 0, \tag{4.18}$$

and function F satisfy condition (4.1). Then for any fixed $T > 0$ the problem (1.1), (1.2), (1.3) has at least one weak generalized solution $u \in \overset{\circ}{W}{}^{2,2k+1}_{0}(D_T)$ in the sense of the Definition 1.1.

Proof. First let us consider a priori estimate for a solution $u \in \overset{\circ}{W}{}^{2,2k+1}_0(D_T)$ of the problem (1.1), (1.2), (1.3). Since $f_0 \in C(\mathbb{R})$ then due to (4.18) for any $\varepsilon > 0$ there exists a number K_ε such that

$$s_0 f(s_0, s_1, ..., s_{n+1}) \le K_\varepsilon + \varepsilon s_0^2 \ \ \forall (s_0, s_1, ..., s_{n+1}) \in \mathbb{R}^{n+2}. \tag{4.19}$$

Assuming that $\varphi = u \in W^{2,2k+1}_0(D_T)$ in equality (1.16) and taking into account (4.19) and (2.4) for any ε we get

$$\begin{aligned} ||u||_1^2 &= \int\limits_{D_T} u f(u, \nabla u) dxdt - \int\limits_{D_T} F u dxdt \le \\ &\le K_\varepsilon \text{mes} D_T + \varepsilon \int_{D_T} u^2 dxdt + \int_{D_T} \Big(\frac{1}{4\varepsilon} F^2 + \varepsilon u^2\Big) dxdt = \\ &= \frac{1}{4\varepsilon} ||F||^2_{L_2(D_T)} + K_\varepsilon \text{mes} D_T + 2\varepsilon ||u||^2_{L_2(D_T)} \le \\ &\le \frac{1}{4\varepsilon} ||F||^2_{L_2(D_T)} + K_\varepsilon \text{mes} D_T + 2\varepsilon ||u||_0^2. \end{aligned} \tag{4.20}$$

From (4.20) by virtue of (2.5) we have

$$c_1^2 ||u||_0^2 \le ||u||_1^2 \le \frac{1}{4\varepsilon} ||F||^2_{L_2(D_T)} + K_\varepsilon \text{mes} D_T + 2\varepsilon ||u||_0^2,$$

whence for $\varepsilon = \frac{1}{4} c_1^2$ we obtain

$$||u||_0^2 \le 2c_1^{-4} ||F||^2_{L_2(D_T)} + 2c_1^{-2} K_\varepsilon \text{mes} D_T.$$

From the last inequality follows the following a priori estimate for the solution $u \in W^{2,2k+1}_0(D_T)$ of the problem (1.1), (1.2), (1.3)

$$||u||_0 = ||u||_{W^{2,2k+1}_0(D_T)} \le c_3 ||F||_{L_2(D_T)} + c_4 \tag{4.21}$$

with constants $c_3 = (2c_1^{-4})^{1/2}$ and $c_4 = (2c_1^{-2} K_\varepsilon \text{mes} D_T)^{1/2}$, not dependent on u and F, where $\varepsilon = \frac{1}{4} c_1^2$.

According to the Remark 3.1 the problem (1.1), (1.2), (1.3) is equivalent to the functional equation (3.6), where the operator A, acting in the Hilbert space $W^{2,2k+1}_0(D_T)$, is continuous and compact. At the same time according to the a priori estimate (4.21) for the solution of equation $u = Au$ from (3.6) is valid and for the solution of equation $u = \tau A u$ with parameter $\tau \in [0, 1]$ with the same constants c_3 and c_4. Therefore, by the Schaefer's fixed point theorem [3] equation (3.6), and hence the problem (1.1), (1.2), (1.3), has at least one weak generalized solution u from the space $W^{2,2k+1}_0(D_T)$ in the sense of the Definition 1.1. This concludes the proof of the theorem. □

5. The uniqueness of a solution of the problem (1.1), (1.2), (1.3)

Theorem 5.1. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ with a boundary $\partial\Omega$ of the class C^2, function $a = a(x, t), (x, t) \in \Gamma$, satisfy condition (2.2), a nonlinear function $f = f(s_0)$, dependent only on variable s_0, be monotonically decreasing and satisfy conditions (1.10), (1.11), (1.12). Then for any $F \in L_2(D_T)$ the problem (1.1), (1.2), (1.3) cannot have more than one weak generalized solution in the space $W^{2,2k+1}_0(D_T)$.*

Proof. Let $F \in L_2(D_T)$ and u_1, u_2 be two weak generalized solutions of the problem (1.1), (1.2), (1.3) from the space $W_0^{1,2k}(D_T)$, i.e., according to (1.16) the equalities

$$\int\limits_{D_T} \Big[\frac{\partial^{2k+1}u_i}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1}\varphi}{\partial t^{2k+1}} + \Delta u_i \cdot \Delta\varphi\Big] dxdt + \int\limits_{\Gamma} a \frac{\partial u_i}{\partial \boldsymbol{\nu}} \cdot \frac{\partial \varphi}{\partial \boldsymbol{\nu}} ds -$$

$$- \int\limits_{D_T} f(u_i)\varphi dxdt = - \int_{D_T} F\varphi dxdt \ \ \forall \varphi \in W_0^{2,2k+1}(D_T), \ i = 1, 2, \tag{5.1}$$

are valid.

From (5.1), for the difference $v = u_2 - u_1$ we have

$$\int\limits_{D_T} \Big[\frac{\partial^{2k+1}v}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1}\varphi}{\partial t^{2k+1}} + \Delta v \cdot \Delta\varphi\Big] dxdt + \int\limits_{\Gamma} a \frac{\partial v}{\partial \boldsymbol{\nu}} \cdot \frac{\partial \varphi}{\partial \boldsymbol{\nu}} ds =$$

$$= \int\limits_{D_T} \big(f(u_2) - f(u_1)\big)\varphi dxdt \ \ \forall \varphi \in W_0^{2,2k+1}(D_T). \tag{5.2}$$

Putting $\varphi = v \in W_0^{2,2k+1}(D_T)$ in the equality (5.2), due to (2.4) we obtain

$$||v||_1 = \int\limits_{D_T} \big(f(u_2) - f(u_1)\big)(u_2 - u_1) dxdt. \tag{5.3}$$

Since f is the monotonically decreasing function, we have

$$\big(f(s_2) - f(s_1)\big)(s_2 - s_1) \le 0 \quad \forall s_1, s_2 \in \mathbb{R}. \tag{5.4}$$

From (2.5), (5.3) and (5.4) it follows that

$$c_1||v||_0 \le ||v||_1 \le 0,$$

whence we find that $v = 0$, i.e., $u_2 = u_1$, and hence the proof of the Theorem 5.1 is complete. □

From Theorems 4.3 and 5.1, in its turn, follows

Theorem 5.2. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ with a boundary $\partial\Omega$ of the class C^2, function $a = a(x,t), (x,t) \in \Gamma$, satisfy condition (2.2), a nonlinear function $f = f(s_0)$, dependent only on variable s_0, be monotonically decreasing and satisfy conditions (1.10), (1.11), (1.12) and*

$$\lim_{|s_0|\to\infty} \sup \frac{f(s_0)}{s_0} \le 0.$$

Then for any $F \in L_2(D_T)$ the problem (1.1), (1.2), (1.3) has a unique weak generalized solution in the space $W_0^{2,2k+1}(D_T)$ in the sense of the Definition 1.1.

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