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NEW BEREZIN RADIUS UPPER BOUNDS

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Abstract. In this paper, we provide the new Berezin radius inequalities on the space of operators defined on a functional Hilbert space. By using these inequalities, we obtain various upper bounds for the Berezin radius of functional Hilbert space operators. We prove, in particular, the following sharp upper bound

$$\operatorname{ber}^2(S^*T) \leq \frac{1}{2\xi+2}\operatorname{ber}(S^*T)\left\| |T|^2+|S|^2 \right\|_{\operatorname{ber}} + \frac{\xi}{2\xi+2}\left\| |T|^4+|S|^4 \right\|_{\operatorname{ber}}$$

for arbitrary $T, S \in \mathbb{B}(\mathcal{H})$ and $\xi \geq 0$. Other related issues are also discussed.

1. Introduction

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle .,. \rangle)$ with the identity operator $1_{\mathcal{H}}$ in $\mathbb{B}(\mathcal{H})$. An operator $T \in \mathbb{B}(\mathcal{H})$ is called positive if $\langle Tx, x\rangle \geq 0$ for any $x \in \mathcal{H}$. We work in functional Hilbert space (FHS) throughout this paper. These are complete inner-product spaces made up of complex-valued functions defined on a set Υ with bounded point evaluation. Formally, if Υ is a set and $\mathcal{H} = \mathcal{H}(\Upsilon)$ is a subset of all functions $\Upsilon \to \mathbb{C}$, then $\mathcal{H}$ is a FHS on Υ if it is a complete inner product space and point evaluation at each $\tau \in \Upsilon$ is a bounded linear functional on $\mathcal{H}$. We know from the traditional Riesz representation theorem that if $\mathcal{H}$ is a FHS on Υ, there is a unique element $k_\tau \in \mathcal{H}$ such that $h(\tau) = \langle h, k_\tau\rangle_{\mathcal{H}}$ for every $\tau \in \Upsilon$ and all $h \in \mathcal{H}$. The reproducing kernel is denoted by the element k_τ. In addition, we will designate the normalized reproducing kernel at τ as $\widehat{k}_\tau := \frac{k_\tau}{\|k_\tau\|_{\mathcal{H}}}$. For a bounded linear operator T on $\mathcal{H}$, the function $\widetilde{T}$ defined on Υ by $\widetilde{T}(\tau) := \left\langle T\widehat{k}_\tau, \widehat{k}_\tau \right\rangle$ is the Berezin symbol of T, which firstly have been introduced by Berezin [8]. The Berezin set (or range) and the Berezin number (or radius) of the operator T are defined by

$$\operatorname{Ber}(T) := \operatorname{Range}(\widetilde{T}) = \left\{ \widetilde{T}(\tau) : \tau \in \Upsilon \right\} \text{ and } \operatorname{ber}(T) := \sup_{\tau\in\Upsilon} \left| \widetilde{T}(\tau) \right|,$$

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respectively, (see [18, 19]). We also define the following so-called Berezin norm of operators $T \in \mathbb{B}(\mathcal{H})$:

$$\|T\|_{\text{Ber}} := \sup_{\tau \in \Upsilon} \left\| T\widehat{k}_\tau \right\|.$$

It is easy to see that actually $\|T\|_{\text{Ber}}$ determines a new operator norm in $\mathbb{B}(\mathcal{H}(\Upsilon))$ (since the set of reproducing kernels $\{k_\tau : \tau \in \Upsilon\}$ span the space $\mathcal{H}(\Upsilon)$). It is also trivial that $\operatorname{ber}(T) \leq \|T\|_{\text{Ber}} \leq \|T\|$. For the basic properties and facts on these new concepts, see [5, 6, 7, 15, 24].

In an FHS, the Berezin range of an operator T is a subset of the numerical range of T,

$$W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}(\Omega) \text{ and } \|x\| = 1\}.$$

Hence

$$\operatorname{ber}(T) \leq w(T) := \sup\{|\langle Tx, x\rangle| : x \in \mathcal{H}(\Omega) \text{ and } \|x\| = 1\}$$

(the numerical radius of operator T). The numerical range of an operator has some interesting properties. For example, it is well known that the spectrum of an operator is contained in the closure of its numerical range. For some results about the numerical radius inequalities and their applications, we refer to see [2, 4, 9, 21, 22, 23]. The Berezin radius of T satisfies the following: $\operatorname{ber}(\xi T) = |\xi| \operatorname{ber}(T)$ for all $\xi \in \mathbb{C}$ and $\operatorname{ber}(T+S) \leq \operatorname{ber}(T) + \operatorname{ber}(S)$ for bounded linear operators $T, S \in \mathcal{H}$. It is well-known that

$$\operatorname{ber}(T) \leq w(T) \leq \|T\| \tag{1.1}$$

and

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|, \tag{1.2}$$

for $T \in \mathbb{B}(\mathcal{H})$ where the sharpness holds when $T^2 = 0$ and T is normal for the first and second inequalities respectively.

In [17, Theorem 3.1], Huban et al. significantly improved the upper bound in (1.1) by demonstrating that if $T \in \mathbb{B}(\mathcal{H})$, then

$$\operatorname{ber}(T) \leq \frac{1}{2}\||T| + |T^*|\|_{\text{ber}}. \tag{1.3}$$

Another improvement for the inequality (1.2) was provided by Huban et al. [16, Corollary 3.3.] as

$$\operatorname{ber}^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|_{\text{ber}}, \tag{1.4}$$

which was further improved in [7] by Başaran and Gürdal as

$$\operatorname{ber}^2(T) \leq \frac{1}{6}\left\||T|^2 + |T^*|^2\right\|_{\text{ber}} + \frac{1}{3}\operatorname{ber}(T)\||T| + |T^*|\|_{\text{ber}}. \tag{1.5}$$

In [16, Theorem 3.11], Huban et al. proved the following estimate of the Berezin radius of the product of two functional Hilbert space operators

$$\operatorname{ber}^p(S^*T) \leq \frac{1}{2}\left\||T|^{2p} + |S|^{2p}\right\| \ (p \geq 1). \tag{1.6}$$

Başaran and Gürdal [7, p. 6] improved the last inequality for the case $p = 2$ by demonstrating that

$$\operatorname{ber}^2(S^*T) \leq \frac{1}{6}\left\||T|^4 + |T^*|^4\right\| + \frac{1}{3}\operatorname{ber}(S^*T)\left\||T|^2 + |T^*|^2\right\|. \tag{1.7}$$

The Berezin transform has been studied in details for Toeplitz and Hankel operators on Hardy and Bergman spaces. The Berezin transform and Berezin radius have been studied by many mathematicians over the years, a few of them are [10, 11, 12, 13, 20, 25]. The purpose of this paper is to establish several refinements of the above Berezin radius inequalities for functional Hilbert space operators. Our results, in particular, refine inequalities (1.4) and (1.6) for the situation $p = 2$ and provide specific instances for inequalities (1.5) and (1.7). Other related issues are also discussed.

2. Known Lemmas

In order to achieve our goal, we need the following sequence of corollaries.

The Cauchy-Schwarz inequality states that for all vectors u and v in an inner product space

$$|\langle u,v\rangle| \leq \|u\| \, \|v\| \tag{2.1}$$

where $\langle .,.\rangle$ is the inner product and $\|u\| = \sqrt{\langle u,u\rangle}$.

Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is called positive if $\langle Tu,u\rangle \geq 0$ for all $u \in \mathcal{H}$. In this case we will write $T \geq 0$. The classical operator Jensen inequality for the positive operators $T \in \mathbb{B}(\mathcal{H})$ is

$$\langle Tu,u\rangle^p \leq (\geq) \langle T^p u,u\rangle, \ p \geq 1 \ \ (0 \leq p \leq 1) \tag{2.2}$$

for any unit vector $u \in \mathcal{H}$.

The following lemma which can be found in [1, Theorem 2.3] gives a norm inequality involving convex function of positive operators.

Lemma 2.1. *Let f be a non-negative non-decreasing convex function on $[0,\infty)$ and $T, S \in \mathbb{B}(\mathcal{H})$ be two positive operators. Then we have*

$$\left\| f\left(\frac{T+S}{2}\right)\right\| \leq \left\| \frac{f(T)+f(S)}{2}\right\|. \tag{2.3}$$

In particular,

$$\left\| \left(\frac{T+S}{2}\right)^p\right\| \leq \left\| \frac{T^p+S^p}{2}\right\|, \ \ (p \geq 1).$$

The next lemma is found in [21].

Lemma 2.2. *If $T \in \mathbb{B}(\mathcal{H})$, then we have*

$$|\langle Tu,v\rangle| \leq \sqrt{\langle |T|\, u,u\rangle \langle |T^*|\, v,v\rangle} \tag{2.4}$$

for any $u, v \in \mathcal{H}$.

Lemma 2.3 ([2]). *If $x, y \in \mathcal{H}$, then we have*

$$|\langle x,y\rangle|^2 \leq \frac{1}{\xi+1}\|x\| \, \|y\| \, |\langle x,y\rangle| + \frac{\xi}{\xi+1}\|x\|^2 \|y\|^2 \leq \|x\|^2 \|y\|^2. \tag{2.5}$$

for any $\xi \geq 0$.

3. The Main Results

In this part, we will present our key conclusions on Berezin radius upper bounds, which are based on a novel refinement of the famous Cauchy-Schwarz inequality.

Now, we are ready to present the main results of this section. This result gives the refinement of the inequality (1.6).

Theorem 3.1. *Let $T, S \in \mathbb{B}(\mathcal{H})$. Then for any $\xi \geq 0$,*

$$\operatorname{ber}^2(S^*T) \leq \frac{1}{2\xi+2}\operatorname{ber}(S^*T)\left\||T|^2+|S|^2\right\|_{\mathrm{ber}} + \frac{\xi}{2\xi+2}\left\||T|^4+|S|^4\right\|_{\mathrm{ber}}. \quad (3.1)$$

Proof. Let $\tau \in \Upsilon$ be an arbitrary. Putting $x = T\widehat{k}_\tau$ and $y = S\widehat{k}_\tau$ in the inequality (2.5), we have

$$\begin{aligned}
&\left|\left\langle T\widehat{k}_\tau, S\widehat{k}_\tau\right\rangle\right|^2 \leq \frac{1}{\xi+1}\left\|T\widehat{k}_\tau\right\|\left\|S\widehat{k}_\tau\right\|\left|\left\langle T\widehat{k}_\tau, S\widehat{k}_\tau\right\rangle\right| + \frac{\xi}{\xi+1}\left\|T\widehat{k}_\tau\right\|^2\left\|S\widehat{k}_\tau\right\|^2 \\
&= \frac{1}{\xi+1}\left|\left\langle S^*T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|\sqrt{\left\langle |T|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle}\sqrt{\left\langle |S|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle} \\
&+ \frac{\xi}{\xi+1}\left\langle |T|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\left\langle |S|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\leq \frac{1}{2\xi+2}\left|\left\langle S^*T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|\left\langle\left(|T|^2+|S|^2\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&+ \frac{\xi}{2\xi+2}\left(\left\langle |T|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^2 + \left\langle |S|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^2\right) \\
&\leq \frac{1}{2\xi+2}\left|\left\langle S^*T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|\left\langle\left(|T|^2+|S|^2\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \frac{\xi}{2\xi+2}\left\langle\left(|T|^4+|S|^4\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle
\end{aligned}$$

where the third inequality follows from the arithmetic-geometric mean inequality and the fourth inequality follows from the inequality (2.2). So, we get

$$\begin{aligned}
\left|\left\langle T\widehat{k}_\tau, S\widehat{k}_\tau\right\rangle\right|^2 &\leq \frac{1}{2\xi+2}\left|\left\langle S^*T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|\left\langle\left(|T|^2+|S|^2\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&+ \frac{\xi}{2\xi+2}\left\langle\left(|T|^4+|S|^4\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle.
\end{aligned}$$

Taking the supremum over $\tau \in \Upsilon$ in the above inequality, we deduce

$$\begin{aligned}
&\sup_{\tau\in\Upsilon}\left|\left\langle S^*T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^2 \\
&\leq \sup_{\tau\in\Upsilon}\left\{\frac{1}{2\xi+2}\left|\left\langle S^*T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|\left\langle\left(|T|^2+|S|^2\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \frac{\xi}{2\xi+2}\left\langle\left(|T|^4+|S|^4\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right\}
\end{aligned}$$

which is equivalent to

$$\operatorname{ber}^2(S^*T) \leq \frac{1}{2\xi+2}\operatorname{ber}(S^*T)\left\||T|^2+|S|^2\right\|_{\mathrm{ber}} + \frac{\xi}{2\xi+2}\left\||T|^4+|S|^4\right\|_{\mathrm{ber}}.$$

This completes the proof. □

From Theorem 3.1 we get the following result, which is new refinement of inequality (1.6) given by Huban et al. [16, Theorem 3.11].

Corollary 3.1. *If* $T,S\in\mathbb{B}(\mathcal{H})$, *then*

$$\begin{aligned}\operatorname{ber}^2(S^*T) &\leq \frac{1}{2\xi+2}\operatorname{ber}(S^*T)\left\||T|^2+|S|^2\right\|_{\text{ber}}+\frac{\xi}{2\xi+2}\left\||T|^4+|S|^4\right\|_{\text{ber}}\\ &\leq \frac{1}{2}\left\||T|^4+|S|^4\right\|_{\text{ber}}\end{aligned}$$

for any $\xi\geq 0$.

Proof. Let $\tau\in\Upsilon$ be an arbitrary. Then, it follows from Theorem 3.1 and the inequalities (1.6) and (2.3) that

$$\begin{aligned}\operatorname{ber}^2(S^*T) &\leq \frac{1}{2\xi+2}\operatorname{ber}(S^*T)\left\||T|^2+|S|^2\right\|_{\text{ber}}+\frac{\xi}{2\xi+2}\left\||T|^4+|S|^4\right\|_{\text{ber}}\\ &\leq \frac{1}{4\xi+4}\left\||T|^2+|S|^2\right\|^2+\frac{\xi}{2\xi+2}\left\||T|^4+|S|^4\right\|_{\text{ber}}\\ &= \frac{1}{4\xi+4}\left\|\left(|T|^2+|S|^2\right)^2\right\|_{\text{ber}}+\frac{\xi}{2\xi+2}\left\||T|^4+|S|^4\right\|_{\text{ber}}\\ &\leq \frac{1}{2\xi+2}\left\||T|^4+|S|^4\right\|_{\text{ber}}+\frac{\xi}{2\xi+2}\left\||T|^4+|S|^4\right\|_{\text{ber}}\\ &= \frac{1}{2}\left\||T|^4+|S|^4\right\|_{\text{ber}}.\end{aligned}$$

This inequality gives required inequality. □

Remark 3.1. By choosing $\xi=\frac{1}{2}$ in (3.1), we have

$$\operatorname{ber}^2(S^*T)\leq\frac{1}{3}\operatorname{ber}(S^*T)\left\||T|^2+|S|^2\right\|_{\text{ber}}+\frac{1}{6}\left\||T|^4+|S|^4\right\|_{\text{ber}}$$

which was given in [7, p. 6]. Additionally, for particular values of ξ, the upper bound of (3.1) provides a more precise estimate than the upper bound of (1.7).

The following theorem offers an updated upper bound that will be used to improve inequality (1.4) given by Huban et al. [16, Corollary 3.3.].

Theorem 3.2. *If* $T\in\mathbb{B}(\mathcal{H})$ *and* $p\in[0,1]$, *then we have*

$$\operatorname{ber}^2(T)\leq\frac{p}{2}\left\||T|^2+|T^*|^2\right\|_{\text{ber}}+\frac{1-p}{2}\operatorname{ber}(T)\left\||T|+|T^*|\right\|_{\text{ber}}. \tag{3.2}$$

Proof. Let $\tau\in\Upsilon$ be an arbitrary. Using the arithmetic-geometric mean inequality and the inequality (2.4), we get

$$\begin{aligned}\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^2 &= p\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^2+(1-p)\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^2\\ &\leq p\left\langle |T|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle |T^*|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\\ &+(1-p)\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|\sqrt{\left\langle |T|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle}\sqrt{\left\langle |T^*|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle}\\ &\leq \frac{p}{2}\left\langle\left(|T|^2+|T^*|^2\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\frac{1-p}{2}\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|\left\langle\left(|T|+|T^*|\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle.\end{aligned}$$

Taking the supremum over $\tau\in\Upsilon$ in the above inequality, we get

$$\operatorname{ber}^2(T)\leq\frac{p}{2}\left\||T|^2+|T^*|^2\right\|_{\text{ber}}+\frac{1-p}{2}\operatorname{ber}(T)\left\||T|+|T^*|\right\|_{\text{ber}}.$$

Then the desired result has been obtained. □

By Theorem 3.2, our conclusion is as follows.

Corollary 3.2. *If* $T \in \mathbb{B}(\mathcal{H})$ *and* $p \in [0,1]$, *then we have*

$$\begin{aligned}\operatorname{ber}^2(T) &\leq \frac{p}{2}\left\| |T|^2 + |T^*|^2 \right\|_{\text{ber}} + \frac{1-p}{2}\operatorname{ber}(T)\, \| |T| + |T^*| \|_{\text{ber}} \\ &\leq \frac{1}{2}\left\| |T|^2 + |T^*|^2 \right\|_{\text{ber}}.\end{aligned}$$

Proof. Let $\tau \in \mathcal{H}$ be arbitrary. Then, it can be proved by using Theorem 3.2 and the inequalities (1.3) and (2.3) that

$$\begin{aligned}\operatorname{ber}^2(T) &\leq \frac{p}{2}\left\| |T|^2 + |T^*|^2 \right\|_{\text{ber}} + \frac{1-p}{2}\operatorname{ber}(T)\, \| |T| + |T^*| \|_{\text{ber}} \\ &\leq \frac{p}{2}\left\| |T|^2 + |T^*|^2 \right\|_{\text{ber}} + \frac{1-p}{4}\, \| |T| + |T^*| \|_{\text{ber}}^2 \\ &\leq \frac{p}{2}\left\| |T|^2 + |T^*|^2 \right\|_{\text{ber}} + \frac{1-p}{2}\left\| |T|^2 + |T^*|^2 \right\|_{\text{ber}} \\ &= \frac{1}{2}\left\| |T|^2 + |T^*|^2 \right\|_{\text{ber}}.\end{aligned}$$

This completes the proof. □

As a special case of Theorem 3.2 we have the following remark.

Remark 3.2. If we take $p = \frac{1}{3}$ in Theorem 3.2, we get [7, p. 13]

$$\operatorname{ber}^2(T) \leq \frac{1}{6}\left\| |T|^2 + |T^*|^2 \right\|_{\text{ber}} + \frac{1}{3}\operatorname{ber}(T)\, \| |T| + |T^*| \|_{\text{ber}}.$$

On the other hand, this inequality in the above is a special case of (3.2). Furthermore, for certain values of p, the new upper bound (3.2) offers a more accurate estimate than the one in (1.5).

The following lemma in [2] is needed for our next result.

Lemma 3.1. *If* $u, v, e \in \mathcal{H}$ *with* $\|e\| = 1$ *and* $\xi \geq 0$, *then we have*

$$|\langle u, e\rangle \langle e, v\rangle|^2 \leq \frac{2\xi + 1}{2\xi + 2}\|u\|^2 \|v\|^2 + \frac{1}{2\xi + 2}\|u\| \|v\| |\langle u, v\rangle|. \tag{3.3}$$

Now, in the following theorem we obtain a new upper bound.

Theorem 3.3. *If* $T \in \mathbb{B}(\mathcal{H})$ *and* $\xi \geq 0$, *then we have*

$$\operatorname{ber}^4(T) \leq \frac{2\xi + 1}{4\xi + 4}\left\| |T|^4 + |T^*|^4 \right\|_{\text{ber}} + \frac{1}{4\xi + 4}\operatorname{ber}\left(T^2\right)\left\| |T|^2 + |T^*|^2 \right\|_{\text{ber}}.$$

Proof. Let $\widehat{k}_\eta$ be a normalized reproducing kernel. By taking $e=\widehat{k}_\tau$, $u=T\widehat{k}_\tau$ and $v=T^*\widehat{k}_\tau$ in the inequality (3.3), then we get

$$\begin{aligned}
&\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^4\\
&\leq \frac{2\xi+1}{2\xi+2}\left\|T\widehat{k}_\tau\right\|^2\left\|T^*\widehat{k}_\tau\right\|^2+\frac{1}{2\xi+2}\left\|T\widehat{k}_\tau\right\|\left\|T^*\widehat{k}_\tau\right\|\left|\left\langle T^2\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|\\
&=\frac{2\xi+1}{2\xi+2}\left\langle |T|^2\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle |T^*|^2\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\\
&+\frac{1}{2\xi+2}\sqrt{\left\langle |T|^2\widehat{k}_\tau,\widehat{k}_\tau\right\rangle}\sqrt{\left\langle |T^*|^2\widehat{k}_\tau,\widehat{k}_\tau\right\rangle}\left|\left\langle T^2\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|\\
&\leq \frac{2\xi+1}{4\xi+4}\left\langle\left(|T|^4+|T^*|^4\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\frac{1}{4\xi+4}\left\langle\left(|T|^2+|T^*|^2\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left|\left\langle T^2\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^4\\
&\leq \frac{2\xi+1}{4\xi+4}\left\langle\left(|T|^4+|T^*|^4\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\frac{1}{4\xi+4}\left|\left\langle T^2\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|\left\langle\left(|T|^2+|T^*|^2\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle.
\end{aligned}$$

Taking the supremum over $\tau\in\Upsilon$ in the above inequality, we get

$$\operatorname{ber}^4(T)\leq\frac{2\xi+1}{4\xi+4}\left\||T|^4+|T^*|^4\right\|_{\mathrm{ber}}+\frac{1}{4\xi+4}\operatorname{ber}\left(T^2\right)\left\||T|^2+|T^*|^2\right\|_{\mathrm{ber}}.$$

as desired. □

Corollary 3.3. *If $T\in\mathbb{B}(\mathcal{H})$ and $\xi\geq 0$, then we have*

$$\begin{aligned}
\operatorname{ber}^4(T)&\leq\frac{2\xi+1}{4\xi+4}\left\||T|^4+|T^*|^4\right\|_{\mathrm{ber}}+\frac{1}{4\xi+4}\operatorname{ber}\left(T^2\right)\left\||T|^2+|T^*|^2\right\|_{\mathrm{ber}}\\
&\leq\frac{1}{2}\left\||T|^4+|T^*|^4\right\|_{\mathrm{ber}}.
\end{aligned}$$

Proof. Let $\tau\in\mathcal{H}$ be arbitrary. Then, it can be proved by using Theorem 3.3, the inequalities (1.6) and (2.3) that

$$\begin{aligned}
\operatorname{ber}^4(T)&\leq\frac{2\xi+1}{4\xi+4}\left\||T|^4+|T^*|^4\right\|_{\mathrm{ber}}+\frac{1}{4\xi+4}\operatorname{ber}\left(T^2\right)\left\||T|^2+|T^*|^2\right\|_{\mathrm{ber}}\\
&\leq\frac{2\xi+1}{4\xi+4}\left\||T|^4+|T^*|^4\right\|_{\mathrm{ber}}+\frac{1}{8\xi+8}\left\||T|^2+|T^*|^2\right\|_{\mathrm{ber}}^2\\
&=\frac{2\xi+1}{4\xi+4}\left\||T|^4+|T^*|^4\right\|_{\mathrm{ber}}+\frac{1}{8\xi+8}\left\|\left(|T|^2+|T^*|^2\right)^2\right\|_{\mathrm{ber}}\\
&\leq\frac{2\xi+1}{4\xi+4}\left\||T|^4+|T^*|^4\right\|_{\mathrm{ber}}+\frac{1}{4\xi+4}\left\||T|^4+|T^*|^4\right\|_{\mathrm{ber}}\\
&=\frac{1}{2}\left\||T|^4+|T^*|^4\right\|_{\mathrm{ber}}.
\end{aligned}$$

Hence, we get the required inequality. □

Next inequality reads as follows:

Corollary 3.4. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$\operatorname{ber}^4(T) \leq \frac{3}{8}\left\||T|^4+|T^*|^4\right\|_{\text{ber}}+\frac{1}{8}\operatorname{ber}\left(T^2\right)\left\||T|^2+|T^*|^2\right\|_{\text{ber}}.$$

Proof. To put $\xi = 1$ in Theorem 3.3. □

Remark 3.3. (i) It should be noted that Corollary 3.4 is same as the one obtained in [14, Theorem 2.5] by Gürdal et al.

(ii) Also, if T is normal (i.e. $TT^* = T^*T$), the inequalities in Theorem 3.2 and Theorem 3.3 become equalities.

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