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ALGORITHMS FOR RECOVERY OF DIFFUSION AND STURM–LIOUVILLE OPERATORS WITH SEMI-SEPARATED BOUNDARY CONDITIONS

IBRAHIM M. NABIEV, LEYLA I. MAMMADOVA, AND GULDANE S. MAMMEDZADEH

Abstract. Diffusion and Sturm-Liouville equations under semi-separated boundary conditions, one of which contains a quadratic function of the spectral parameter are considered in this paper. Some properties of the spectrum of the operators under consideration are given, uniqueness theorems are proved, and algorithms for solving inverse recovery problems for the corresponding boundary value problems are constructed.

1. Introduction

The spectral theory of differential operators finds numerous and varied applications in various issues of science, technology, and natural science. As is known [10, 24, 2], the problems of constructing systems for protecting devices from impact, obtaining the desired timbre of the sound of musical instruments, torsional vibrations of a rod with pulleys at the ends, the propagation of heat in the rod, at the ends of which concentrated heat capacities are placed, and electrical vibrations in a wire closed through lumped resistances, self-induction and capacitance lead to the need to solve spectral problems with a spectral parameter in the boundary conditions.

Direct and inverse problems of spectral analysis associated with problems of this type also play an important role in the study of some nonlinear evolutionary equations of mathematical physics. A huge number of papers on spectral theory have been published, among which we can mention [20, 26, 15, 12, 21, 13, 27, 4], which also contain extensive lists of references in this area. Since the 1970s, researchers have been intensively studying boundary value problems with non-separated boundary conditions. Currently, the range of such tasks is expanding in various directions. Among them, a special place is occupied by problems with a spectral parameter in the boundary condition (see [23, 11, 1, 16, 19, 22]).

In this paper we consider the diffusion and Sturm-Liouville equations for semi-separated (one separated, the other non-separated) boundary conditions. The non-separated boundary condition contains a quadratic function of the spectral parameter. Some properties of the spectrum of the operators under consideration

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are given, uniqueness theorems are proved, and algorithms for solving inverse recovery problems for the corresponding boundary value problems are constructed.

2. Asymptotics of the spectrum of boundary value problems

Consider the boundary value problem generated on the interval $[0,\pi]$ by the diffusion differential equation

$$y''(x)+\left[\lambda^2-2\lambda p\ (x)-q\ (x)\right]\ y(x)=0 \tag{2.1}$$

and boundary conditions

$$\begin{aligned} &y\,(0)=0,\\ &y'(0)+\left(m\lambda^2+\alpha\lambda+\beta\right)y(\pi)+y'(\pi)=0, \end{aligned} \tag{2.2}$$

where $p\ (x)\in W_2^1\,[0,\pi]\ ,\ q\ (x)\in L_2\,[0,\pi]$ are complex-valued functions, $m,\ \alpha,\ \beta$ are complex numbers, and $m\neq 0$. We denote by $W_2^n\,[0,\pi]$ the Sobolev space of complex-valued functions on the interval $[0,\pi]$ which have $n-1$ absolutely continuous derivatives and square-summable nth derivative on $[0,\pi]$. Problem (2.1) - (2.2) will be denoted by P.

Let $s(x,\lambda)$ be the solution of the equation (2.1), satisfying the initial conditions

$$s(0,\lambda)=0, s'(0,\lambda)=1.$$

For any x the functions $s(x,\lambda)$ and $s'(x,\lambda)$ are entire functions of exponential type of the variable λ. The eigenvalues of the problem P are zeros of the characteristic function

$$\Delta(\lambda)=1+\left(m\lambda^2+\alpha\lambda+\beta\right)s(\pi,\lambda)+s'(\pi,\lambda). \tag{2.3}$$

Using the known formulas [15]

$$s\ (\pi,\ \lambda)=\frac{\sin\pi\ (\lambda-a)}{\lambda}+c_0\frac{\sin\pi\ (\lambda-a)}{\lambda^2}-Q\frac{\cos\pi\ (\lambda-a)}{\lambda^2}+\frac{1}{\lambda^2}\int_{-\pi}^{\pi}\psi_1\ (t)\ e^{i\lambda t}dt,$$

$$s'\ (\pi,\ \lambda)=\cos\pi\ (\lambda-a)+c_1\frac{\cos\pi\ (\lambda-a)}{\lambda}+Q\frac{\sin\pi\ (\lambda-a)}{\lambda}+\frac{1}{\lambda}\int_{-\pi}^{\pi}\psi_2\ (t)\ e^{i\lambda t}dt,$$

where $a=\frac{1}{\pi}\int_0^{\pi}p\ (x)\ dx,\quad Q=\frac{1}{2}\int_0^{\pi}\left[q\ (x)+p^2\ (x)\right]\ dx,$

$$c_0=\frac{1}{2}\left[p\ (0)+p\ (\pi)\right],\quad c_1=\frac{1}{2}\left[p\ (0)-p\ (\pi)\right],\psi_m\,(t)\in L_2\,[-\pi,\pi]\,,\ m=1,\,2$$

and the Paley–Wiener theorem, from (2.3) we obtain the following representation for the function $\Delta\,(\lambda)$:

$$\begin{aligned} \Delta\,(\lambda)=1+m\,(\lambda-a)\sin\pi\,(\lambda-a)+(1-mQ)\cos\pi\,(\lambda-a)+\\ +(ma+\alpha+mc_0)\sin\pi\,(\lambda-a)+f\,(\lambda)\,, \end{aligned} \tag{2.4}$$

where$f\,(\lambda)=\int_{-\pi}^{\pi}\tilde{f}\,(t)\,e^{i\lambda t}dt,\ \tilde{f}\,(t)\in L_2\,[-\pi,\pi].$

By virtue of Lemma 1.3 in [15], for the sequence of zeros $\{\gamma_k\}$ $(k=0,\pm1,\pm2,...)$ of an entire function $\Delta\,(\lambda)$, representable in the form (2.4), for $|k|\to\infty$ the following asymptotic formula holds:

$$\gamma_k=k+a+\frac{(-1)^{k+1}-1+mQ}{m\pi k}+\frac{\tau_k}{k}, \tag{2.5}$$

where $\{\tau_k\}\in l_2$.

Along with the problem P, we also consider the boundary value problem D generated by the same equation (2.1) and the Dirichlet boundary conditions

$$y(0) = y(\pi) = 0. \tag{2.6}$$

The spectrum of this problem will be denoted by $\{\lambda_k\}$ $(k = \pm 1, \pm 2, ...)$. According to [15], this spectrum satisfies the asymptotic formula

$$\lambda_k = k + a + \frac{Q}{\pi k} + \frac{\eta_k}{k} \tag{2.7}$$

for $|k| \to \infty$, where $\{\eta_k\} \in l_2$.

The sequences $\{\gamma_k\}$, $\{\lambda_k\}$ and the number β will be called the spectral data of the pair of boundary value problems P and D.

3. Statement of the inverse problem. Uniqueness theorem

Consider the following inverse problem.

Inverse problem B. Based on the given spectral data of the boundary value problems P and D, construct the functions $p(x)$ and $q(x)$ in equation (2.1) and the coefficients m, α in the boundary conditions (2.2).

The following uniqueness theorem is valid.

Theorem 3.1. *If $p(0) = -p(\pi)$, then the boundary-value problems P and D are uniquely determined by their spectral data.*

Proof. According to the given spectra of $\{\gamma_k\}$ and $\{\lambda_k\}$ of the boundary value problems P and D, it is possible to uniquely determine the values a and Q, since according to asymptotic formulas (2.5) and (2.7) we have

$$a = \lim_{k\to\infty} (\gamma_k - k), Q = \pi \lim_{k\to\infty} k(\lambda_k - k - a). \tag{3.1}$$

From relation (2.5) it also easily follows that

$$\gamma_{2k} = 2k + a + \frac{mQ - 2}{2m\pi k} + \frac{\tau_{2k}}{2k}.$$

Hence, the first coefficient m of the quadratic function in (2.2) can be determined as follows:

$$m = \frac{2}{Q - 2\pi \lim\limits_{k\to\infty} k(\gamma_{2k} - 2k - a)}. \tag{3.2}$$

By virtue of Lemma 1.3 in [15], for an entire function $\frac{\Delta(\lambda)}{m}$, representable as

$$\frac{\Delta(\lambda)}{m} = \frac{1}{m} + (\lambda - a)\sin\pi(\lambda - a) + \left(\frac{1}{m} - Q\right)\cos\pi(\lambda - a) +$$

$$+\left(c_0 + \frac{\alpha}{m} + a\right)\sin\pi(\lambda - a) + \frac{f(\lambda)}{m}$$

the following expansion into an infinite product takes place:

$$\frac{\Delta(\lambda)}{m} == \pi(\gamma_{-0} - \lambda)(\gamma_{+0} - \lambda) \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\gamma_k - k}{k}.$$

Therefore,

$$\Delta(\lambda) = m\pi\left(\gamma_{-0} - \lambda\right)\left(\gamma_{+0} - \lambda\right) \prod_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\gamma_k - k}{k}. \tag{3.3}$$

Substituting $\lambda = \frac{1}{2} + a + 2k$ into (2.4) and taking into account that $c_0 = 0$ (due to the conditions of the theorem), we obtain

$$\Delta\left(\frac{1}{2} + a + 2k\right) = 1 + m\left(2k + \frac{1}{2}\right) + \alpha + ma + f\left(2k + \frac{1}{2} + a\right).$$

Then the parameter $\alpha$is recovered by the formula

$$\alpha = \lim_{k\to\infty}\left[\Delta\left(2k + a + \frac{1}{2}\right) - 2mk\right] - m\left(a + \frac{1}{2}\right) - 1, \tag{3.4}$$

since, by virtue of the Riemann-Lebesgue lemma $\lim\limits_{k\to\infty} f\left(2k + \frac{1}{2} + a\right) = 0$.

According to a given sequence $\{\lambda_k\}$, the characteristic function of the boundary value problem D can be recovered as an infinite product by the formula

$$s(\pi, \lambda) = \pi \prod_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\lambda_k - \lambda}{k}. \tag{3.5}$$

Knowing, $m,\ \alpha,\ \beta,\ s(\pi, \lambda)$ and $\Delta(\lambda)$, the characteristic function of the boundary value problem generated by equation (2.1) and boundary conditions

$$y(0) = y'(\pi) = 0 \tag{3.6}$$

we determine from relation (2.3):

$$s'(\pi, \lambda) = \Delta(\lambda) - \left(m\lambda^2 + \alpha\lambda + \beta\right) s(\pi, \lambda) - 1. \tag{3.7}$$

It is known [9] that the coefficients of equation (2.1) are uniquely determined from the zeros of this function and the sequence $\{\lambda_k\}$.

Thus, from the given values of $\{\gamma_k\}$, $\{\lambda_k\}$ and β, both the coefficients $p(x)$ and $q(x)$ of equation (2.1) and the parameters $m,\ \alpha$ of the boundary conditions (2.2) are uniquely recovered. Theorem is proved. □

4. Case $p(x) \equiv 0$

Let us consider a special case of equation (2.1) for $p(x) \equiv 0$ i.e., Sturm-Liouville equation

$$-y''(x) + q(x)\, y(x) = \lambda^2 y(x), 0 \leq x \leq \pi \tag{4.1}$$

under boundary conditions (2.2), where $m\alpha \neq 0$. This case has its own specific features. The amount of spectral data for reconstructing problem (4.1), (2.2), which will be denoted in what follows by P_0, is reduced by one spectrum, i.e. one spectrum and some number are already used as spectral data.

Let $s_0(x,\lambda)$ be a solution of equation (4.1) satisfying the initial conditions $s_0(0,\lambda)=0$, $s_0'(0,\lambda)=1$. For the functions $s_0(\pi,\lambda)$, $s_0'(\pi,\lambda)$ and the characteristic function $\Delta_0(\lambda)$ of the boundary value problem P_0, the following relations are valid:

$$s_0(\pi,\lambda)=\frac{\sin\lambda\pi}{\lambda}-Q_0\frac{\cos\lambda\pi}{\lambda^2}+\frac{1}{\lambda^2}\int_0^\pi\varphi_1(t)\cos\lambda t dt, \tag{4.2}$$

$$s_0'(\pi,\lambda)=\cos\lambda\pi+Q_0\frac{\sin\lambda\pi}{\lambda}+\frac{1}{\lambda}\int_0^\pi\varphi_2(t)\sin\lambda t dt, \tag{4.3}$$

$$\begin{aligned}\Delta_0(\lambda)&=1+\left(m\lambda^2+\alpha\lambda+\beta\right)s_0(\pi,\lambda)+s_0'(\pi,\lambda)=\\&=1+m\lambda\sin\pi\lambda+(1-mQ_0)\cos\pi\lambda+\alpha\sin\pi\lambda+f_0(\lambda),\end{aligned} \tag{4.4}$$

where $Q_0=\frac{1}{2}\int_0^\pi q(x)\,dx$, $f_0(\lambda)=\int_{-\pi}^\pi\tilde f_0(t)e^{i\lambda t}dt$, $\tilde f_0(t)\in L_2[-\pi,\pi]$,

$$\varphi_1(t),\ \varphi_2(t)\in L_2[0,\pi].$$

Inverse problem B_0. The spectrum $\left\{\gamma_k^{(0)}\right\}$ of the boundary value problem P_0 and the free constant β of the quadratic function of the spectral parameter contained in the boundary condition are given. Construct the function $q(x)$ in equation (4.1) and the coefficients m, α of the quadratic function in (2.2).

The following uniqueness theorem holds.

Theorem 4.1. *A boundary value problem P_0 can be uniquely recovered if its spectrum and the number β are known.*

Proof. According to relation (2.5), the spectrum $\left\{\gamma_k^{(0)}\right\}$ of the problem P_0 obeys the asymptotics

$$\gamma_k^{(0)}=k+\frac{(-1)^{k+1}-1+mQ_0}{m\pi k}+\frac{\xi_k}{k},$$

where $\{\xi_k\}\in l_2$. Since

$$\gamma_{2k}^{(0)}=2k+\frac{mQ_0-2}{2m\pi k}+\frac{\xi_{2k}}{2k},$$

$$\gamma_{2k+1}^{(0)}=2k+1+\frac{Q_0}{2\pi k}+\frac{\eta_k}{k},\ \{\eta_k\}\in l_2,$$

then the values Q_0 and m can be determined by the formulas

$$Q_0=2\pi\lim_{k\to\infty}k\left(\gamma_{2k+1}^{(0)}-2k-1\right), \tag{4.5}$$

$$m=\frac{2}{Q_0-2\pi\lim\limits_{k\to\infty}k\left(\gamma_{2k}^{(0)}-2k\right)}. \tag{4.6}$$

Knowing the spectrum $\left\{\gamma_k^{(0)}\right\}$ and the parameter m, the function $\Delta_0(\lambda)$ (as an entire function of exponential type) can be restored in the form of an infinite product as follows:

$$\Delta_0(\lambda)=m\pi\left(\gamma_{-0}^{(0)}-\lambda\right)\left(\gamma_{+0}^{(0)}-\lambda\right)\prod_{\substack{k=-\infty\\k\neq0}}^{\infty}\frac{\gamma_k^{(0)}-\lambda}{k}. \tag{4.7}$$

Similarly to relation (3.4), the parameter is determined by the formula

$$\alpha = \lim_{k\to\infty}\left[\Delta_0\left(2k+\frac{1}{2}\right)-\left(2k+\frac{1}{2}\right)m-1\right]. \tag{4.8}$$

Since the functions $s_0(\pi,\lambda)$ and $s_0'(\pi,\lambda)$ are even (see representations (4.2) and (4.3)), then by virtue of (4.4)

$$\Delta_0(-\lambda) = 1 + \left(m\lambda^2 - \alpha\lambda + \beta\right) s_0(\pi,\lambda) + s_0'(\pi,\lambda).$$

According to this formula and equality (4.4), the characteristic function $s_0(\pi,\lambda)$ of the boundary value problem generated by equation (4.1) and boundary conditions (2.6) is recovered by the formula

$$s_0(\pi,\lambda) = \frac{\Delta_0(\lambda)-\Delta_0(-\lambda)}{2\alpha\lambda}. \tag{4.9}$$

From here we find the zeros $\lambda_n,\ \ n=\pm1,\pm2,...$ of the function $s_0(\pi,\lambda)$, squares of which are the eigenvalues of the boundary value problem (4.1), (2.6).

Finally, knowing the values $\Delta_0(\lambda)$, $s_0(\pi,\lambda)$, m, α, β, using (4.4), the characteristic function $s_0'(\pi,\lambda)$ of problem (4.1), (3.6) is reconstructed by the following formula:

$$s_0'(\pi,\lambda) = \Delta_0(\lambda) - \left(m\lambda^2+\alpha\lambda+\beta\right) s_0(\pi,\lambda) - 1. \tag{4.10}$$

It is known [26] (see also [17, 8]) that the coefficient $q(x)$ of equation (4.1) is uniquely determined from the zeros ν_n, $n = 1, 2, ...$ of this function and the sequence $\{\lambda_n\}$.

Theorem is proved. □

5. Algorithms for recovery of boundary problem

According to the proof of Theorem 3.1, the inverse problem B can be solved using the following algorithm.

Algorithm 1. Let the sequences $\{\gamma_k\}$, $\{\lambda_k\}$ and number β (spectra of the problems P and D) be given.

1. With the help of the sequences $\{\gamma_k\}$ and $\{\lambda_k\}$, we determine the values of a and Q using formulas (3.1).
2. The first coefficient m of the quadratic trinomial in (2.2) is found from relation (3.2).
3. With the help of the sequence $\{\gamma_k\}$ and the parameter m, construct the characteristic function $\Delta(\lambda)$ in the form of an infinite product (3.3).
4. Find the parameter α from (3.4).
5. Let us reconstruct the characteristic function $s(\pi,\lambda)$ of the boundary value problem D from (3.5).
6. Knowing m, α, β, $s(\pi,\lambda)$ and $\Delta(\lambda)$, we determine the characteristic function $s'(\pi,\lambda)$ of the boundary value problem (2.1), (3.6) by formula (3.7) and find the zeros ν_k $(k=\pm1,\pm2,...)$ of this function.
7. Determine the coefficients $p(x)$ and $q(x)$ from the sequences of zeros $\{\lambda_k\}$ and $\{\nu_k\}$ of the functions $s(\pi,\lambda)$ and $s'(\lambda,\pi)$ by a well-known procedure (see, for example, [9]).

We now present the main steps of the algorithm for solving the inverse problem B_0, based on the proof of Theorem 4.1.

Algorithm 2. Let the sequences $\{\gamma_k\}$, $\{\lambda_k\}$ and number β (spectra of the problems P and D) be given.

1. The values Q_0 and $m$are determined by formulas (4.5) and (4.6).

2. From the spectrum $\left\{\gamma_k^{(0)}\right\}$ and the parameter m, the function $\Delta_0(\lambda)$ can be recovered using relation (4.7).

3. The parameter α of the boundary condition is found from relation (4.8).

4. Let us construct the characteristic function $s_0(\pi,\lambda)$ of the boundary value problem (4.1), (2.6) using (4.9).

5. We recover the characteristic function of problem (4.1), (3.6) by formula (4.10).

6. Using the well-known algorithm (see, for example, [26]), we construct the coefficient of equation (4.1) from the zeros of the functions $s_0(\pi,\lambda)$ and $s_0'(\pi,\lambda)$.

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Ibrahim M. Nabiev
Baku State University, AZ1148, Z. Khalilov st. 23, Baku, Azerbaijan
Institute of Mathematics and Mechanics of the Ministry of Education and Science of Azerbaijan, AZ1141, B. Vahabzade st. 9, Baku, Azerbaijan
Khazar University, AZ1096, Mahsati st. 11, Baku, Azerbaijan
E-mail address: nabievim@yahoo.com

Leyla I. Mammadova
Azerbaijan State Oil and Industry University, AZ1010, Azadlig Ave. 20, Baku, Azerbaijan
E-mail address: leylaimae@yahoo.com

Guldane S. Mammedzadeh
Azerbaijan State University of Economics (UNEC), AZ1001, Istiglaliyyat st. 6, Baku, Azerbaijan
E-mail address: guldane.mammedzadeh@mail.ru

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