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NONLOCAL PROBLEM FOR A SECOND ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATION WITH DEGENERATE KERNEL AND REAL PARAMETERS

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Abstract. The questions of solvability and construction of solutions of a homogeneous nonlocal boundary value problem for a second-order homogeneous Fredholm integro-differential equation with a degenerate kernel and two real parameters are considered. The degenerate kernel method was developed. The features that have arisen in the construction of solutions and are associated with the determination of the integration coefficients are studied. The values of the parameters are calculated for which the solvability of the boundary value problem is established and the corresponding solutions are constructed.

1. Introduction. Problem statement

Differential and integro-differential equations are one of the basic equations of mathematical physics and mechanics. There are a large number of works devoted to the study of the properties of solutions of nonlocal boundary problems for differential and integro-differential equations (see, for example, [1, 2, 3, 4, 5, 6, 12, 13, 19, 21, 22, 28, 29]). Spectral problems for differential equations are considered in [7, 8, 9, 14, 15, 16, 17, 18, 20]. Integro-differential equations with a degenerate kernel were considered earlier in [10, 11, 23, 24, 25]. In cases, where the boundary of the region of a physical process is not available for measurements, nonlocal conditions in integral form can serve as additional information sufficient for the unambiguous solvability of the problem. The papers [26, 27] considered nonlocal problems for a second-order integro-differential equation with the real parameters and an integral condition.

In this paper, we study a nonlocal homogeneous problem for a second-order ordinary Fredholm integro-differential equation with a degenerate kernel and two parameters. The regular and irregular values of the parameters are calculated, under which the existence of only trivial solutions, the existence of a unique solution, and the existence of an infinite set of solutions of the problem are established, and the corresponding solutions are constructed. The method used

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in this paper is unique known for us method, which helps us to obtain interesting results.

Problem. It is required to find a function $u(t)$ on the interval $(0,T)$, that satisfies the equation

$$u''(t)+\lambda^2 u(t)=\nu\int_0^T K(t,s)\left[s\,u(s)+(T-s)u'(s)\right]\,ds \tag{1.1}$$

with the following homogeneous boundary conditions:

$$u(T)-\int_0^T s\,u(s)\,ds=0,\quad u'(T)-\int_0^T (T-s)\,u'(s)\,ds=0, \tag{1.2}$$

where $T>\sqrt{2}$, λ is positive parameter, ν is nonzero real parameter, $K(t,s)=\sum_{i=1}^k a_i(t)\,b_i(s)\neq 0$, $a_i(t),\,b_i(s)\in C[0,T]$. It is assumed that the functions $a_i(t)$ and $b_i(s)$ are linearly independent.

Since the boundary conditions (1.2) are homogeneous, the homogeneous integro-differential equation (1.1) always has trivial solutions. Therefore, we investigate the existence of nontrivial solutions. Let us determine that for what values of the parameters λ and ν the problem has nontrivial solutions and construct these solutions. We especially note that due to the homogeneity of the problem (1.1), (1.2), it is relevant to establish the fact that for some values of the parameters λ and ν this problem has only a trivial solution.

Present work differs from [27] not only in the research method, but also in the content of the obtained results.

2. Integration of the boundary value problem (1.1), (1.2)

Taking into account the degeneracy of the kernel, equation (1.1) is written in the following form

$$u''(t)+\lambda^2 u(t)=\nu\sum_{i=1}^k a_i(t)\,\tau_i,$$

where

$$\tau_i=\int_0^T b_i(s)\left[s\,u(s)+(T-s)u'(s)\right]\,ds. \tag{2.1}$$

This equation is solved by the method of variation of arbitrary constants

$$u(t)=A_1\cos\lambda\,t+A_2\sin\lambda\,t+\frac{\nu}{\lambda}\sum_{i=1}^k\tau_i\int_0^t \sin\lambda\,(t-s)\,a_i(s)\,ds, \tag{2.2}$$

where A_1 and A_2 are arbitrary constants of integration. By differentiation from (2.2) we obtain

$$u'(t)=-\lambda\,A_1\sin\lambda\,t+\lambda\,A_2\cos\lambda\,t+\nu\sum_{i=1}^k\tau_i\int_0^t\cos\lambda\,(t-s)\,a_i(s)\,ds. \tag{2.3}$$

To find the unknown coefficients A_1 and A_2 in (2.2), we use the homogeneous integral conditions (1.2) and arrive at the following transcendental system of linear algebraic equations

$$\begin{cases} A_1\chi_{11}(\lambda) + A_2\chi_{12}(\lambda) = \chi_{13}(\lambda), \\ A_1\chi_{21}(\lambda) + A_2\chi_{22}(\lambda) = \chi_{23}(\lambda), \end{cases} \tag{2.4}$$

where

$$\chi_{11}(\lambda) = \cos\lambda T - \frac{T}{\lambda}\sin\lambda T + \frac{1}{\lambda^2}(1-\cos\lambda T), \quad \chi_{12}(\lambda) = \sin\lambda T + \frac{T}{\lambda}\cos\lambda T - \frac{1}{\lambda^2}\sin\lambda T,$$

$$\chi_{21}(\lambda) = -\lambda\,\sin\lambda T - T - \frac{1}{\lambda}\sin\lambda T, \quad \chi_{22}(\lambda) = \lambda\,\cos\lambda T - \frac{1}{\lambda}(1-\cos\lambda T),$$

$$\chi_{13}(\lambda) = -\eta(T,\lambda) + \int\limits_0^T s\cdot\eta(s,\lambda)\,ds, \quad \chi_{23}(\lambda) = -\eta'(T,\lambda) + \int_0^T (T-s)\cdot\eta'(s,\lambda)\,ds,$$

$$\eta\,(t,\lambda) = \frac{\nu}{\lambda}\sum_{i=1}^{k}\tau_i h_i(t,\lambda), \quad h_i(t,\lambda) = \int_0^t \sin\lambda\,(t-s)\,a_i(s)\,ds, \quad i = \overline{1,k}.$$

Compute values of parameter λ in the following two cases:
1) $\chi_{11}(\lambda) = \chi_{12}(\lambda) = 0$, 2) $\chi_{21}(\lambda) = \chi_{22}(\lambda) = 0$.
If such values of the parameter λ exist, then for these values of the parameter we additionally check the correctness of the formulated problem (1.1), (1.2). If the problem is correct, then we construct all solutions of this problem. If the problem is not correct, then there are only trivial solutions.

Compute values of parameter λ in the following case:
3) $\chi_{11}(\lambda)\,\chi_{22}(\lambda) - \chi_{12}(\lambda)\chi_{21}(\lambda) = 0$,
for which the uniqueness of the solution of the problem posed is violated.

We find out that in the following case

$$\chi_{11}(\lambda)\,\chi_{22}(\lambda) - \chi_{12}(\lambda)\,\chi_{21}(\lambda) \neq 0$$

the uniqueness of the solution of the stated problem (1.1), (1.2) is not violated. In this case, we should find sufficient conditions for the existence of a unique solution and construct this solution.

First, we analyze the solvability of the following four transcendental equations: $\chi_{ij}(\lambda) = 0, \quad i,j = 1,2.$

2.1. First equation. The set of solutions of the equation $\chi_{11}(\lambda) = 0$ coincides with the solutions of the equation

$$(x^2 - 2)\,\tan^2\frac{T}{2}x + 2Tx\,\tan\frac{T}{2}x - x^2 = 0, \quad x = \lambda > 0. \tag{2.5}$$

The solvability of the transcendental equation (2.5) is illustrated in figure 1 below:

2.2. Second equation. The set of solutions of the equations $\chi_{12}(\lambda) = 0$ coincides with the solutions of the equation

$$\tan Tx = \frac{Tx}{1-x^2}, \quad x = \lambda > 0. \tag{2.6}$$

The solvability of the transcendental equation (2.6) is illustrated in figure 2.

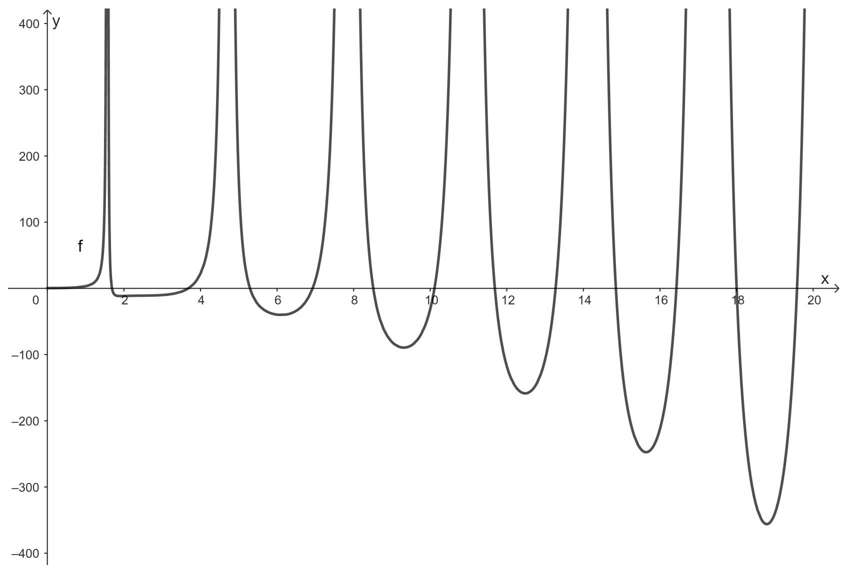

FIGURE 1. Graph of the function in the left-hand side of (2.5). T=2

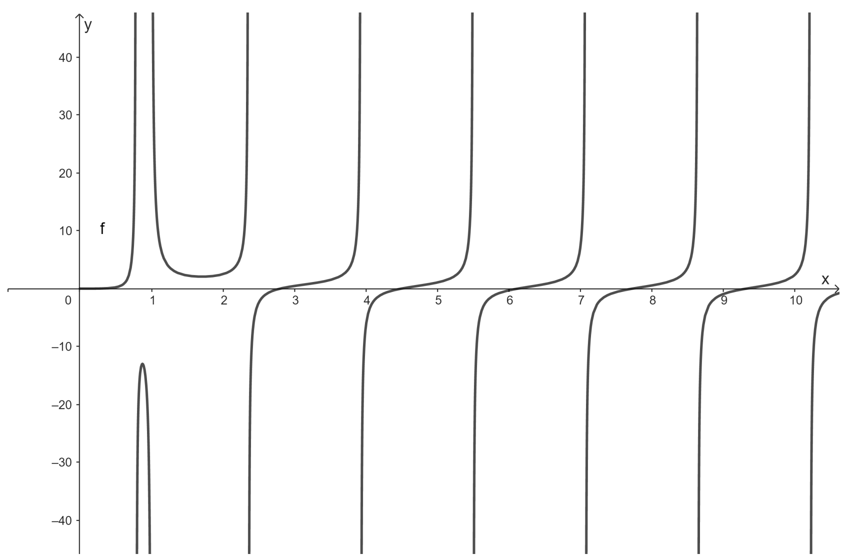

FIGURE 2. Graph of the function in (2.6). T=2

2.3. Third equation. The set of solutions of the equations $\chi_{21}(\lambda) = 0$ coincides with the solutions of the equation

$$\sin Tx = -\frac{Tx}{1+x^2}, \quad x = \lambda > 0. \tag{2.7}$$

The solvability of the transcendental equation (2.7) is illustrated in figure 3.

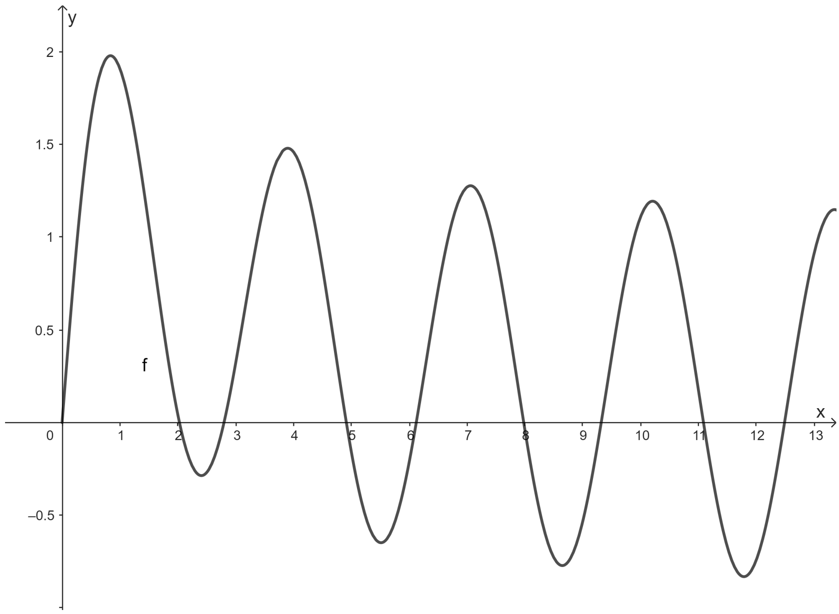

FIGURE 3. Graph of the function in (2.7). T=2

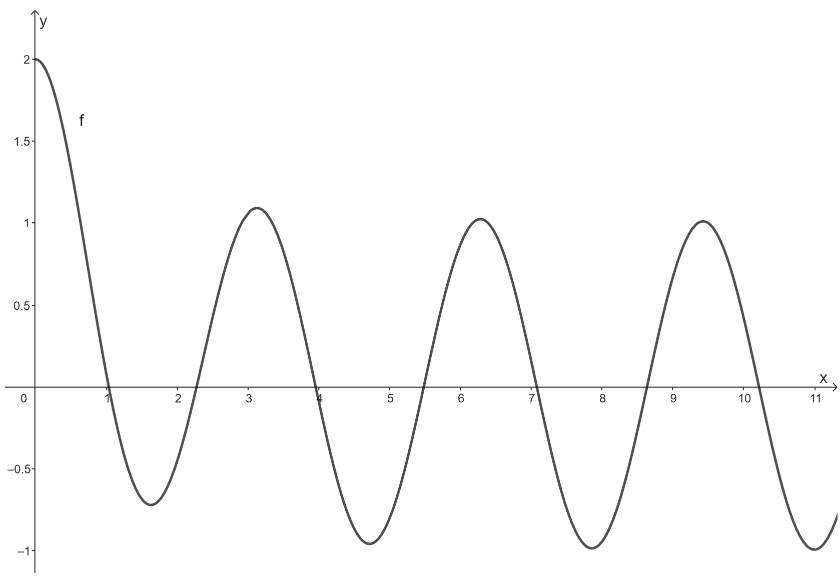

FIGURE 4. Graph of the function in (2.8). T=2

2.4. Fourth equation. The set of solutions of the equations $\chi_{22}(\lambda) = 0$ coincides with the solutions of the equation

$$\cos Tx = \frac{1}{1+x^2}, \quad x = \lambda > 0. \tag{2.8}$$

The solvability of the transcendental equation (2.8) is illustrated in figure 4.

The sets of solutions of transcendental equations (2.5)–(2.8) will be denoted by $\Im_i$ $(i = 1, 2, 3, 4)$, respectively. So, we introduce some new denotations $\Lambda_i = (0, \infty) \setminus \Im_i,\ \ i = 1, 2, 3, 4.$

Case 2.1: $\chi_{11}(\lambda) = \chi_{12}(\lambda) = 0$. Let us determine at what values of the parameter λ this case takes place. Indeed, if this case takes place, then the algebraic equation

$$\chi_{11}^2(\lambda) + \chi_{12}^2(\lambda) = 0 \tag{2.9}$$

has a solution. Equation (2.9) is equivalent to the following transcendental equation

$$\cos(\lambda T + \varphi) = -\frac{\sqrt{(\lambda^2-1)^2+\lambda^2T^2}}{2} - \frac{1}{2\sqrt{(\lambda^2-1)^2+\lambda^2T^2}}, \tag{2.10}$$

where $\varphi = \arccos \frac{\lambda^2-1}{\sqrt{(\lambda^2-1)^2+\lambda^2T^2}}$.

Equation (2.10) has a solution, if its right-hand side belongs to the interval $[-1, 1]$. We assume the opposite:

$$\frac{\sqrt{(\lambda^2-1)^2+\lambda^2T^2}}{2} + \frac{1}{2\sqrt{(\lambda^2-1)^2+\lambda^2T^2}} > 1.$$

Then we obtain $\left(\sqrt{(\lambda^2-1)^2+\lambda^2T^2} - 1\right)^2 > 0$ or $\lambda > \sqrt{2-T^2}$. We have arrived at an incorrect inequality. Since the parameter λ is positive and $T > \sqrt{2}$, then the inequality $\lambda > \sqrt{2-T^2}$ does not make sense and equation (2.10) has a solution. Indeed, to verify this fact, we construct graphs of the equations $\chi_{11}(\lambda) = 0$, $\chi_{12}(\lambda) = 0$ in one coordinate plane. From figure 5, it can be seen that this case is possible for small values of the parameter λ.

The problem can have nontrivial solutions, if $\chi_{13}(\lambda) = 0$ for these values of the parameter λ. However, this equation has a solution only for large values of the parameter λ. Hence, in this case, problem (1.1), (1.2) has only trivial solutions.

Case 2.2. We consider the case $\chi_{21}(\lambda) = \chi_{22}(\lambda) = 0$. Let us determine at what values of the parameter λ this case takes place. If we suppose that this case takes place, then the algebraic equation

$$\chi_{21}^2(\lambda) + \chi_{22}^2(\lambda) = 0 \tag{2.11}$$

has a solution. Equation (2.11) is equivalent to the following transcendental equation

$$\cos(\lambda T + \varphi) = \frac{(\lambda^2+1)^2 + 1 + \lambda^2T^2}{2(\lambda^2+1)\sqrt{1+\lambda^2T^2}}, \tag{2.12}$$

where $\varphi = \arccos \frac{1}{1+\lambda^2T^2}$.

Equation (2.12) has a solution, if its right-hand side belongs to the interval $[-1, 1]$. If we assume the opposite

$$\frac{(\lambda^2+1)^2 + 1 + \lambda^2T^2}{2(\lambda^2+1)\sqrt{1+\lambda^2T^2}} > 1,$$

then we obtain that $\left(\lambda^2 + 1 - \sqrt{1+\lambda^2T^2}\right)^2 > 0$ or $\lambda^2 > T^2 - 2$. Since $T > \sqrt{2}$, then we obtain correct inequality. Hence, we deduce that equation (2.12) has no solution. We build graphs of the equations $\chi_{21}(\lambda) = 0,\ \chi_{22}(\lambda) = 0$ in one

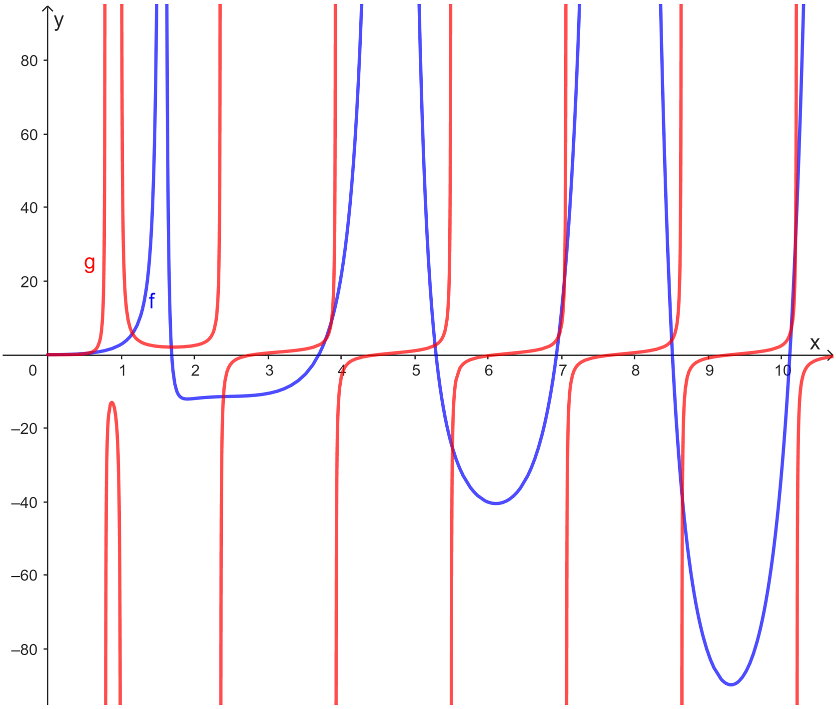

FIGURE 5. Graph of the function in (2.5) is blue line. Graph of the function in (2.6) – red. T=2

coordinate plane. From figure 6, it can be seen that these equations have no common solutions. So, this case is impossible.

Case 2.3. This is the case when the main determinant of the system vanishes: $\Delta = \chi_{11}(\lambda) \cdot \chi_{22}(\lambda) - \chi_{12}(\lambda) \cdot \chi_{21}(\lambda) = 0$. Let us determine at what values of the parameter λ this case takes place. Indeed, if this case takes place, then the algebraic equation

$$\chi_{11}(\lambda) \cdot \chi_{22}(\lambda) - \chi_{12}(\lambda) \cdot \chi_{21}(\lambda) = 0 \tag{2.13}$$

has a solution. Equation (2.13) is equivalently reduced to solving the following transcendental equation

$$(2 + x^2T^2) \cos Tx + x^3T \sin xT + x^4 - 2 = 0, \;\; x = \lambda > 0. \tag{2.14}$$

The solution of this equation (2.14) is illustrated in figure 7.

The set of values of the parameter λ, for which equation (2.14) has solutions will be denoted by $\Im_5$. Using this notation, we adopt a new notation $\Lambda_5 = \cup_{i=1}^{4} \Lambda_i \backslash \Im_5$. For values of the parameter λ from the set Λ_5, we study the influence of the second parameter ν on the solvability of problem (1.1), (1.2) and, for certain values of this parameter ν, construct a unique solution to the problem (1.1), (1.2). The values of the parameter λ from the set Λ_5 are called regular.

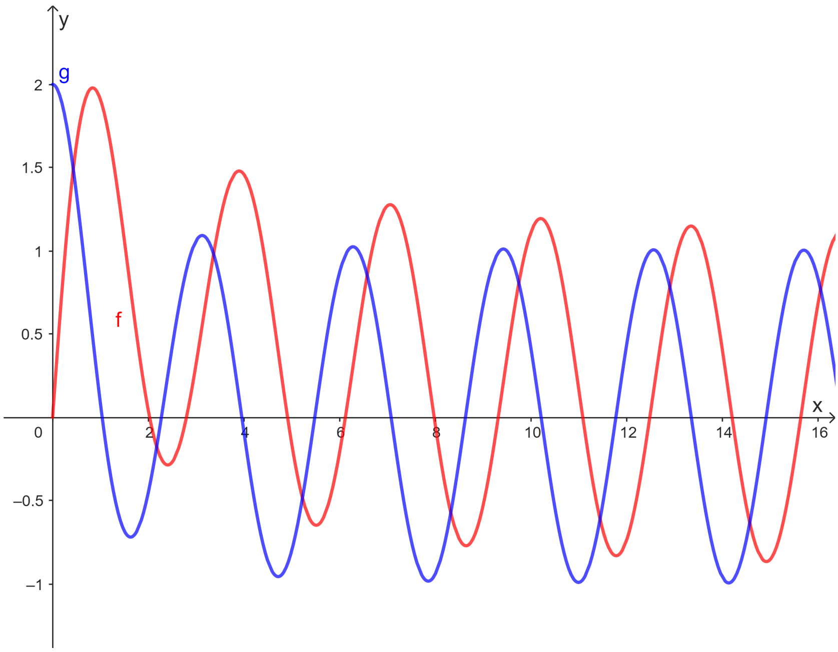

FIGURE 6. Graph of the function in (2.7) is red line. Graph of the function in (2.8) – blue. T=2

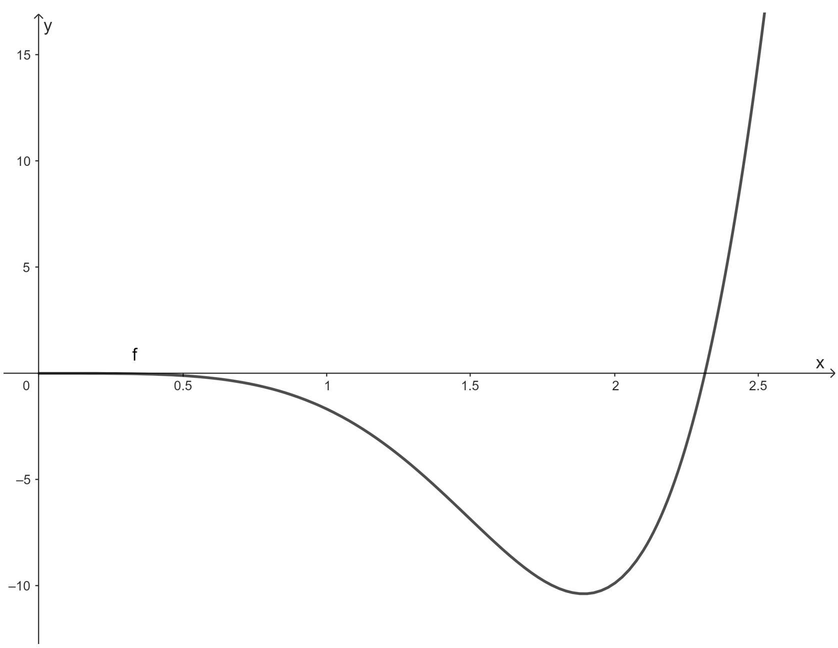

FIGURE 7. Graph of the function in (2.14). T=2

3. Regular values of the parameter λ

We consider the case of values of the parameter λ from the set Λ_5. In this case we have $D(\lambda) = \chi_{11}(\lambda)\,\chi_{22}(\lambda) - \chi_{12}(\lambda)\,\chi_{21}(\lambda) \neq 0$. Then we solve the system of equations (2.4) using the standard Cramer method and uniquely determine A_1 and A_2:

$$A_1 = \frac{1}{D(\lambda)} \begin{vmatrix} \chi_{13}(\lambda) & \chi_{12}(\lambda) \\ \chi_{23}(\lambda) & \chi_{22}(\lambda) \end{vmatrix} =$$

$$= \frac{1}{D(\lambda)} \left[\chi_{13}(\lambda)\,\chi_{22}(\lambda) - \chi_{12}(\lambda)\,\chi_{23}(\lambda)\right] = \chi_{01}(\lambda), \tag{3.1}$$

$$A_2 = \frac{1}{D(\lambda)} \begin{vmatrix} \chi_{11}(\lambda) & \chi_{13}(\lambda) \\ \chi_{21}(\lambda) & \chi_{23}(\lambda) \end{vmatrix} =$$

$$= \frac{1}{D(\lambda)} \left[\chi_{11}(\lambda)\,\chi_{23}(\lambda) - \chi_{13}(\lambda)\,\chi_{21}(\lambda)\right] = \chi_{02}(\lambda). \tag{3.2}$$

Substituting the values A_1 and A_2 from (3.1) and (3.2) into representations (2.2) and (2.3), we obtain for $\lambda \in \Lambda_5$

$$u(t, \lambda) = \chi_{01}(\lambda) \cos \lambda\, t + \chi_{02}(\lambda) \sin \lambda\, t + \frac{\nu}{\lambda} \sum_{i=1}^{k} \tau_i \int_0^t \sin \lambda\,(t - s)\, a_i(s)\, ds, \tag{3.3}$$

$$u'(t, \lambda) = -\lambda\, \chi_{01}(\lambda)\, \sin \lambda\, t + \lambda \chi_{02}(\lambda) \cos \lambda\, t + \nu \sum_{i=1}^{k} \tau_i \int_0^t \cos \lambda\,(t - s)\, a_i(s) ds. \tag{3.4}$$

Substituting representations (3.3), (3.4) into notation (2.1), we arrive at a linear system of algebraic equations (SAE)

$$\tau_i - \frac{\nu}{\lambda} \sum_{j=1}^{k} \tau_j \Phi_{ij} = \Psi_i, \quad i = \overline{1, k}, \tag{3.5}$$

where

$$\Phi_{ij} = \int_0^T b_i(s) \left[\int_0^s \left[s \cdot \sin \lambda\,(s - \theta) + \lambda(T - s) \cos \lambda\,(s - \theta)\right]\, a_j(\theta)\, d\theta\right] ds,$$

$$\Psi_i = \int_0^T b_i(s) \Big[s\,(\chi_{01}(\lambda) \cos \lambda\, s + \chi_{02}(\lambda) \sin \lambda\, s) -$$

$$-\lambda(T - s)\ \left(\chi_{01}(\lambda) \sin \lambda\, s - \chi_{02}(\lambda) \cos \lambda\, s\right) \Big] ds.$$

SAE (3.5) is uniquely solvable for any finite right-hand side Ψ_i, if the following Fredholm condition is satisfied

$$\Delta_\Phi(\nu, \lambda) = \begin{vmatrix} 1 - \frac{\nu}{\lambda}\Phi_{11} & \frac{\nu}{\lambda}\Phi_{12} & \dots & \frac{\nu}{\lambda}\Phi_{1k} \\ \frac{\nu}{\lambda}\Phi_{21} & 1 - \frac{\nu}{\lambda}\Phi_{22} & \dots & \frac{\nu}{\lambda}\Phi_{2k} \\ \dots & \dots & \dots & \dots \\ \frac{\nu}{\lambda}\Phi_{k1} & \frac{\nu}{\lambda}\Phi_{k2} & \dots & 1 - \frac{\nu}{\lambda}\Phi_{kk} \end{vmatrix} \neq 0. \tag{3.6}$$

The determinant $\Delta_\Phi(\nu,\, \lambda)$ in (3.6) is a polynomial with respect to $\frac{\nu}{\lambda}$ of degree at most k. The equation $\Delta_\Phi(\nu, \lambda) = 0$ has at most k distinct real roots. We denote them by μ_r, $1 \leq r \leq k$. Then $\nu = \lambda\, \mu_r$ are eigenvalues of kernel of the integro-differential equation (1.1). For other values $\nu \neq \lambda\,\mu_r$ of the parameter, the condition $\Delta_\Phi(\nu,\lambda) \neq 0$ holds.

We consider the following two sets

$$\Omega_5(\nu,\lambda) = \{\lambda \in \Lambda_5,\ \nu = \lambda\,\mu_r\}\,,\ \ \tilde{\Omega}_5(\nu,\lambda) = \{\lambda \in \Lambda_5,\ \nu \neq \lambda\,\mu_r\}\,.$$

On the set $\tilde{\Omega}_5(\nu,\lambda)$ solution of SAE (3.5) has the form

$$\tau_i = \frac{\Delta_{\Psi_i}(\nu,\lambda)}{\Delta_\Phi(\nu,\lambda)}, \quad i = \overline{1,k}, \tag{3.7}$$

where

$$\Delta_{\Psi_i}(\nu,\lambda) = \begin{vmatrix} 1-\frac{\nu}{\lambda}\Phi_{11} & \dots & \frac{\nu}{\lambda}\Phi_{1(i-1)} & \Psi_1 & \frac{\nu}{\lambda}\Phi_{1(i+1)} & \dots & \frac{\nu}{\lambda}\Phi_{1k} \\ \frac{\nu}{\lambda}\Phi_{21} & \dots & \frac{\nu}{\lambda}\Phi_{2(i-1)} & \Psi_2 & \frac{\nu}{\lambda}\Phi_{2(i+1)} & \dots & \frac{\nu}{\lambda}\Phi_{2k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\nu}{\lambda}\Phi_{k1} & \dots & \frac{\nu}{\lambda}\Phi_{k(i-1)} & \Psi_k & \frac{\nu}{\lambda}\Phi_{k(i+1)} & \dots & 1-\frac{\nu}{\lambda}\Phi_{kk} \end{vmatrix}.$$

Substituting (3.7) into (3.3), we derive

$$u(t,\nu,\lambda) = \chi_{01}(\lambda)\cos\lambda\,t + \chi_{02}(\lambda)\sin\lambda\,t +$$
$$+\frac{\nu}{\lambda}\sum_{i=1}^{k}\frac{\Delta_{\Psi_i}(\nu,\lambda)}{\Delta_\Phi(\nu,\lambda)}\int_0^t \sin\lambda\,(t-s)\,a_i(s)\,ds. \tag{3.8}$$

Function (3.8) is the unique solution to the nonlocal problem (1.1), (1.2) for parameter values from the set $\tilde{\Omega}_5(\nu,\lambda)$.

On the set $\Omega_5(\nu,\lambda)$ the solution of problem (1.1), (1.2) is reduced to consideration of the following homogeneous system of algebraic equations (HSAE), if the orthogonality condition $\Psi_i = 0$ is fulfilled:

$$\tau_i - \frac{\nu}{\lambda}\sum_{j=1}^{k}\tau_j\Phi_{ij} = 0, \quad i = \overline{1,k}. \tag{3.9}$$

Orthogonality condition is reduced to the form

$$\int_0^T \Big[\,(s\chi_{01}(\lambda) + \lambda(T-s)\chi_{02}(\lambda))\cos\lambda s\,ds +$$
$$+\,(s\chi_{02}(\lambda) - \lambda(T-s)\chi_{01}(\lambda))\sin\lambda s\Big]\,ds = 0, \quad \lambda \in \Lambda_5. \tag{3.10}$$

Let us check the fulfillment of condition (3.10) for the values of the parameter λ from the sets Λ_5. The fulfillment of condition (3.10) is reduced to solving the following equation:

$$[\chi_{02} + \lambda(1+T)\chi_{01}]\sin xT +$$
$$+\,[\chi_{01} - \lambda(1+T)]\,\cos xT + x\,\chi_{02} - (1+xT)\,\chi_{01} = 0. \tag{3.11}$$

However, the set of solutions to equation (3.11) cannot intersect with the set of solutions to equation (2.14). Recall that the set of solutions to equation (2.14) is denoted by $\Im_5$. By the set Λ_5 is denoted $\cup_{i=1}^4\Lambda_i\setminus\Im_5$. Equation (3.11) is based on the assumption that $\lambda \notin \Im_5$. Therefore, the values of the parameter λ, for which the orthogonality conditions (3.11) are satisfied, lie in the set Λ_5. Let us compose a new set $\Omega_6 = \{(\nu,\lambda) : \lambda \in \Im_6,\ \nu = \lambda\,\mu_r\}$, where the set $\Im_6$ denotes the values of the parameter, for which the orthogonality condition (3.11) is satisfied. We construct an infinite set of solutions to problem (1.1), (1.2) on the set Ω_6.

HSAE (3.9) has some number p $(1 \leq p < k)$ of linear independent nonzero vector-solutions $\left\{\tau_1^{(l)},\ \tau_2^{(l)},\ \ldots,\ \tau_k^{(l)}\right\}$, $l = \overline{1, p}$. The following functions

$$u_l(t, \lambda) = \frac{\nu}{\lambda} \sum_{i=1}^{k} \tau_i^{(l)}\, \xi_i(t), \quad l = \overline{1, p}$$

are nontrivial solutions of the corresponding homogeneous equation

$$u(t, \lambda) = \frac{\nu}{\lambda} \sum_{i=1}^{k} \xi_i(t) \int_0^T b_i(s)\, u(s, \lambda)\, ds, \tag{3.12}$$

where

$$\xi_i(t) = \int_0^t \left[t \sin\lambda\,(t - s) + \lambda(T - t)\cos\lambda\,(t - s)\right]\, a_i(s)\, ds.$$

Therefore, the general solution of the homogeneous integral equation (3.12) can be written as

$$u(t, \lambda) = \sum_{l=1}^{p} \alpha_l u_l(t, \lambda), \tag{3.13}$$

where α_l is arbitrary constants.

On the set $\Omega_7 = \Big\{(\nu, \lambda) \,:\, \lambda \in \Lambda_5 \setminus \Im_6,\ \ \nu = \lambda\, \mu_r\Big\}$, it is obvious that the orthogonality condition (3.10) is not satisfied and, therefore, problem (1.1), (1.2) has only a trivial solution.

4. Irregular values of parameter λ

Now consider the case of irregular values of parameter λ from the set $\Im_5$. In this case, the main determinant of the transcendental system of algebraic equations (2.4) vanishes: $D(\lambda) = \chi_{11}(\lambda)\, \chi_{22}(\lambda) - \chi_{12}(\lambda)\chi_{21}(\lambda) = 0$. Then we cannot solve the system of equations (2.4) using the standard Cramer method and we cannot uniquely determine the coefficients A_1 and A_2. Therefore, problem (1.1), (1.2) can have an infinite set of solutions. We will study sufficient conditions for the existence of an infinite set of solutions. Substituting representations (2.2) and (2.3) into notation (2.1), we arrive at a new system of algebraic equations (SAE):

$$\tau_i - \frac{\nu}{\lambda} \sum_{j=1}^{k} \tau_j \Phi_{ij} = Q_i, \quad i = \overline{1, k}, \tag{4.1}$$

where

$$\Phi_{ij} = \int_0^T b_i(s) \left[\int_0^s \left[s \cdot \sin\lambda\,(s - \theta) + \lambda(T - s)\cos\lambda\,(s - \theta)\right]\, a_j(\theta)\, d\theta\right] ds,$$

$$Q_i = \int_0^T b_i(s) \left[s\,(A_1 \cos\lambda\, s + A_2 \sin\lambda\, s) - \lambda(T - s)\,(A_1 \sin\lambda\, s - A_2 \cos\lambda\, s)\right] ds,$$

A_1, A_2 are arbitrary constants. It is known that SAE (4.1) is uniquely solvable for any finite right-hand sides Q_i, if the Fredholm condition (3.6) is satisfied.

Note that the unique solvability of SAE (4.1) does not guarantee the uniqueness of the solution to problem (1.1), (1.2). We construct only an infinite set of solutions to problem (1.1), (1.2) for the values $\lambda \in \Im_5$. The determinant $\Delta_\Phi(\nu,\, \lambda)$

in (3.6) is a polynomial with respect to $\frac{\nu}{\lambda}$ degree at most k. The equation has no more than k different real roots. They were denoted above by μ_r, $1 \leq r \leq k$. The values $\nu = \lambda\,\mu_r$ are called above eigenvalues of the kernel of the integro-differential equation (1.1) or irregular values of the parameter ν. Other values $\nu \neq \lambda\,\mu_r$ were called regular values of the parameter ν, and condition (3.6) is satisfied for them, i.e. $\Delta_\Phi(\nu,\lambda) \neq 0$.

We consider the following two sets

$$\Omega_{8,1} = \Big\{(\nu,\lambda) : \lambda \in \Im_5,\ \nu \neq \lambda\,\mu_r\Big\}, \quad \Omega_{8,2} = \Big\{(\nu,\lambda) : \lambda \in \Im_5,\ \nu = \lambda\,\mu_r\Big\}.$$

On the set $(\nu,\lambda) \in \Omega_{8,1}$ solution of SAE (4.1) is written as

$$\tau_i = \frac{\Delta_{Q_i}(\nu,\lambda)}{\Delta_\Phi(\nu,\lambda)}, \quad i = \overline{1,k},, \tag{4.2}$$

where

$$\Delta_{Q_i}(\nu,\lambda) = \begin{vmatrix} 1-\frac{\nu}{\lambda}\Phi_{11} & \dots & \frac{\nu}{\lambda}\Phi_{1(i-1)} & Q_1 & \frac{\nu}{\lambda}\Phi_{1(i+1)} & \dots & \frac{\nu}{\lambda}\Phi_{1k} \\ \frac{\nu}{\lambda}\Phi_{21} & \dots & \frac{\nu}{\lambda}\Phi_{2(i-1)} & Q_2 & \frac{\nu}{\lambda}\Phi_{2(i+1)} & \dots & \frac{\nu}{\lambda}\Phi_{2k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\nu}{\lambda}\Phi_{k1} & \dots & \frac{\nu}{\lambda}\Phi_{k(i-1)} & Q_k & \frac{\nu}{\lambda}\Phi_{k(i+1)} & \dots & 1-\frac{\nu}{\lambda}\Phi_{kk} \end{vmatrix}.$$

Substituting (4.2) into (2.2), we obtain an infinite set of solutions to problem (1.1), (1.2) on the set $(\nu,\lambda) \in \Omega_{8,1}$:

$$u(t,\nu,\lambda) = A_1\cos\lambda\,t + A_2\sin\lambda\,t + \frac{\nu}{\lambda}\sum_{i=1}^{k}\frac{\Delta_{Q_i}(\nu,\lambda)}{\Delta_\Phi(\nu,\lambda)}\int_0^t \sin\lambda\,(t-s)\,a_i(s)\,ds, \tag{4.3}$$

where A_1 and A_2 are arbitrary constants.

On the set $(\nu,\lambda) \in \Omega_{8,2}$ the process of solving problem (1.1), (1.2) reduces to considering the following HSAE (3.9), if the orthogonality conditions are satisfied for $\lambda \in \Im_5$

$$\int_0^T [s\,(A_1\cos\lambda\,s + A_2\sin\lambda\,s) - \lambda(T-s)\,(A_1\sin\lambda\,s - A_2\cos\lambda\,s)] = 0. \tag{4.4}$$

Fulfillment of the condition (4.4) reduces to solving the equation for $\lambda \in \Im_5$

$$(A_2 + x(1+T)A_1)\sin xT + (A_1 - x(1+T))\cos xT = (1+xT)A_1 - xA_2. \tag{4.5}$$

Equation (4.5) has no solutions for the values of the parameter $\lambda \in \Im_5$. Therefore, HSAE (3.9) is meaningless in this case. Consequently, problem (1.1), (1.2) has only a trivial solution on the set $(\nu,\lambda) \in \Omega_{8,2}$.

Remark. By the aid of similar way can be studied the following cases:
1) $\chi_{11}(\lambda) = 0,\ \chi_{12}(\lambda) \neq 0,\ \chi_{21}(\lambda) \neq 0,\ \chi_{22}(\lambda) \neq 0;$
2) $\chi_{11}(\lambda) \neq 0,\ \chi_{12}(\lambda) = 0,\ \chi_{21}(\lambda) \neq 0,\ \chi_{22}(\lambda) \neq 0;$
3) $\chi_{11}(\lambda) \neq 0,\ \chi_{12}(\lambda) \neq 0,\ \chi_{21}(\lambda) = 0,\ \chi_{22}(\lambda) \neq 0;$
4) $\chi_{11}(\lambda) \neq 0,\ \chi_{12}(\lambda) \neq 0,\ \chi_{21}(\lambda) \neq 0,\ \chi_{22}(\lambda) = 0.$

5. Conclusion

In this paper, we consider the issues of solvability and construction of solutions for the second order homogeneous Fredholm integro-differential equation (1.1) with homogeneous nonlocal boundary conditions (1.2). The degenerate kernel method was developed. The features that have arisen in the construction of solutions and are associated with the determination of the integration coefficients are studied. The values of the parameters are calculated and for which the solvability of the boundary value problem is established and the corresponding solutions are constructed.

It has been proved that the following theorem is true.

Theorem 5.1. *In the questions of the solvability of problem* (1.1), (1.2), *the following propositions take place.*
1. *For the values of the parameter* λ, *for which* $\chi_{11}(\lambda) = \chi_{12}(\lambda) = 0$, *problem* (1.1), (1.2) *has only a trivial solution.*
2. *On the sets* Ω_7 *and* $\Omega_{8,2}$ *of parameters* (ν, λ), *problem* (1.1), (1.2) *has only a trivial solution.*
3. *On the set* $\tilde{\Omega}_5$ *of parameters* (ν, λ), *problem* (1.1), (1.2) *has a unique solution, and this solution is represented by formula* (3.8).
4. *On the sets* Ω_6 *and* $\Omega_{8,1}$ *of parameters* (ν, λ), *problem* (1.1), (1.2) *has an infinite set of solutions, and these sets of solutions are represented by the formulas* (3.13) *and* (4.3), *respectively.*

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