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THE NEW ASYMPTOTICS FOR SOLUTIONS OF THE STURM–LIOUVILLE EQUATION

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Abstract. In this paper, we show the development of a method that allows one to construct asymptotics for solutions to ordinary differential equations of arbitrary order with oscillating coefficients on the semiaxis. The idea of the method is presented on the example of studying the asymptotics of the Sturm-Liouville equation solutions.

1. Introduction

A significant number of papers are devoted to the study of the asymptotic properties of solutions of singular Sturm-Liouville equations and differential equations of arbitrary order (see [1, 2, 3, 14] and references to them). However, as a rule, when studying the asymptotic behavior of solutions of linear ordinary differential equations, it was essentially used that their coefficients have the correct growth at infinity and the possibility of applying Levinson's classical result for L-diagonal systems of linear differential equations.

Recently, a number of papers [4, 5, 8, 9, 10, 11, 12, 13] have been published, in which the asymptotic properties of solutions to ordinary differential equations were studied for equations with coefficients from wider classes, in particular, those that do not satisfy the Titchmarsh-Levitan conditions.

In this paper, we develop methods and approaches for studying linear differential equations with "regularly oscillating" coefficients, described in [9]-[12]. In these studies, classes of coefficients of linear differential equations are described for which it is algorithmically possible to construct asymptotic formulas for large values of the independent variable. Of special interest are the cases of nontrivial asymptotics (see Example 2 in [11]).

In [9]-[12] the cases of rapidly oscillating coefficients were considered. In [13], an approach was proposed to study the asymptotics of solutions of the Sturm-Liouville equation with coefficients of the form: $\mu^2 + q(x)/x^\alpha$, $\alpha > 0$, where $q(x)$ is an almost periodic function, but the main result is formulated for the case when the perturbation $q(x)/x^\alpha$ does not affect the leading term of the asymptotics.

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In this article, we apply the main ideas of [11, 12] to construct asymptotic formulas for solving the equation

$$y'' + \left(\mu^2 + \frac{\sin x}{x^\alpha}\right) y = 0, \quad x_0 < x < \infty, \quad \mu \in \mathbb{C},$$

for the case when the main part of the asymptotics is affected by the oscillating potential $\sin x/x^\alpha$.

Note that in R. Bellman's monograph [1] the problem of studying the asymptotic behavior of the solution of this equation for $x \to \infty$, when $\alpha < 1$ was posed. The case when $\alpha > 1$ is of no interest, since the perturbation is a summable function on the interval $[x_0, \infty)$. The result obtained in this article answers the question for $1/3 < \alpha < 1/2$, and the method of constructing asymptotic formulas is also suitable for coefficients of a more general form from classes of regularly oscillating functions.

We also note that the approaches developed by us can be used to study linear differential equations of arbitrary orders with regularly oscillating coefficients.

2. Construction of asymptotic formulas

Consider the model Sturm-Liouville equation

$$y'' + \left(\mu^2 + \frac{\sin x}{x^\alpha}\right) y = 0, \quad x_0 \le x < \infty, \quad \mu \in \mathbb{R}\backslash\{0\}, \quad \alpha > 0. \tag{2.1}$$

The main result of the article is the following assertion

Theorem 2.1. *Let $1/3 < \alpha < 1/2$ and the parameter μ be such that $\mu \neq \pm 1, \pm 1/2$.*
Then for the fundamental system of solutions of equation (2.1) for $x \to +\infty$ the following asymptotic relations are true:

$$\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = \begin{pmatrix} ie^{-ip(x)} + o(1) & -ie^{ip(x)} + o(1) \\ e^{-ip(x)} + o(1) & e^{ip(x)} + o(1) \end{pmatrix},$$

$$p(x) = \frac{x^{1-2\alpha}}{4(1-2\alpha)(4\mu^3 - \mu)} + \mu x. \tag{2.2}$$

Proof. Let us give a sketch of the proof of the theorem. We reduce equation (2.1) to an equivalent system of equations.

Let us introduce the vector function $\vec{z}(x, \mu) = colon(z_1, z_2):\ z_1 = y,\ z_2 = y'$.

Then equation (2.1) can be written in the form

$$\vec{z}\,' = \begin{pmatrix} 0 & 1 \\ -\mu^2 - \sin x/x^\alpha & 0 \end{pmatrix} \vec{z}.$$

By replacing

$$\vec{z}(x) = T\vec{u}, \quad T = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \tag{2.3}$$

let's go to the system:

$$\vec{u}\,' = i\mu L_0 \vec{u} + \frac{1}{x^\alpha} D(x)\vec{u},$$

$$L_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(x) = \frac{i \sin x}{2\mu} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Let's make another substitution:

$$\vec{u} = B(x)\vec{v}, \quad B(x) = B_0(x) + \frac{1}{x^\alpha} B_1(x). \tag{2.4}$$

Replacement (2.4) leads to the system:

$$B'(x)\vec{v} + B(x)\vec{v}' = i\mu L_0 B(x)\vec{v} + \frac{1}{x^\alpha} D(x)B(x)\vec{v}. \tag{2.5}$$

Considering $x^{-\alpha}$ as a small parameter for $x \to \infty$, we will look for the matrices $B_0(x)$, $B_1(x)$ from the following system of matrix equations:

$$\begin{cases} B_0' = i\mu L_0 B_0, \\ B_1' = i\mu L_0 B_1 + DB_0. \end{cases} \tag{2.6}$$

From the first equation of this system

$$B_0 = e^{i\mu L_0 x} = \begin{pmatrix} e^{-i\mu x} & 0 \\ 0 & e^{i\mu x} \end{pmatrix}.$$

Here is the formula for the B_1 matrix:

$$B_1 = B_0 - B_0 \cdot D_1, \quad D_1' = D_0 = B_0^{-1} D B_0,$$

$$D_0(x) = \frac{i \sin x}{2\mu} \begin{pmatrix} -1 & e^{2i\mu x} \\ -e^{-2i\mu x} & 1 \end{pmatrix}.$$

Now for $B(x)$ we get the following representation:

$$B(x) = B_0 \cdot (E - \frac{1}{x^\alpha} D_1) = (E - \frac{1}{x^\alpha} B_0^{-1} D_1) \cdot B_0,$$

which, among other things, implies the nondegeneracy of the matrix $B(x)$. By virtue of the condition of the matrix B_0, B_1 are limited.

To calculate the matrix D_1, it is required to integrate the elements of the matrix D_0:

$$D_1(x) = \frac{i}{4\mu} \begin{pmatrix} 2\cos x & -\frac{1}{1-2\mu} e^{-ix(1-2\mu)} - \frac{1}{1+2\mu} e^{ix(1+2\mu)} \\ \frac{1}{1+2\mu} e^{-ix(1+2\mu)} + \frac{1}{1-2\mu} e^{ix(1-2\mu)} & -2\cos x \end{pmatrix}.$$

The condition $2\mu \neq \pm 1$ provides a non-zero imaginary part for all elements of the matrix D_0, i.e. the absence of parametric resonance of resonance (for the effect of parametric resonance, see, for example, in [6]).

The boundedness of the matrix $B^{-1}DB_1$ and the condition $1/3 < \alpha < 1/2$ imply summability of the elements of the matrix $\frac{1}{x^{\alpha+1}} B^{-1} D B_1$.

Taking into account relations (2.6), we write system (2.5) in the form:

$$\vec{v}' = \frac{1}{x^{2\alpha}} B^{-1} D B_1 \vec{v} + \frac{1}{x^{\alpha+1}} B^{-1} D B_1 \vec{v} =$$

$$= \frac{1}{x^{2\alpha}} B_0^{-1} (E - \frac{1}{x^\alpha} D_1)^{-1} D B_1 \vec{v} + \frac{1}{x^{\alpha+1}} B^{-1} D B_1 \vec{v}.$$

Since $2\alpha < 1$, the asymptotic relation $x^{\alpha+1} = o(x^{3\alpha})$, $x \to \infty$. Then imagine the last system like this:

$$\vec{v}' = \frac{1}{x^{2\alpha}} B_0^{-1} D B_1 \vec{v} + \frac{1}{x^{3\alpha}} F\vec{v}, \tag{2.7}$$

where F is a matrix bounded on $[x_0, \infty)$.

The matrix $B_0^{-1}DB_1$ can be represented as the sum of two matrices, the first of which is constant, and the second consists of oscillating elements:

$$B_0^{-1}DB_1 = \frac{i}{4\mu(4\mu^2-1)}L_0 + C(x,\mu),$$

where the matrix $C(x,\mu)$ has the following structure:

$$C(x,\mu) = \sum_{\gamma\in\Omega} C_\gamma e^{i\gamma x}, \quad \Omega = \{\pm 1, \pm 2, \pm 1 \pm 2\mu, \pm 2 \pm 2\mu\},$$

here C_γ are certain constant matrices.

Thus, system (2.7) takes the form:

$$\vec{v}' = \frac{i}{ax^{2\alpha}}L_0\vec{v} + \frac{1}{x^{2\alpha}}C\vec{v} + \frac{1}{x^{3\alpha}}F\vec{v}, \quad a = 4\mu(4\mu^2-1). \tag{2.8}$$

Then the first and second terms on the right side of this system are not summable matrices, and the third term is a summable matrix. Let's make a substitution change of the independent variable:

$$\xi = \phi(x), \;\; \phi'(x) = \frac{1}{x^{2\alpha}}, \;\; \vec{v}(x) = \vec{w}(\xi), \;\; \tilde{C}(\xi,\mu) = C(x,\mu), \;\; \tilde{F}(\xi,\mu) = F(x,\mu),$$

whence

$$\xi = \frac{x^{1-2\alpha}}{1-2\alpha}, \quad x = ((1-2\alpha)\xi)^{\frac{1}{1-2\alpha}} = \left(\frac{\xi}{1+\beta}\right)^{1+\beta}, \quad \beta = \frac{1}{1-2\alpha} - 1 = \frac{2\alpha}{1-2\alpha}.$$

Note that the condition $1/3 < \alpha < 1/2$ implies that $2 < \beta$. Taking into account the introduced notation, we write system (2.8):

$$\frac{d}{d\xi}\vec{w}(\xi) = \frac{i}{a}L_0\vec{w}(\xi) + \tilde{C}(\xi,\mu)\vec{w}(\xi) + \xi^{-\beta/2}\tilde{F}(\xi,\mu)\vec{w}(\xi). \tag{2.9}$$

Note that system (2.9) is a system with rapidly oscillating coefficients. Indeed, the elements of the matrix $\tilde{C}(\xi,\mu)$ have the form

$$\tilde{c}_{ij}(\xi,\mu) = \sum_{\gamma\in\Omega} (C_\gamma)_{ij} \exp\left\{ i\gamma \left(\frac{\xi}{1+\beta}\right)^{1+\beta} \right\}, \; \beta > 2.$$

Whence it follows that

$$\left\| \int_\xi^\infty \tilde{C}(\tau,\mu)d\tau \right\| \in L[\xi_0, \infty).$$

The term $\xi^{-\beta/2}\tilde{F}(\xi,\mu)$ is obviously a matrix with coefficients summable on $[\xi_0,\infty)$.

Applying the ideas and approaches to the study of such systems presented in [7], [11], we pass to an equivalent system of integral equations and apply the method of successive approximations to the resulting system. Whence, in view of the rapid oscillation of the elements of the matrix $\tilde{C}$, it follows that the main term of the asymptotics of system (2.9) can be written in terms of the matrix $\frac{i}{a}L_0$.

Omitting a number of cumbersome standard calculations, we write down the final result for the fundamental system of solutions of the system (2.9):

$$\begin{pmatrix} w_1^1(\xi) & w_1^2(\xi) \\ w_2^1(\xi) & w_2^2(\xi) \end{pmatrix} = \begin{pmatrix} e^{-i/a\xi} + o(1) & o(1) \\ o(1) & e^{i/a\xi} + o(1) \end{pmatrix}, \; x \to \infty.$$

Making the reverse substitutions (2.3) and (2.4), we finally obtain the required asymptotic formulas (2.2) for the fundamental system of solutions of equation (2.1):

$$\vec{z}(x) = T(B_0(x) + \frac{1}{x^\alpha}B_1(x))\vec{v}(x) = T(B_0(x) + o(1)B_1(x))\vec{v}(x) =$$

$$= T(B_0(x) + o(1)B_1(x))\vec{w}(\xi), \quad \xi = \phi(x),$$

whence for the leading term of the asymptotics we obtain:

$$\left(\begin{array}{cc} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{array} \right) = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\mu x} & 0 \\ 0 & e^{i\mu x} \end{pmatrix} \begin{pmatrix} e^{-i/a\xi} + o(1) & o(1) \\ o(1) & e^{i/a\xi} + o(1) \end{pmatrix}.$$

Given that

$$\xi = \frac{x^{1-2\alpha}}{1-2\alpha}, \quad p(x) = \frac{x^{1-2\alpha}}{4(1-2\alpha)(4\mu^3 - \mu)} + \mu x,$$

we obtain the asymptotic formulas (2.2).

The theorem is proved.

Remark. The studies carried out in the articles [11, 12] and in this work can be applied to the study of the asymptotic behavior for the fundamental system of solutions of arbitrary order equations with regularly oscillating coefficients.

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