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FIRST ORDER NONCONVEX SWEEPING PROCESS WITH SUBSMOOTH SETS

IMENE MECEMMA, SABRINA LOUNIS, AND MUSTAPHA FATEH YAROU

Abstract. In this paper, using a discretization approach, we discuss the existence of solutions for a class of first order nonconvex sweeping process depending on both time and state, with an unbounded perturbation in an infinite dimensional Hilbert space. We state a new existence result according to a class of subsmooth sets. An application is given on complementarity systems.

1. Introduction

Consider a nonnegative real number T, a separable Hilbert space H and a set-valued mapping $D : [0,T]\times H \rightrightarrows H$ with nonempty closed and unbounded values. The aim of this paper is to study an evolution differential inclusion governed by a normal cone to a moving set depending on both time and state of the form

$$(\mathcal{S})\begin{cases}-\dot{u}(t) \in N_{D(t,u(t))}(u(t)) + G(t,u(t)) + f(t,u(t)) \text{ a.e. } t \in [0,T],\\ u(t) \in D(t,u(t)) \ for\ all\ t \in [0,T],\\ u(0) = u_0,\end{cases}$$

where $G : [0,T] \times H \rightrightarrows H$ is a scalarly upper semicontinuous set-valued mapping with nonempty closed unbounded values and $f : [0,T] \times H \to H$ is a Carathéodory function. $N_{D(t,u(t))}(u(t))$ denotes the Clark normal cone to the moving set $D(t,u(t))$ at $u(t)$. This kind of problem is called first order perturbed sweeping process, such problems have been introduced and studied for the first time by J.J. Moreau for convex sets $D(t)$ and $G, f \equiv 0$, with motivation in elastoplasticity, mechanical systems. Since then, several authors have been interested in the study of the sweeping processes by weakening the assumptions, see for example [3, 6, 10, 12] and the references therein.

The nonconvex case has been considered for uniformly prox regular sets by [1, 2, 11] among others and then for uniformly subsmooth sets (see for instance [7]). Other extensions include the state-dependent case, that is when the moving set depends separately on time and state. [5] has considered the problem $(\mathcal{S})$ without f for a class of subsmooth moving sets and the perturbation unnecessarily

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bounded in finite dimensional space. The same problem has been studied by [25] with delayed perturbation. For other approaches, see for instance [4, 15, 16].

Recently, the existence of solutions for $(\mathcal{S})$ has been obtained in [22] where the perturbation G satisfies a linear growth condition and the moving set D is bounded, uniformly prox-regular and there exist real constants $L_1 \geq 0$, $L_2 \in [0,1[$ such that for any t, $s \in [0,T]$ with $s < t$ one has

$$exc(D(t,u), D(s,v)) \leq L_1(t-s) + l_2\|u-v\|,$$

for any u, $v \in H$. As an extension of this result, in [8] it has been proved the existence results for first order Mixed partially BV sweeping process, which gives the existence of solution for the problem $(\mathcal{S})$ when the moving set is bounded subsmooth, relatively compact and the set valued mapping G is unbounded and only the element of minimal norm satisfies a linear growth condition. We refer to [23] for another BV version of the problem $(\mathcal{S})$ for a prox regular ball compact moving set and unbounded perturbation. In the present paper, we establish the existence of solutions for $(\mathcal{S})$ in the infinite-dimensional setting, for the general class of equi-uniformly subsmooth sets, which generalizes the convex and the uniform prox-regularity cases, moreover the moving set D is not necessarily bounded. We weaken the hypothesis on the perturbation by taking a sum of a Carathéodory function f satisfying a linear growth condition and an unbounded set-valued mapping for which only the element of minimal norm satisfies a linear growth condition.

The paper is organized as follows. In Section 2, we recall some definitions, notations and auxiliary results that we need. Section 3 is devoted to the presentation and the proof of the main result. Finally, we give an application to a class of complementarity dynamical systems.

2. Notation and Preliminaries

Throughout the paper, H will denote a real separable Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$, and the associated norm by $\| \cdot \|$ and T is a positive real number. For any $a \in H$ and $r > 0$ the open (resp. closed) ball centered at a with radius r is denoted by $B(a,r)$(resp. $\overline{B}(a,r)$). For $a = 0$ and $r = 1$ we will use the standard notation $\mathbb{B}$ the unit closed ball. The set of nearest points of S to x, $\mathrm{Proj}_S(x)$ is given by

$$\mathrm{Proj}_S(x) := \{y \in S : d_S(x) = \| x - y \|\},$$

where $d_S(x) := \inf\limits_{y \in S} \| x - y \|$ represent the usual distance function to a nonempty subset $S \subset H$.

The excess distance between two nonempty subsets S and S' in H is defined by

$$\mathrm{exc}(S, S') := \sup_{x \in S} d_{S'}(x).$$

The Hausdorff distance between two nonempty subsets S and S' in H is defined by

$$\mathcal{H}(S, S') = \max\Big\{\mathrm{exc}(S, S'), \mathrm{exc}(S', S)\Big\}.$$

It is easy to see that

$$\mathcal{H}(S,S') = \sup_{x\in H}|d_{S'}(x) - d_S(x)|.$$

The closed convex hull of S is characterized by

$$\overline{co}(S) := \{x \in H : \forall x' \in H, < x', x > \leq \sigma(x', S)\},$$

where $\sigma(x', S) := \sup_{x\in S}\langle x', x\rangle$ stands for the support function of S at $x' \in H$.
A subset S is said to be ball compact if, for any closed ball $\overline{B}(x, r)$ of H, the set $\overline{B}(x, r) \cap S$ is compact in H. Recall that if a subset S is nonempty and ball compact then the projection of x onto S is nonempty.
Let $\phi : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $x \in H$ be a point where $\phi(x)$ is finite. A vector $\zeta \in H$ is said to be in the Fréchet subdifferential $\partial^F\phi(x)$ of ϕ at x (see [9]), if for every $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\langle\zeta, y - x\rangle \leq \phi(y) - \phi(x) + \varepsilon\|y - x\| \qquad \text{for all } y \in \overline{B}(x, \eta).$$

The Clarke subdifferential $\partial\phi(x)$ of a locally Lipschitz function ϕ at x is the nonempty convex closed subset of H given by

$$\partial\phi(x) = \{\zeta \in H : \phi^\circ(x; h) \geq \langle\zeta, h\rangle, \forall h \in H\},$$

where

$$\phi^\circ(x; h) = \limsup_{u\to x, t\downarrow 0} t^{-1}\left(\phi(u + th) - \phi(u)\right) \quad for\ all\ h \in H.$$

The Fréchet normal cone (resp. the Clarke normal cone) of S at $x \in S$ is denoted by $N^F_S(x)$ (resp. $N_S(x)$) and is given by $N^F_S(x) = \partial^F\psi_S(x)$ (resp. $N_S(x) = \partial\psi_S(x)$) where $\psi_S(x)$ denotes the indicator function of the set S, i.e., $\psi_S(x) = 0$ if $x \in S$ and $\psi_S(x) = \infty$, otherwise.
It is also known, for any nonempty closed subset S of H and $x \in S$, the following relations hold true

$$\partial^F d_S(x) = N^F_S(x) \cap \mathbb{B} \quad and \quad \partial d_S(x) \subset N_S(x) \cap \mathbb{B}. \tag{2.1}$$

We say that $\varphi : [0, T] \times H \longrightarrow H$ is a Carathéodory function, if $\varphi(\cdot, x)$ is measurable for each $x \in H$ and $\varphi(t, \cdot)$ is continuous for each $t \in [0, T]$. We recall the definition of subsmoothness concept, introduced in [9].

Definition 2.1. We say that a nonempty closed subset S of H is subsmooth at $x_0 \in H$, if, for every $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x_1, x_2 \in B(x_0, \delta) \cap S$ and all $y_1 \in N_S(x_1) \cap \mathbb{B}$, $y_2 \in N_S(x_2) \cap \mathbb{B}$, we have

$$\langle y_1 - y_2, x_1 - x_2\rangle \geq -\epsilon\|x_1 - x_2\|. \tag{2.2}$$

The set S is subsmooth, if it is subsmooth at each point of S. We further say that S is uniformly subsmooth, if, for every $\epsilon > 0$, there exists $\delta > 0$ such that (2.2) holds for all $x_1, x_2 \in S$ satisfying $\|x_1 - x_2\| < \delta$ and all $y_i \in N_S(x_i) \cap \mathbb{B}$ $(i \in \{1, 2\})$.

The following result gives us the normal regularity of the subsmooth sets as well the subdifferential regularity of the distance function (see [26]).

Proposition 2.1. *Let S be a closed subset of H. If S is subsmooth at $x_0 \in S$ then*

$$N_S(x_0) = N^F_S(x_0) \qquad and \qquad \partial d_S(x_0) = \partial^F d_S(x_0).$$

Next, we give the definition of the equi-uniform subsmoothness for a family of sets, introduced in [26].

Definition 2.2. Let $(S(p))_{p\in Q}$ be a family of closed sets of H with parameter $p \in Q$. This family is called equi-uniformly subsmooth, if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for each $p \in Q$, the inequality (2.2) holds, for all $x_1, x_2 \in S(p)$ satisfying $\|x_1 - x_2\| < \delta$ and for all $y_i \in N_S(x_i) \cap \mathbb{B}, (i \in \{1, 2\})$.

The following proposition corresponds to the scalar upper semicontinuous property of distance function. For the proof, we refer the reader to [17].

Proposition 2.2. *Let $\{C(t,u) : \ (t,u) \in [0,T] \times H\}$ be a family of nonempty and closed sets of H which is equi-uniformly subsmooth and let ρ be a positive real number. Assume that there exist real constants $L_1, L_2 \geq 0$ such that, for any $u, v \in H$ and $t, s \in [0,T]$ with $s \leq t$.*

$$\operatorname{exc}(C(t,u), C(s,v)) \leq L_1|t-s| + L_2 \parallel u - v \parallel .$$

Then, the following assertions hold

(a) *For all $(s,u,y) \in Gph(C)$, we have $\rho\partial d_{C(s,u)}(y) \subset \rho\mathbb{B}$.*
(b) *The convex weakly compact set valued mapping $(s,u) \longrightarrow \partial d_{C(s,u)}(y)$ satisfies the upper semicontinuous property : for any sequence $(s_n)_n$ in $[0,T]$ converging to s, any sequence $(u_n)_n$ converging to u, any sequence $(y_n)_n$ converging to $y \in C(s,u)$ with $y_n \in C(s_n,u_n)$ and any $\zeta \in H$, we have*

$$\limsup_{n\longrightarrow\infty} \sigma(\zeta, \rho\partial d_{C(s_n,u_n)}(y_n)) \leq \sigma(\zeta, \rho\partial d_{C(s,u)}(y)).$$

3. Main result

Let us consider two set valued mappings $D : [0,T] \times H \rightrightarrows H$ and $G : [0,T] \times H \rightrightarrows H$ with nonempty closed values and a mapping $f : [0,T] \times H \longrightarrow H$ such that we have the following assumptions

$(\mathcal{A}_{D_1})$ The family $\{D(t,x); (t,x) \in [0,T] \times H\}$ is equi-uniformly subsmooth.
$(\mathcal{A}_{D_2})$ There exists a real $L_1 \geq 0$ and $0 \leq L_2 < 1$ such that, for every $(t,y), (\tau,x) \in [0,T] \times H$ with $\tau < t$,

$$\operatorname{exc}(D(\tau,x), D(t,y)) \leq L_1(t-\tau) + L_2\|x-y\|.$$

$(\mathcal{A}_{D_3})$ For every bounded subset A of H, the set $D([0,T] \times A)$ is ball compact.
$(\mathcal{A}_{G_1})$ The set valued mapping $G(t,\cdot)$ is scalarly upper semicontinuous (i.e., for each $y \in H$, the function $(t,u) \to \sigma(y, G(t,u))$ is upper semicontinuous).
$(\mathcal{A}_{G_2})$ For each $u \in H$ the mapping $\operatorname{Proj}_{G(\cdot,u)}(0) : [0,T] \to H$ is measurable and there exists a real $\beta > 0$, such that

$$d_{G(t,u)}(0) \leq \beta(1 + \|u\|) \quad \text{for all } t \in [0,T].$$

$(\mathcal{A}_f)$ f is a Carathéodory function and for some real constant $\alpha > 0$,

$$\|f(t,u)\| \leq \alpha(1 + \|u\|).$$

Theorem 3.1. *Suppose that assumptions $(\mathcal{A}_{D_1})$, $(\mathcal{A}_{D_2})$, $(\mathcal{A}_{D_3})$, $(\mathcal{A}_{G_1})$, $(\mathcal{A}_{G_2})$ and $(\mathcal{A}_f)$ are satisfied, then for every $u_0 \in H$ with $u_0 \in D(0,u_0)$, there exists a*

Lipschitz continuous mapping $u : [0,T] \longrightarrow H$ *satisfying* $(\mathcal{S})$. *Furthermore, for every* $t \in [0,T]$

$$\|\dot{u}(t)\| \leq \frac{L_1 + 2(\alpha+\beta)(1+\Delta)}{1-L_2}$$

where $\Delta := \left\{ \|u_0\| + T\frac{L_1+2(\alpha+\beta)}{1-L_2} \right\} e^{2T\dfrac{\alpha+\beta}{1-L_2}}$.

Proof. For each $n \in \mathbb{N}$, we consider the standard partition of $I = [0,T]$,

$$t_i^n = ih_n, \; h_n = \frac{T}{n} \;\text{ if }\; i \in \{0,...,n\}, \tag{3.1}$$

and set

$$I_i^n = [t_i^n, t_{i+1}^n[, \quad \text{for } \; i \in \{0,...,n-1\}.$$

For each $(t,u) \in I \times H$, take $g(t,u)$ the element of minimal norm of the closed convex set $G(t,u)$ of H defined by $g(t,u) := \mathrm{Proj}_{G(t,u)}(0)$ and set $h(t,u) = g(t,u) + f(t,u)$. Then it follows that

$$\| h(t,u) \| \leq (\alpha+\beta)(1+ \| u \|). \tag{3.2}$$

Step 1: Let us construct u_0^n, u_1^n, .., u_{n-1}^n in H such that, for each $i = \{0, ..., n\}$ the following inclusions hold

$$u_{i+1}^n \in D(t_{i+1}^n, u_i^n), \tag{3.3}$$

$$u_{i+1}^n \in \mathrm{Proj}_{D(t_{i+1}^n, u_i^n)} \left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} h(\tau, u_i^n) d\tau \right),$$

with the following inequalities

$$\|u_{i+1}^n\| \leq \Delta, \tag{3.4}$$

and

$$\|u_{i+1}^n - u_i^n\| \leq h_n \left(\frac{L_1 + 2(\alpha+\beta)(1+\Delta)}{1-L_2} \right).$$

Indeed, using the ball compactness of $D(t_1^n, u_0^n)$, we can choose $u_1^n \in \mathrm{Proj}_{D(t_1^n,u_0^n)} \left(u_0^n - \int_{t_0^n}^{t_1^n} h(\tau, u_0^n) d\tau \right)$ and hence $u_1^n \in D(t_1^n, u_0^n)$. From $(\mathcal{A}_{D_2})$, (3.1) and (3.2), we have

$$\begin{aligned}
\|u_1^n - u_0^n\| &\leq \left\| u_1^n - \left(u_0^n - \int_{t_0^n}^{t_1^n} h(\tau, u_0^n) d\tau \right) \right\| + \int_{t_0^n}^{t_1^n} \|h(\tau, u_0^n)\| d\tau \\
&\leq d_{D(t_1^n, u_0^n)} \left(u_0^n - \int_{t_0^n}^{t_1^n} h(\tau, u_0^n) d\tau \right) + (t_1^n - t_0^n)(\alpha+\beta)(1 + \|u_0^n\|) \\
&\leq d_{D(t_1^n,u_0^n)}(u_0^n) + \int_{t_0^n}^{t_1^n} \|h(\tau, u_0^n)\| d\tau + (t_1^n - t_0^n)(\alpha+\beta)(1+\|u_0^n\|) \\
&\leq \mathrm{exc}(D(t_0^n, u_0^n), D(t_1^n, u_0^n)) + 2(t_1^n - t_0^n)(\alpha+\beta)(1+\|u_0^n\|),
\end{aligned}$$

which implies that

$$\|u_1^n - u_0^n\| \leq (t_1^n - t_0^n)(L_1 + 2(\alpha+\beta)(1+\|u_0^n\|)), \tag{3.5}$$

then
$$\|u_1^n - u_0^n\| \le (t_1^n - t_0^n)\frac{L_1 + 2(\alpha+\beta)(1+\|u_0^n\|)}{1-L_2},$$
and hence
$$\begin{aligned}\|u_1^n\| &\le \|u_1^n - u_0^n\| + \|u_0^n\| \le h_n \frac{L_1 + 2(\alpha+\beta)(1+\|u_0^n\|)}{1-L_2} + \|u_0^n\| \\ &\le h_n \frac{L_1 + 2(\alpha+\beta)}{1-L_2} + h_n\frac{2(\alpha+\beta)}{1-L_2}\|u_0^n\| + \|u_0^n\| \\ &\le \left\{\|u_0^n\| + h_n\frac{L_1+2(\alpha+\beta)}{1-L_2}\right\}\left(1 + 2h_n\frac{\alpha+\beta}{1-L_2}\right) \\ &\le \left\{\|u_0^n\| + T\frac{L_1+2(\alpha+\beta)}{1-L_2}\right\} e^{2T\dfrac{\alpha+\beta}{1-L_2}} = \Delta.\end{aligned}$$
Suppose now that the points $u_0^n, ..., u_i^n$ for $0, ..., i$, with $i \le n-1$ have been constructed. Using the ball compactness of the set $D(t_{i+1}^n, u_i^n)$, ensures that we can choose $u_{i+1}^n \in \mathrm{Proj}_{D(t_{i+1}^n, u_i^n)}\left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} h(\tau, u_i^n)d\tau\right)$.
From $(\mathcal{A}_{D_2})$, (3.1) and (3.2), we get
$$\begin{aligned}\|u_{i+1}^n - u_i^n\| &\le \left\| u_{i+1}^n - \left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} h(\tau,u_i^n)d\tau\right)\right\| + \int_{t_i^n}^{t_{i+1}^n}\|h(\tau,u_i^n)\|d\tau \\ &\le d_{D(t_{i+1}^n,u_i^n)}\left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} h(\tau,u_i^n)d\tau\right) + (t_{i+1}^n - t_i^n)(\alpha+\beta)(1+\|u_i^n\|) \\ &\le d_{D(t_{i+1}^n,u_i^n)}(u_i^n) + \int_{t_i^n}^{t_{i+1}^n}\|h(\tau,u_i^n)\|d\tau + (t_{i+1}^n - t_i^n)(\alpha+\beta)(1+\|u_i^n\|) \\ &\le \mathrm{exc}(D(t_{i+1}^n,u_i^n), D(t_i^n,u_{i-1}^n)) + 2(t_{i+1}^n - t_i^n)(\alpha+\beta)(1+\|u_i^n\|) \\ &\le h_n(L_1 + 2(\alpha+\beta)(1+\|u_i^n\|)) + L_2\|u_i^n - u_{i-1}^n\|.\end{aligned}$$
It implies that
$$\begin{aligned}\|u_{i+1}^n - u_i^n\| \le h_n(L_1 + 2(\alpha+\beta)(1+\|u_i^n\|)) + h_nL_2(L_1 + 2(\alpha+\beta)(1+\|u_{i-1}^n\|)) \\ +L_2^2\|u_{i-1}^n - u_{i-2}^n\|.\end{aligned}$$
Thus, we deduce that
$$\|u_{i+1}^n - u_i^n\| \le h_n\sum_{m=1}^{i} L_2^{i-m}(L_1 + 2(\alpha+\beta)(1+\|u_m^n\|)) + L_2^i\|u_1^n - u_0^n\|.$$
By (3.5) it follows that
$$\begin{aligned}\|u_{i+1}^n - u_i^n\| &\le h_n\sum_{m=1}^{i} L_2^{i-m}(L_1 + 2(\alpha+\beta)(1+\|u_m^n\|)) + L_2^i h_n(L_1 + 2(\alpha+\beta)(1+\|u_0^n\|)) \\ &\le h_n\sum_{m=0}^{i} L_2^{i-m}(L_1 + 2(\alpha+\beta)(1+\|u_m^n\|))\end{aligned}$$

$$\leq h_n \left(\frac{L_1 + 2(\alpha+\beta)}{1-L_2} + 2(\alpha+\beta) \sum_{m=0}^{i} L_2^{i-m} \|u_m^n\| \right).$$

On another hand, for $i = 0, \cdots, n-1$, we have

$$\|u_{i+1}^n\| \leq \|u_0^n\| + \sum_{k=0}^{i} \|u_{k+1}^n - u_k^n\|,$$

so

$$\begin{aligned}
\|u_{i+1}^n\| &\leq \|u_0^n\| + h_n(i+1)\frac{L_1+2(\alpha+\beta)}{1-L_2} + 2h_n(\alpha+\beta)\sum_{k=0}^{i}\left(\sum_{m=0}^{k} L_2^{k-m}\right)\|u_m^n\| \\
&= \|u_0^n\| + h_n(i+1)\frac{L_1+2(\alpha+\beta)}{1-L_2} + 2h_n(\alpha+\beta)\sum_{m=0}^{i}\left(\sum_{k=m}^{i} L_2^{k-m}\right)\|u_m^n\| \\
&\leq \left\{\|u_0^n\| + T\frac{L_1+2(\alpha+\beta)}{1-L_2}\right\} + 2h_n\frac{\alpha+\beta}{1-L_2}\sum_{m=0}^{i}\|u_m^n\|.
\end{aligned}$$

Using the discrete version of Gronwall's inequality in [19] we obtain for $i = 0, ..., n-1$

$$\|u_{i+1}^n\| \leq \left\{\|u_0^n\| + T\frac{L_1+2(\alpha+\beta)}{1-L_2}\right\} e^{2T\frac{\alpha+\beta}{1-L_2}} = \Delta,$$

and consequently, we get

$$\|u_{i+1}^n - u_i^n\| \leq h_n \frac{L_1 + 2(\alpha+\beta)(1+\Delta)}{1-L_2} := h_n\Lambda. \tag{3.6}$$

Step 2 : Construction of $(u_n(\cdot))_n, (\theta_n(\cdot))_n, (\gamma_n(\cdot))_n$ from I to H. For every $i = 0, ..., n-1$, we define the sequences $\theta_n(\cdot), \gamma_n(\cdot) : I \longrightarrow H$ by

$$\theta_n(t) = \begin{cases} t_{i+1}^n & if \quad t \in]t_i^n, t_{i+1}^n] \\ t_1^n & if \quad t = 0 \end{cases} \qquad \gamma_n(t) = \begin{cases} t_i^n & if \quad t \in [t_i^n, t_{i+1}^n[\\ t_n^n = T & if \quad t = T, \end{cases}$$

and $u_n : [0, T] \to H$, by

$$u_n(t) = \begin{cases} u_i^n & if \quad t \in [t_i^n, t_{i+1}^n[\\ u_n^n & if \quad t = T. \end{cases} \tag{3.7}$$

Observe first that, for all $t \in I$

$$\lim_{n\to+\infty} |\theta_n(t) - t| = \lim_{n\to+\infty} |\gamma_n(t) - t| = 0.$$

We get from (3.3) that

$$u_n(\theta_n(t)) \in D(\theta_n(t), u_n(\gamma_n(t))). \tag{3.8}$$

Step 3: Let us prove the convergence of $u_n(\cdot)$ to some absolutely continuous mapping $u(\cdot)$.

For all integer $n \in \mathbb{N}$ by (3.4) and (3.7), we remark that

$$\|u_n(t)\| \leq \Delta \quad \text{for all} \quad t \in I,$$

from (3.6), we obtain

$$\operatorname{var}(u_n; I) = \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} \|u_{i+1}^n - u_i^n\| \leq n\Lambda h_n = n\Lambda \frac{T}{n} = T\Lambda.$$

According to Theorem 2.1 in [21], there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$, noted also $(u_n)_n$, and a function u with bounded variation such that

$$u_n(t) \underset{n\to\infty}{\longrightarrow} u(t) \text{ weakly in } H \quad \text{for all } t \in I. \tag{3.9}$$

Now we will show that $u(\cdot)$ is Lipschitz continuous. For any $p, q \in \{0, ..., n\}$ and for all $t, s \in I$, with $t \leq s$. Then $t \in [t_p^n, t_{p+1}^n[$ and $s \in [t_q^n, t_{q+1}^n[$. From (3.6) and (3.7) we obtain

$$\begin{aligned}
\|u_n(t) - u_n(s)\| = \|u_p^n - u_q^n\| &\leq \sum_{k=0}^{q-p-1} \|u_{p+k+1}^n - u_{p+k}^n\| \\
&\leq \Lambda \sum_{k=0}^{q-p+1} (t_{p+k+1}^n - t_{p+k}^n) \leq \Lambda(t_q^n - t_p^n) \\
&\leq \Lambda(|t-s| + |s - t_q^n| + |t_p^n - t|) \leq \Lambda(|t-s| + 2\frac{T}{n}).
\end{aligned} \tag{3.10}$$

Using the weak convergence of u_n to u, we get

$$\|u(t) - u(s)\| \leq \liminf_{n\to+\infty} \|u_n(t) - u_n(s)\| \leq \Lambda(|t-s|),$$

hence, $u(\cdot)$ is Lipschitz continuous on I and we have

$$\|\dot{u}(t)\| \leq \Lambda \quad \text{a.e. } t \in I. \tag{3.11}$$

Next we prove that the set $\Lambda(t) = \{u_n(t) \ : \ n \in \mathbb{N}\}$ is relatively compact in H. From (3.8) and (3.4), we get

$$(u_n(\theta_n(t)))_n \subset D(I \times \Delta\mathbb{B}) \cap \Delta\mathbb{B}.$$

Using the fact that $D(I \times \Delta\mathbb{B}) \cap \Delta\mathbb{B}$ is relatively compact, then $(u_n(\theta_n(t)))_n$ is relatively compact so from (3.10) we have

$$\|u_n(\theta_n(t)) - u_n(t)\| \leq \Lambda\left(\theta_n(t) - t + 2\frac{T}{n}\right) \to 0, \quad as \quad n \to \infty.$$

Thus the set $\{u_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in H. According to the equicontinuity obtained by (3.11), we deduce that $(u_n(t))_{n\in\mathbb{N}}$ is relatively compact in $\mathscr{C}_H(I)$, then we can extract a subsequence of $(u_n)_n$ (that we do not relabel) which converges uniformly to u.

Step 04: We show now that $u(\cdot)$ is a solution of $(\mathcal{S})$. Note first that, from (3.7) and (3.9) we have $u(0) = u_0$, and $u(t) \in D(t, u(t))$. Indeed,

$$\begin{aligned}
d_{D(t,u(t))}(u_n(\theta_n(t))) &\leq \operatorname{exc}(D(\theta_n(t), u_n(\gamma_n(t))), D(t, u(t))) \\
&\leq L_1 \frac{T}{n} + L_2 \|u_n(\gamma_n(t)) - u(t)\|,
\end{aligned}$$

using the fact that $\|u_n(\gamma_n(t)) - u(t)\| \underset{n\to\infty}{\longrightarrow} 0$ and since $D(t,u(t))$ is closed, we obtain $u(t) \in D(t,u(t))$ *for every* $t \in I$.
Let us define the mapping $v_n : I \longrightarrow H$ for each $i \in \{0,...,n-1\}$, by

$$v_n(t) = u_i^n + \frac{t-t_i^n}{t_{i+1}^n - t_i^n}\left(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} h(\tau,u_i^n)d\tau\right) - \int_{t_i^n}^{t} h(\tau,u_i^n)d\tau.$$

Observe that, by the definition of u_n and v_n and the relations (3.2), (3.4) and (3.6), we have for all $t \in [t_i^n, t_{i+1}^n]$

$$\|v_n(t) - u(t)\| \leq \|v_n(t) - u_n(t)\| + \|u_n(t) - u(t)\|$$

$$\leq \frac{t-t_i^n}{t_{i+1}^n - t_i^n}\left(\|u_{i+1}^n - u_i^n\| + \int_{t_i^n}^{t_{i+1}^n} \|h(\tau,u_i^n)\|d\tau\right) + \int_{t_i^n}^{t} \|h(\tau,u_i^n)\|d\tau + \|u_n(t)-u(t)\|$$

$$\leq \frac{t-t_i^n}{t_{i+1}^n - t_i^n}\left(\|u_{i+1}^n - u_i^n\| + (\alpha+\beta)(1+\Delta)\int_{t_i^n}^{t_{i+1}^n} d\tau\right)$$

$$+(\alpha+\beta)(1+\Delta)\int_{t_i^n}^{t} d\tau + \|u_n(t) - u(t)\|$$

$$\leq (t-t_i^n)\left(\Lambda + 2(\beta+\alpha)(1+\Delta)\right) + \|u_n(t) - u(t)\|$$

$$\leq \frac{T}{n}(\Lambda + 2(\beta+\alpha)(1+\Delta)) + \|u_n(t) - u(t)\|,$$

Since u_n converge uniformly to u then, we conclude that

$$v_n(t) \longrightarrow u(t) \quad \text{for all } \ t \in I.$$

For every integer $n \in \mathbb{N}$, $i \in \{0,...,n-1\}$, $v_n(\cdot)$ is differentiable on $]t_i^n, t_{i+1}^n[$ and

$$\dot{v}_n(t) + h(t,u_i^n) = \frac{u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} h(s,u_i^n)ds}{t_{i+1}^n - t_i^n}.$$

Now let's define the mappings ξ_n, $\vartheta_n : I \longrightarrow H$ by

$$\xi_n(t) := g(t,u_n(\delta_n(t))) \quad \text{and} \quad \vartheta_n(t) := f(t,u_n(\delta_n(t))) \quad \text{for all } \ t \in I.$$

Using (3.2) we get, for all $t \in I$

$$\|\xi_n(t)\| \leq \beta(1+\Delta) \quad \text{and} \quad \|\vartheta_n(t)\| \leq \alpha(1+\Delta) \quad \text{for all } \ t \in I. \tag{3.12}$$

It follows that

$$\|\dot{v}_n(t) + \vartheta_n(t) + \xi_n(t)\| \leq \Lambda + (\alpha+\beta)(1+\Delta) = l \ \text{ a.e. } t \in I. \tag{3.13}$$

On another hand, we have

$$\dot{v}_n(t) + \vartheta_n(t) + \xi_n(t) \in -N_{D(\theta_n(t),u_n(\delta_n(t)))}(u_n(\theta_n(t))), \tag{3.14}$$

then

$$\dot{v}_n(t) + \vartheta_n(t) + \xi_n(t) \in -N_{D(\theta_n(t),u_n(\delta_n(t)))}(u_n(\theta_n(t))) \cap l\mathbb{B}.$$

According to (2.1), (3.14) and Proposition 2.1, we get

$$\dot{v}_n(t) + \vartheta_n(t) + \xi_n(t) \in -l\partial d_{D(\theta_n(t),u_n(\gamma_n(t)))}(u_n(\theta_n(t))) \ \text{ a.e. } t \in I.$$

From (3.12), by extracting a subsequence if necessary, we conclude that $\xi_n(\cdot)$ converges weakly in $L^1(I,H)$ to a mapping $\xi(\cdot) \in L^1(I,H)$ such that $\|\xi(t)\| \leq \beta(1+\Delta)$, and by the continuity assumption on $f(\cdot,u(\cdot))$, we deduce that $\vartheta_n(\cdot)$

converges to $\vartheta(\cdot) = f(\cdot, u(\cdot))$ in $L^1(I, H)$.
From (3.13), we have

$$\|\dot{v}_n(t)\| \leq l + (\alpha + \beta)(1 + \Delta) \text{ a.e. } t \in I,$$

which give us the weak convergence in $L^1(I, H)$ of $(\dot{v}_n(\cdot))$ to some $\varphi(\cdot) \in L^1(I, H)$. Using the absolute continuity of each $v_n(\cdot)$, we obtain

$$v_n(t) = v_n(0) + \int_0^t \dot{v}_n(s) ds \quad \text{for all } \ t \in I,$$

passing to the limit, we get $u(t) = u(0) + \int_0^t \varphi(s) ds$, and consequently $\dot{u}(\cdot) = \varphi(\cdot)$ a.e. on I. This yields that $\dot{v}_n(\cdot) \longrightarrow \dot{u}(\cdot)$ weakly in $L^1(I, H)$. Since, $(\dot{v}_n + \vartheta_n + \xi_n, \xi_n)_n$ converge weakly in $L^1(I, H)$ to $(\dot{u} + \vartheta + \xi, \xi)$, by Mazur's Lemma , there is a sequence $(\Lambda_n, \varpi_n)_n$ which converges strongly in $L^1([0, T], H)$ to $(\dot{u} + \vartheta + \xi, \xi)$ with $\Lambda_n \in co\{\dot{v}_m + \vartheta_m + \xi_m : m \geq n\}$ and $\varpi_n \in co\{\xi_m : m \geq n\}$.
As a result, for almost all $t \in [0, T]$

$$\dot{u}(t) + \vartheta(t) + \xi(t) \in \bigcap_{n \geq 0} \overline{co}\{\dot{v}_m(t) + \vartheta_m(t) + \xi_m(t) : \quad m \geq n\},$$

and

$$\xi(t) \in \bigcap_{n \geq 0} \overline{co}\{\xi_m(t) : m \geq n\},$$

which implies that, for every $t \in I$, $n \in \mathbb{N}$, we have

$$\langle y, \dot{u}(t) + \vartheta(t) + \xi(t) \rangle \leq \inf_{n \geq 0} \sup_{m \geq n} \langle y, \dot{v}_m(t) + \vartheta_m(t) + \xi_m(t) \rangle \quad \text{for all } y \in H,$$

and

$$\langle y, \xi(t) \rangle \leq \inf_{n \geq 0} \sup_{m \geq n} \langle y, \xi_m(t) \rangle \quad \text{for all } y \in H.$$

It follows that, for every $t \in I$,

$$\langle y, \dot{u}(t) + \vartheta(t) + \xi(t) \rangle \leq \limsup_{n \longrightarrow +\infty} \sigma\left(y, -l \partial d_{D(\theta_n(t), u_n(\gamma_n(t)))}(u_n(\theta_n(t)))\right) \quad \text{for all } y \in H,$$

and

$$\langle y, \xi(t) \rangle \leq \limsup_{n \longrightarrow +\infty} \sigma\left(y, G(\delta_n(t), u_n(t))\right) \quad \text{for all } y \in H.$$

Hence, according to Proposition 2.2, for every $t \in I$,

$$\langle y, \dot{u}(t) + \vartheta(t) + \xi(t) \rangle \leq \sigma\left(y, l \partial d_{D(t, u(t))}(u(t))\right),$$

using the fact that $G(\cdot, \cdot)$ is scalarly upper semicontinuous, we have

$$\langle y, \xi(t) \rangle \leq \sigma\left(y, G(t, u(t))\right) \quad \text{for all } y \in H,$$

which ensures that

$$\{\dot{u}(t) + \vartheta(t) + \xi(t)\} \subset \overline{co}(-l \partial d_{D(t, u(t))}(u(t))) \ \text{ and } \ \xi(t) \in \overline{co}(G(t, u(t))) \quad \text{a.e. } t \in I,$$

or equivalently

$$\dot{u}(t) + \vartheta(t) + \xi(t) \in -l \partial d_{D(t, u(t))}(u(t)) \subset -N_{D(t, u(t))}(u(t)) \quad \text{a.e. } t \in I,$$

and

$$\xi(t) \in G(t, u(t)).$$

This completes the proof. □

The corollary below is a direct consequence of Theorem 3.1.

Corollary 3.1. *Let $G : [0,T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set valued mapping with convex closed values in $\mathbb{R}^n$ such that $G(t,\cdot)$ is scalary upper semi continuous on Hsatisfying $(\mathcal{A}_{G_2})$. Assume that $(\mathcal{A}_{D_1})$, $(\mathcal{A}_{D_2})$and $(\mathcal{A}_f)$ are satisfies. Then for all $u_0 \in H$ and $u_0 \in D(0,u_0)$, there exist a Lipschitz mapping $u : [0,T] \to \mathbb{R}^n$ satisfies $(\mathcal{S})$ with $\|\dot{u}(t)\| \leq \frac{L_1+2(\alpha+\beta)(1+\Delta)}{1-L_2}$ a.e.*

4. Application to a complementarity problem

In this section, we apply the previous results to prove the existence of solution to the following nonsmooth dynamical system

$$(\mathcal{D}) \begin{cases} \dot{u}(t) = f(t,u(t)) + g(t,u(t)) + [Jh(u(t))]^* x(t) \\ y(t) = h(u(t)) + \phi(t,u(t)) \\ K \ni y(t) \perp x(t) \in K^* \\ u(0) = u_0, \end{cases}$$

where $f(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ verifies $(\mathcal{A}_f)$; $g(t,u(t)) \in G(t,u(t))$ satisfies $(\mathcal{A}_{G_1})$, $(\mathcal{A}_{G_2})$; $K \subset \mathbb{R}^n$ is a closed convex cone with dual cone $K^* = \{y \in \mathbb{R}^n : \langle v,y\rangle \geq 0,\ v \in K\}$. $\phi(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a Lipschitz function with constant $k_d < \frac{1}{k'}$ such that k' is a constant to be chosen later, $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuously differentiable function such that the associated Jacobian matrices Jh is uniformly continuous and satisfies the following assumption

$$\exists k > 0 \text{ such that } \mathbb{B}_{\mathbb{R}^n} \subseteq Jh(u(t))k\mathbb{B}_{\mathbb{R}^n} - K. \tag{4.1}$$

The notation "$\perp$" means orthogonality.

This class of problems is an important class of nonsmooth dynamical systems that is of use in mechanical and electrical engineering as well as in optimization and in other fields. It consists of an ordinary differential equation coupled with a nonlinear complementarity problem in the constraint; for more details, we refer to [13, 14].

The third relation in $(\mathcal{D})$ can be expressed as (see[24])

$$-x(t) \in N_K(y(t)),$$

then

$$-[Jh(u(t))]^* x(t) \in [Jh(u(t))]^* N_K(h(u(t)) + \phi(t,u(t)))$$
$$-f(t,u(t)) - g(t,u(t)) - [Jh(u(t))]^* x(t) \in [Jh(u(t))]^* N_K(h(u(t)) + \phi(t,u(t))) - f(t,u(t)) - g(t,u(t)).$$

According to the first relation in $(\mathcal{D})$ we obtain

$$\dot{u}(t) \in -[Jh(u(t))]^* N_K(h(u(t)) + \phi(t,u(t))) + f(t,u(t)) + g(t,u(t)).$$

From Proposition 1 in [18], we get

$$\dot{u}(t) \in -N_{D(t,u(t))}(u(t)) + f(t,u(t)) + g(t,u(t))$$

where $D(t,u(t)) = h^{-1}(K - \phi(t,u(t)))$. In order to prove that the set $D(t,u(t)) = h^{-1}(K - \phi(t,u(t)))$ is uniformly subsmooth we have used Proposition 3.14 in [20], therefore it is sufficient to prove that $h^{-1}(K - \phi(t,u(t)))$ satisfies the uniform normal cone inverse image property, that is

$$N(h^{-1}(K - \phi(t,u(t))),x) \cap \mathbb{B}_{\mathbb{R}^n} \subseteq Jh(x)^* \Big[N(K - \phi(t,u(t)),x) \cap \beta\mathbb{B}_{\mathbb{R}^n}\Big],$$

for some $\beta > 0$. it is known that there exists $k > 0$ such that for all $x \in \mathbb{R}^n$

$$\mathbb{B}_{\mathbb{R}^n} \subseteq kJh(x)\mathbb{B}_{\mathbb{R}^n} - K,$$

so by Proposition 3.13 in [20] we get the uniform normal cone inverse image property with $\beta = k$ and consequently $h^{-1}(K-\phi(t, u(t)))$ is uniformly subsmooth. Let us now prove that $D(t, u(t))$ satisfies the hypothesis $(\mathcal{A}_{D_2})$. For this purpose, we must prove the following result

Proposition 4.1. *Let $h : \mathbb{B}_{\mathbb{R}^n} \to \mathbb{B}_{\mathbb{R}^n}$ be a continuously differentiable function such that Jh is uniformly continuous and let K be a closed convex cone such that* (4.1) *is satisfied. Then, the set valued map $v \rightrightarrows h^{-1}(K - v)$ is $k'-$Lipschitz continuous for every $k' > k$.*

Since K is a cone, then from (4.1) and Proposition 3.13 in [20], for all $x \in \mathbb{R}^n$ and for all $w \in \mathbb{R}^n$ we have

$$d_{h^{-1}(K-v)}(x) \leq kd_K(h(x) + v).$$

Let v_1, $v_2 \in \mathbb{R}^n$ and $x \in h^{-1}(K - v_1)$. Then,

$$d_{h^{-1}(K-v_2)}(x) \leq kd_K(h(x) + v_2) \leq k\|v_2 - v_1\|.$$

Therefore, for $\epsilon < k' - k$, there exists $x_\epsilon \in h^{-1}(K - v_2)$ such that

$$\|x - x_\epsilon\| \leq d_{h^{-1}(K-v_2)}(x) + \epsilon\|v_2 - v_1\|,$$

and thus

$$\|x - x_\epsilon\| \leq (k + \epsilon)\|v_2 - v_1\| \leq k'\|v_2 - v_1\|,$$

there ensues

$$h^{-1}(K - v_1) \subset h^{-1}(K - v_2) + k'\|v_2 - v_1\|,$$

which completes the proof of the proposition.■

Let (τ, u), $(t, w) \in [0, T] \times \mathbb{R}^n$ with $\tau < t$

$$\begin{aligned} exc\Big(D(\tau, u), D(t, w)\Big) &\leq \mathcal{H}\Big(D(\tau, u), D(t, w)\Big) \\ &= \sup_{v\in\mathbb{R}^n} |d_{D(\tau,u)}(v) - d_{D(t,w)}(v)| \\ &\leq k'\Big[\|\phi(\tau, u) - \phi(t, w)\|\Big] \\ &\leq k'k_d\Big[(t - \tau) + \|u - w\|\Big] \end{aligned}$$

It follows that $(\mathcal{A}_{D_2})$ is satisfied. Then, according to Corollary 3.1 the problem $(\mathcal{D})$ has a solution.

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Imene Mecemma
Laboratory of LMPA, Department of Mathematics, Faculty of exact sciences and computer science, University of Jijel, 18000 Jijel, Algeria.
E-mail address: `mecemma.imene@yahoo.com`

Sabrina Lounis
Laboratory of LMPA, Department of Mathematics, Faculty of exact sciences and computer science, University of Jijel, 18000 Jijel, Algeria.
E-mail address: `lounis_18sabrina@yahoo.fr`

Mustapha Fateh Yarou
Laboratory of LMPA, Department of Mathematics, Faculty of exact sciences and computer science, University of Jijel, 18000 Jijel, Algeria.
E-mail address: `mfyarou@yahoo.com`

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