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SOME RESULTS OF HARMONICITY ON TANGENT BUNDLES WITH φ-SASAKIAN METRICS OVER PARA-KÄHLER-NORDEN MANIFOLD

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Abstract. In this paper, we introduce the φ-Sasaki metric on the tangent bundle TM over a para-Kähler-Norden manifold (M^{2m}, φ, g). First, we study the harmonicity of a vector field with respect to this metric, we also construct some examples of harmonic vector fields. Secondly, we study the harmonicity of a vector field along a map between Riemannian manifolds, the target manifold being para-Kähler-Norden equipped with the φ-Sasaki metric on its tangent bundle. Thirdly, we discuss the harmonicity of the composition of the projection map of the tangent bundle of a Riemannian manifold with a map from this manifold into another Riemannian manifold, the source manifold being para-Kähler-Norden whose tangent bundle is endowed with the φ-Sasaki metric. Finally, we study the harmonicity of the tangent map of smooth map between para-Kähler-Norden manifolds.

1. Introduction

On the tangent bundle of a Riemannian manifold one can define natural Riemannian metrics. Their construction makes use of the Levi-Civita connection. Among them, the so called Sasaki metric [20] is of particular interest. That is why the geometry of tangent bundle equipped with the Sasaki metric has been studied by many authors such as Yano and Ishihara [22], Dombrowski [4], Salimov and his collaborators [17] etc. The rigidity of Sasaki metric has incited some researchers to construct and study other metrics on tangent bundle. This is the reason why they have attempted to search for different metrics on the tangent bundle which are different deformations of the Sasaki metric. Musso and Tricerri have introduced the Cheeger-Gromoll metric [16], which has been studied also by many authors see [10, 21].

In a previous work, [24], we proposed the φ-Sasaki metric on the tangent bundle where we studied the para-Kahler-Norden properties on the tangent bundle with this metric. In this paper, we investigate some harmonicity properties for the φ-Sasaki metric on the tangent bundle. Firstly, we study the harmonicity of a vector field with respect to this metric (Theorem 4.2 and Theorem 4.3). Secondly,

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we study the harmonicity of a vector field along a map between Riemannian manifolds, the target manifold being anti-paraKähler equipped with the φ-Sasaki metric on its tangent bundle (Proposition 4.2 and Theorem 4.6). Thirdly, we discuss the harmonicity of the composition of the projection map of the tangent bundle of a Riemannian manifold with a map from this manifold into another Riemannian manifold, the source manifold being anti-para-Kähler whose tangent bundle is endowed with the φ-Sasaki metric (Theorem 4.7 and Theorem 4.8). Finally, we study the harmonicity of the tangent map of smooth map between para-Kähler-Norden manifolds (Theorem 4.9 and Theorem 4.10).

2. Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, u^i)_{i=\overline{1,m}}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and $\Im_0^1(M)$ be the module over $C^\infty(M)$ of C^∞ vector fields on M.

The Levi Civita connection ∇ defines a direct sum decomposition

$$T_{(x,u)}TM = V_{(x,u)}TM \oplus H_{(x,u)}TM. \tag{2.1}$$

of the tangent bundle to TM at any $(x,u) \in TM$ into vertical subspace

$$V_{(x,u)}TM \;=\; Ker(d\pi_{(x,u)}) = \{\xi^i \frac{\partial}{\partial u^i}|_{(x,u)},\, \xi^i \in \mathbb{R}\}, \tag{2.2}$$

and the horizontal subspace

$$H_{(x,u)}TM \;=\; \{\xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}|_{(x,u)},\, \xi^i \in \mathbb{R}\}. \tag{2.3}$$

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$^VX \;=\; X^i \frac{\partial}{\partial u^i}, \tag{2.4}$$

$$^HX \;=\; X^i (\frac{\partial}{\partial x^i} - u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}). \tag{2.5}$$

We have $^H(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}$ and $^V(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial u^i}$, then $(^H(\frac{\partial}{\partial x^i}), {}^V(\frac{\partial}{\partial x^i}))_{i=\overline{1,m}}$ is a local adapted frame on TTM.

The bracket operation of vertical and horizontal vector fields is given by the formulas: [4, 22]

$$\begin{cases} [^HX, {}^HY] = {}^H[X,Y] - {}^V(R(X,Y)u), \\ [^HX, {}^VY] = {}^V(\nabla_X Y), \\ [^VX, {}^VY] = 0, \end{cases} \tag{2.6}$$

for all vector fields $X, Y \in \Im_0^1(M)$, where R is the Riemannian curvature of g.

3. φ-Sasaki metric

An almost product structure φ on a manifold M is a $(1,1)$ tensor field on M such that $\varphi^2 = id_M$, $\varphi \neq \pm id_M$ (id_M is the identity tensor field of type $(1,1)$ on M). The pair (M, φ) is called an almost product manifold.

An almost para-complex manifold is an almost product manifold (M, φ), such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even [2].

An almost para-complex structure φ is integrable if the Nijenhuis tensor:

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + [X,Y] \tag{3.1}$$

vanishes identically on M for all vector fields $X, Y \in \Im_0^1(M)$. Moreover, for an almost para-complex structure φ to be integrable, it is necessary and sufficient that we can introduce a torsion free linear connection ∇ such that $\nabla\varphi = 0$ [19].

An almost para-complex Norden manifold (M^{2m}, φ, g) is a $2m$-dimensional differentiable manifold M with an almost para-complex structure φ and a Riemannian metric g such that:

$$g(\varphi X, Y) = g(X, \varphi Y) \quad \Leftrightarrow \quad g(\varphi X, \varphi Y) = g(X,Y), \tag{3.2}$$

for any vector fields $X, Y \in \Im_0^1(M)$, in this case g is called a pure metric with respect to φ or para-Norden metric (B-metric)[19].

Also note that

$$G(X,Y) = g(\varphi X, Y), \tag{3.3}$$

is a bilinear, symmetric tensor field of type $(0,2)$ on (M, φ) and pure with respect to the paracomplex structure φ, which is called the twin (or dual) metric of g, and it plays a role similar to the Kähler form in Hermitian Geometry. Some properties of twin Norden metric are investigated in [11, 19].

A para-Kähler-Norden manifold is an almost para-complex Norden manifold (M^{2m}, φ, g) such that $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g [18, 19].

It is well known that if (M^{2m}, φ, g) is a para-Kähler-Norden manifold, the Riemannian curvature tensor is pure [19]. Moreover, we have

$$\begin{cases} R(\varphi Y, Z) = R(Y, \varphi Z) = R(Y,Z)\varphi = \varphi R(Y,Z), \\ R(\varphi Y, \varphi Z) = R(Y,Z), \end{cases} \tag{3.4}$$

for any vector fields $Y, Z \in \Im_0^1(M)$.

Definition 3.1. [24] Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold. On the tangent bundle TM, we define a φ-Sasaki metric noted g^φ by

$$\begin{aligned} (1) \quad g^\varphi({}^HX, {}^HY)_{(x,u)} &= g_x(X,Y), \\ (2) \quad g^\varphi({}^HX, {}^VY)_{(x,u)} &= 0, \\ (3) \quad g^\varphi({}^VX, {}^VY)_{(x,u)} &= g_x(X, \varphi Y) = G(X,Y), \end{aligned}$$

for any vector fields $X, Y \in \Im_0^1(M)$ and $(x,u) \in TM$, where G is the twin Norden metric of g defined by (3.3).

The Levi-Civita connection $\widetilde{\nabla}$ of TM with φ-Sasaki metric g^φ is given by the following theorem.

Theorem 3.1. [24] *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^{φ}. If ∇ (resp $\widetilde{\nabla}$) denote the Levi-Civita connection of (M^{2m},φ,g) (resp (TM,g^{φ})), then we have*

$$\begin{array}{lrcl}
(1) & (\widetilde{\nabla}_{{}^HX}{}^HY)_{(x,u)} & = & {}^H(\nabla_XY)_{(x,u)}-\frac{1}{2}{}^V(R_x(X,Y)u),\\
(2) & (\widetilde{\nabla}_{{}^HX}{}^VY)_{(x,u)} & = & {}^V(\nabla_XY)_{(x,u)}+\frac{1}{2}{}^H(R_x(\varphi u,Y)X),\\
(3) & (\widetilde{\nabla}_{{}^VX}{}^HY)_{(x,u)} & = & \frac{1}{2}{}^H(R_x(\varphi u,X)Y),\\
(4) & (\widetilde{\nabla}_{{}^VX}{}^VY)_{(x,u)} & = & 0,
\end{array}$$

for all vector fields $X,Y\in\Im^1_0(M)$ and $(x,u)\in TM$, where R denote the curvature tensor of (M^{2m},φ,g).

4. φ-Sasaki metric and harmonicity

Let $\phi:(M^m,g)\to(N^n,h)$ be a smooth map between two Riemannian manifolds. The map ϕ is said to be harmonic if it is a critical point of the energy functional

$$E(\phi,K)=\int_K e(\phi)\,v^g, \tag{4.1}$$

for any compact domain $K\subseteq M$. Here

$$e(\phi):=\frac{1}{2}Tr_g\,h(d\phi,d\phi) \tag{4.2}$$

is the energy density of ϕ, Tr_g stands for the trace with respect to g and v^g is the Riemannian volume form on M. For any smooth 1-parameter variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0=\phi$ and $V=\frac{d}{dt}\phi_t\Big|_{t=0}$ [13], we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0}=-\int_K h(\tau(\phi),V)v^g. \tag{4.3}$$

Then, ϕ is harmonic if it satisfies the associated Euler-Lagrange equations given by the following formula:

$$0=\tau(\phi):=Tr_g\nabla d\phi, \tag{4.4}$$

where $\tau(\phi)$ is the tension field of ϕ. For more details see [7, 8, 9, 12, 14]. In recent years, this theme has been widely developed even on the tangent bundle and on the cotangent bundle has been done by many authors [1, 3, 23, 25, 26, 27].

4.1. Harmonicity of a vector field $X:(M^{2m},g,\varphi)\to(TM,g^{\varphi})$.

A vector field $X\in\Im^1_0(M)$ on (M^{2m},g,φ) can be regarded as the immersion

$$\begin{array}{rcl}
X:(M^{2m},g,\varphi) & \to & (TM,g^{\varphi})\\
x & \mapsto & (x,X_x)
\end{array}$$

into its tangent bundle TM equipped with the φ-Sasaki metric g^{φ}.

Lemma 4.1. [14, 15] *Let (M^m,g) be a Riemannian manifold. If $X,Y\in\Im^1_0(M)$ are vector fields and $(x,u)\in TM$ such that $Y_x=u$, then we have:*

$$d_xY(X_x)={}^HX_{(x,u)}+{}^V(\nabla_XY)_{(x,u)}. \tag{4.5}$$

Lemma 4.2. *Let(M^{2m}, φ, g) be a para-Kähler-Norden manifold, $X \in \Im_0^1(M)$ be a vector field. Then the following equation is satisfied:*

$$g(\bar{\Delta}X, X) = |\nabla X|^2 - \frac{1}{2}\Delta|X|^2. \tag{4.6}$$

where $\bar{\Delta}X := -Tr_g\nabla^2 X = -Tr_g(\nabla_*\nabla_* - \nabla_{\nabla_**})X$ is the rough Laplacian of X and Δ is the ordinary Laplace-Beltrami operator acting on functions.

Proof. Let $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M, then we have

$$\begin{aligned}
g(\bar{\Delta}X, X) &= -g(Tr_g(\nabla_*\nabla_* - \nabla_{\nabla_**})X, X) \\
&= -\sum_{i=1}^{2m}\big(g(\nabla_{e_i}\nabla_{e_i}X, X) - g(\nabla_{\nabla_{e_i}e_i}X, X)\big) \\
&= -\sum_{i=1}^{2m}\big(e_i(g(\nabla_{e_i}X, X)) - g(\nabla_{e_i}X, \nabla_{e_i}X) - \frac{1}{2}\nabla_{e_i}e_i(g(X, X))\big) \\
&= -\sum_{i=1}^{2m}\big(\frac{1}{2}e_ie_i(|X|^2) - |\nabla_{e_i}X|^2 - \frac{1}{2}\nabla_{e_i}e_i(|X|^2)\big) \\
&= |\nabla X|^2 - \frac{1}{2}\Delta|X|^2.
\end{aligned}$$

□

Lemma 4.3. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and $X \in \Im_0^1(M)$ be a vector fields. Then the following equation is satisfied:*

$$\bar{\Delta}(fX) = f\bar{\Delta}X - (\Delta f)X - 2\nabla_{gradf}X, \tag{4.7}$$

where f being a smooth function of M and $gradf$ the gradient of f.

Proof. Let $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M, then we have

$$\begin{aligned}
\bar{\Delta}(fX) &= -\sum_{i=1}^{2m}\big(\nabla_{e_i}\nabla_{e_i}(fX) - \nabla_{\nabla_{e_i}e_i}(fX)\big) \\
&= -\sum_{i=1}^{2m}\big(\nabla_{e_i}(e_i(f)X + f\nabla_{e_i}X) - \nabla_{e_i}e_i(f)X - f\nabla_{\nabla_{e_i}e_i}X\big) \\
&= -\sum_{i=1}^{2m}\big(e_ie_i(f)X + e_i(f)\nabla_{e_i}X + e_i(f)\nabla_{e_i}X + f\nabla_{e_i}\nabla_{e_i}X \\
&\quad -\nabla_{e_i}e_i(f)X - f\nabla_{\nabla_{e_i}e_i}X\big) \\
&= -\sum_{i=1}^{2m}\big((e_ie_i(f) - \nabla_{e_i}e_i(f))X + 2\nabla_{e_i(f)e_i}X + f(\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i})X\big) \\
&= f\bar{\Delta}X - (\Delta f)X - 2\nabla_{gradf}X.
\end{aligned}$$

□

Lemma 4.4. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g^{φ}) its tangent bundle equipped with the φ-Sasaki metric and $X \in \Im_0^1(M)$ be a vector* *field. Then the energy density associated to X is given by:*

$$e(X) = m + \frac{1}{2} Tr_g g(\nabla_* X, \varphi \nabla_* X). \tag{4.8}$$

Proof. Let $(x,u) \in TM$, X be a vector field on M, $X_x = u$ and $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M, from (4.2), we have

$$e(X)_x = \frac{1}{2}\sum_{i=1}^{2m} g^{\varphi}(dX(e_i), dX(e_i))_{(x,u)}.$$

Using (4.5), we obtain:

$$\begin{aligned}
e(X) &= \frac{1}{2}\sum_{i=1}^{2m} g^{\varphi}({}^He_i + {}^V(\nabla_{e_i}X), {}^He_i + {}^V(\nabla_{e_i}X)) \\
&= \frac{1}{2}\sum_{i=1}^{2m} \big(g^{\varphi}({}^He_i, {}^He_i) + g^{\varphi}({}^V(\nabla_{e_i}X), {}^V(\nabla_{e_i}X))\big) \\
&= \frac{1}{2}\sum_{i=1}^{2m} \big(g(e_i, e_i) + g(\nabla_{e_i}X, \varphi\nabla_{e_i}X)\big) \\
&= m + \frac{1}{2} Tr_g g(\nabla_* X, \varphi \nabla_* X).
\end{aligned}$$

□

Theorem 4.1. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, (TM, g^{φ}) its tangent bundle equipped with the φ-Sasaki metric and $X \in \Im^1_0(M)$ be a vector field. The tension field associated to X is given by:*

$$\tau(X) = {}^H\big(Tr_g R(\varphi X, \nabla_* X)*\big) - {}^V\bar{\Delta} X. \tag{4.9}$$

Proof. Let $(x,u) \in TM$ and $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M at x and $X_x = u$. Using (4.4) and (4.5), we have

$$\begin{aligned}
\tau(X)_x &= Tr_g \nabla dX \\
&= \sum_{i=1}^{2m} \big(\nabla^X_{e_i} dX(e_i) - dX\nabla_{e_i}e_i\big)_x \\
&= \sum_{i=1}^{2m} \big(\widetilde{\nabla}_{dX(e_i)} dX(e_i) - {}^H(\nabla_{e_i}e_i) - {}^V(\nabla_{\nabla_{e_i}e_i}X)\big)_{(x,u)} \\
&= \sum_{i=1}^{2m} \big(\widetilde{\nabla}_{({}^He_i + {}^V(\nabla_{e_i}X))}({}^He_i + {}^V(\nabla_{e_i}X)) - {}^H(\nabla_{e_i}e_i) - {}^V(\nabla_{\nabla_{e_i}e_i}X)\big)_{(x,u)} \\
&= \sum_{i=1}^{2m} \big(\widetilde{\nabla}_{{}^He_i}{}^He_i + \widetilde{\nabla}_{{}^He_i}{}^V(\nabla_{e_i}X) + \widetilde{\nabla}_{{}^V(\nabla_{e_i}X)}{}^He_i + \widetilde{\nabla}_{{}^V(\nabla_{e_i}X)}{}^V(\nabla_{e_i}X) \\
&\quad - {}^H(\nabla_{e_i}e_i) - {}^V(\nabla_{\nabla_{e_i}e_i}X)\big)_{(x,u)}.
\end{aligned}$$

Using Theorem 3.1, we obtain

$$\begin{aligned}\tau(X) &= \sum_{i=1}^{2m}\Big({}^H(\nabla_{e_i}e_i) - \frac{1}{2}{}^V(R(e_i,e_i)X) + \frac{1}{2}{}^H(R(\varphi X,\nabla_{e_i}X)e_i)\\ &\quad + {}^V(\nabla_{e_i}\nabla_{e_i}X) + \frac{1}{2}{}^H(R(\varphi X,\nabla_{e_i}X)e_i) - {}^H(\nabla_{e_i}e_i) - {}^V(\nabla_{\nabla_{e_i}e_i}X)\Big)\\ &= \sum_{i=1}^{2m}\big({}^H(R(\varphi X,\nabla_{e_i}X)e_i) + {}^V(\nabla_{e_i}\nabla_{e_i}X - \nabla_{\nabla_{e_i}e_i}X)\big)\\ &= {}^H\big(Tr_g R(\varphi X,\nabla_* X)*\big) + {}^V\big(Tr_g\nabla^2 X\big)\\ &= {}^H\big(Tr_g R(\varphi X,\nabla_* X)*\big) - {}^V\bar{\Delta}X.\end{aligned}$$

□

Theorem 4.2. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, (TM,g^φ) its tangent bundle equipped with the φ-Sasaki metric and $X\in\Im^1_0(M)$ be a vector field. Then X is a harmonic map if and only if the following conditions are verified*

$$Tr_g R(\varphi X,\nabla_* X)* = 0,\ \textit{and}\ \bar{\Delta}X = 0. \tag{4.10}$$

Proof. The proof is a direct consequence of Theorem 4.1. □

Let (M^{2m},φ,g) be a compact oriented para-Kähler-Norden manifold, (TM,g^φ) its tangent bundle equipped with the φ-Sasaki metric and $X\in\Im^1_0(M)$ be a vector field. The energy $E(X)$ of X is defined to be the energy of the corresponding map $X:(M^{2m},\varphi,g)\to(TM,g^\varphi)$. More precisely, from (4.8), we get

$$\begin{aligned}E(X) &= \int_M e(X)v^g\\ &= \int_M\big(m+\frac{1}{2}Tr_g g(\nabla_* X,\varphi\nabla_* X)\big)v^g\\ &= m\,Vol(M) + \frac{1}{2}\int_M Tr_g g(\nabla_* X,\varphi\nabla_* X)v^g.\end{aligned} \tag{4.11}$$

Definition 4.1. Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, (TM,g^φ) its tangent bundle equipped with the φ-Sasaki metric and $X\in\Im^1_0(M)$ be a vector field. We say that X is harmonic vector field if the corresponding map $X:(M^{2m},\varphi,g)\to(TM,g^\varphi)$ is a critical point for the energy functional E, only considering variations among maps defined by vector fields.

In the following theorem, we determine the first variation of the energy restricted to the space $\Im^1_0(M)$.

Theorem 4.3. *Let (M^{2m},φ,g) be a compact oriented para-Kähler-Norden manifold, (TM,g^φ) its tangent bundle equipped with the φ-Sasaki metric, $X\in\Im^1_0(M)$ be a vector field and $E:\Im^1_0(M)\to[0,+\infty)$ the energy functional restricted to the space of all vector fields. Then*

$$\frac{d}{dt}E(X_t)\big|_{t=0} = \int_M g\big(\bar{\Delta}X,\varphi V\big)v^g, \tag{4.12}$$

for any smooth 1*-parameter variation* $\Phi : M \times (-\epsilon, \epsilon) \to TM$ *of* X *through vector fields i.e.,* $\Phi(x,t) = X_t(x) \in TM$ *for any* $x \in M$ *and any* $|t| < \epsilon$, $(\epsilon > 0)$*, or equivalently* $X_t \in \Im^1_0(M)$ *for any* $|t| < \epsilon$*. Also,* $V \in \Im^1_0(M)$ *is the vector field on* M *given by*

$$V(x) = \lim_{t\to 0} \frac{1}{t}(X_t(x) - X(x)) = \frac{d}{dt}X_x(0), \quad x \in M,$$

where $X_x(t) = \Phi(x,t), (x,t) \in M \times (-\epsilon, \epsilon)$*.*

Proof. We consider the smooth 1-parameter variation $\Phi : M \times (-\epsilon, \epsilon) \to TM$ of X, i.e. $\Phi(x,t) = X_t(x) \in T_xM$ for any $(x,t) \in M \times (-\epsilon, \epsilon)$ and $\Phi(x,0) = X_0(x) = X(x)$. From (4.1), we have

$$E(X_t) = \int_M e(X_t)v^g.$$

Then, as well known the theory of harmonic maps [13]

$$\frac{d}{dt}E(X_t)\Big|_{t=0} = -\int_M g^{\varphi}(\mathcal{V}, \tau(X))v^g, \tag{4.13}$$

where $\mathcal{V}$ is the infinitesimal variation induced by Φ, i.e.

$$\mathcal{V}(x) = d_{(x,0)}\Phi(0, \frac{d}{dt})\big|_{t=0} = dX_x(\frac{d}{dt})\big|_{t=0} = \frac{d}{dt}X_t(x)\big|_{t=0} \in T_{X(x)}TM.$$

It is well known that

$$\mathcal{V} = {}^{V}V \circ X, \tag{4.14}$$

which was proven in [5, p.58]. Finally, by taking into account (4.9), (4.13) and (4.14) , we find

$$\frac{d}{dt}E(X_t)\Big|_{t=0} = -\int_M g^{\varphi}({}^{V}V, \tau(X))v^g = \int_M g\big(\bar{\Delta}X, \varphi V\big)v^g.$$

□

Remark 4.1. Theorem 4.3 is holds if (M^{2m}, φ, g) is a non-compact para-Kähler-Norden manifold. Indeed, if M is non-compact, we can take an open subset D in M whose closure is compact, and take an arbitrary V whose support is contained in D. Theorem 4.3 holds under the form:

$$\frac{d}{dt}E(X_t)\big|_{t=0} \;=\; \int_D g\big(\bar{\Delta}X, \varphi V\big)v^g.$$

Corollary 4.1. *Let* (M^{2m}, φ, g) *be a para-Kähler-Norden manifold,* (TM, g^{φ}) *its tangent bundle equipped with the* φ*-Sasaki metric and* $X \in \Im^1_0(M)$ *be a vector field. Then* X *is harmonic vector field if and only if* $\bar{\Delta}X = 0$*.*

From Theorem 4.2 and Corollary 4.1, we get the following corollary

Corollary 4.2. *Let* (M^{2m}, φ, g) *be a para-Kähler-Norden manifold,* (TM, g^{φ}) *its tangent bundle equipped with the* φ*-Sasaki metric and* $X \in \Im^1_0(M)$ *be a vector field. Then* X *is harmonic map if and only if* X *is harmonic vector field and* $Tr_gR(\varphi X, \nabla_* X)* = 0$*.*

Note that if X is parallel, by virtue of Corollary 4.2, X is a harmonic map (resp. harmonic vector field). Conversely we have the following theorem.

Theorem 4.4. *Let (M^{2m}, φ, g) be a compact oriented para-Kähler-Norden manifold, (TM, g^{φ}) its tangent bundle equipped with the φ-Sasaki metric and $X \in \Im^1_0(M)$ be a vector field. Then the following conditions are equivalent*
(i) X is harmonic map on M.
(ii) X is harmonic vector field on M.
(iii) X is parallel.

Proof. 1) If X is harmonic map on M, from Corollary 4.2, we deduce that X is harmonic vector field, i.e. $(i) \Rightarrow (ii)$.
2) We assume that the vector field X is a harmonic vector field on M, from Corollary 4.1, we deduce that $\bar{\Delta} X = 0$. By (4.6), we get $|\nabla X|^2 = \frac{1}{2}\Delta|X|^2$, then

$$\int_M |\nabla X|^2 v^g = \frac{1}{2}\int_M \Delta|X|^2 v^g.$$

Applying the divergence Theorem, we get

$$\int_M \Delta|X|^2 v^g = 0,$$

hence

$$\int_M |\nabla X|^2 v^g = 0.$$

Since $|\nabla X|^2$ is a positive function, We conclude that $\nabla X = 0$, i.e. $(ii) \Rightarrow (iii)$.
3) We assume that the vector field X is a parallel, by virtue of Theorem 4.2, X is a harmonic map, i.e. $(iii) \Rightarrow (i)$. □

Theorem 4.5. *Let $(\mathbb{R}^{2m}, \varphi, <,>)$ be a para-Kähler-Norden real Euclidean space and $T\mathbb{R}^{2m}$ its tangent bundle equipped with the φ-Sasaki metric, (φ is the canonical para-complex structure on $\mathbb{R}^{2m}$ [2]), it is given by the matrix*

$$\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

If $X = (X^1, \cdots, X^{2m})$ is vector field on $\mathbb{R}^{2m}$. Then the following conditions are equivalent
(i) $X = (X^1, \cdots, X^{2m})$ is harmonic vector field on $\mathbb{R}^{2m}$.
(ii) $X = (X^1, \cdots, X^{2m})$ is harmonic map on $\mathbb{R}^{2m}$.
(iii) for all $k = \overline{1, 2m}$, X^k is a real harmonic function on $\mathbb{R}^{2m}$.

Proof. We have $Tr_g R(\varphi X, \nabla_* X)* = 0$, then from Corollary 4.2, we get $(i) \Leftrightarrow (ii)$. Let $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1,2m}}$ be a canonical frame on $\mathbb{R}^{2m}$. Using Theorem 4.2, we have

$$\begin{aligned} \bar{\Delta} X = 0 \quad &\Leftrightarrow \quad Tr_{<,>}\nabla^2 X = 0 \\ &\Leftrightarrow \quad \sum_{i=1}^{2m} \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^i}} X = 0 \\ &\Leftrightarrow \quad \sum_{i,k=1}^{2m} \frac{\partial^2 X^k}{\partial (x^i)^2} \frac{\partial}{\partial x^k} = 0 \\ &\Leftrightarrow \quad \sum_{i=1}^{2m} \frac{\partial^2 X^k}{\partial (x^i)^2} = 0, \quad \textit{for all}, \ k = \overline{1, 2m}. \end{aligned}$$

i.e. for all $k=\overline{1,2m}$, X^k is a real harmonic function, then $(ii)\Leftrightarrow(iii)$. □

Example 4.1. Let $\left(\mathbb{R}^2\times\mathbb{R}^*\times\mathbb{R}^*,g,\varphi\right)$ be a para-Kähler-Norden manifold such that

$$g=e^{2x}dx^2+e^{2y}dy^2+z^2dz^2+t^2dt^2,$$

and

$$\varphi=\begin{pmatrix}0 & e^{y-x} & 0 & 0\\ e^{x-y} & 0 & 0 & 0\\ 0 & 0 & 0 & \dfrac{z}{t}\\ 0 & 0 & \dfrac{t}{z} & 0\end{pmatrix}.$$

Relatively to the orthonormal frame

$$e_1=e^{-x}\partial_x,\quad e_2=e^{-y}\partial_y,\quad e_3=\frac{1}{z}\partial_z,\quad e_4=\frac{1}{t}\partial_t.$$

we have,

$$\varphi e_1=e_2\quad,\quad\varphi e_2=e_1\quad,\quad\varphi e_3=e_4\quad,\quad\varphi e_4=e_3$$

and,

$$\nabla_{e_i}e_j=0,\quad \textit{for all}\, i,j=1,4.$$

We consider the vector field $X=\alpha(x)e_1+\beta(z)e_3$, where α and β are smooth non-zero real functions depending on the variables x and z. Using direct calculations, we find

$$Tr_gR(\varphi X,\nabla_*X)*=0,$$

$$\bar{\Delta}e_1=\bar{\Delta}e_3=0. \tag{4.15}$$

Combining relations (4.7) and (4.15), we obtain

$$\bar{\Delta}X\quad=\quad -e^{-2x}(\alpha''-\alpha')e_1-\frac{1}{z^2}(\beta''-\frac{1}{z}\beta')e_3.$$

$i)$ From Corollary 4.1, we deduce that $X=\alpha(x)e_1+\beta(z)e_3$ is harmonic vector field if and only if $\bar{\Delta}X=0$ or equivalently,

$$\begin{cases}\alpha''-\alpha'=0\\ \beta''-\dfrac{1}{z}\beta'=0\end{cases} \tag{4.16}$$

The general solutions of the ODE System (4.16) are

$$\alpha(x)=ae^x+b,\ \beta(z)=cz^2+d,$$

where a,b,c and b are real constants such that $a\neq 0$ and $c\neq 0$.
Since $Tr_gR(\varphi X,\nabla_*X)*=0$, from Corollary 4.2, the vector fields $X=(ae^x+b)e_1+(cz^2+d)e_3$ are also harmonic maps.
On the other hand $\nabla_{e_1}X=ae^xe_1\neq 0$, then the vector fields $X=(ae^x+b)e_1$ are harmonic but non parallel.

Example 4.2. Let $\mathbb{R}^2$ be endowed with the structure anti-paraKähler (φ,g) in polar coordinate defined by

$$g=dr^2+r^2d\theta^2,$$

$$\varphi\partial_r=\sin 2\theta\partial_r+\frac{1}{r}\cos 2\theta\partial_\theta,\quad \varphi\partial_\theta=r\cos 2\theta\partial_r-\sin 2\theta\partial_\theta,$$

Relatively to the orthonormal frame

$$e_1 = \partial_r, \quad e_2 = \frac{1}{r}\partial_\theta.$$

we have,

$$\varphi e_1 = \sin 2\theta e_1 + \cos 2\theta e_2, \quad \varphi e_2 = \cos 2\theta e_1 - \sin 2\theta e_2,$$

and,

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_1 = \frac{1}{r} e_2, \quad \nabla_{e_2} e_2 = -\frac{1}{r} e_1.$$

(1) We consider the vector field $X = f(r)e_1$, where f is a smooth non-zero real function depending of the variable r. Using direct calculations, we find

$$Tr_g R(\varphi X, \nabla_* X)* = 0,$$

$$\bar{\Delta} e_1 = \frac{1}{r^2} e_1, \tag{4.17}$$

Combining relations (4.7) and (4.17), we obtain

$$\bar{\Delta} X = (-f'' - \frac{1}{r} f' + \frac{1}{r^2} f) e_1.$$

$i)$ From Corollary 4.1, we deduce that $X = f(x)e_1$ is harmonic vector field if and only if $\bar{\Delta} X = 0$ or equivalently, the function f satisfies the following homogeneous second order Euler's equation

$$-f'' - \frac{1}{r} f' + \frac{1}{r^2} f = 0. \tag{4.18}$$

The general solution of differential equation (4.18) is

$$f(r) = \frac{ar^2 + b}{r},$$

where a and b are non-zero real constants.
$ii)$ Since $Tr_g R(\varphi X, \nabla_* X)* = 0$, from Corollary 4.2, the vector fields $X = \dfrac{ar^2 + b}{r} e_1$ are also harmonic maps.

On the other hand $\nabla_{e_1} X = \dfrac{ar^2 - b}{r^2} e_1 \neq 0$, then the vector fields $X = (ax^2 + b)e_1$ are harmonic but non parallel.
(2) On the contrary, the vector fields $Y = \sin\theta e_1 + \cos\theta e_2$ are parallels, then are harmonics.

Proposition 4.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g^φ) its tangent bundle equipped with the φ-Sasaki metric and $X \in \Im^1_0(M)$ be a vector field. Then X is an isometric immersion if and only if X is parallel.*

Proof. Let $Y, Z \in \Im^1_0(M)$ be two vector fields. From Lemma 4.1 we have

$$\begin{aligned} g^\varphi(dX(Y), dX(Z)) &= g^\varphi({}^H Y + {}^V(\nabla_Y X), {}^H Z + {}^V(\nabla_Z X)) \\ &= g^\varphi({}^H Y, {}^H Z) + g^\varphi({}^V(\nabla_Y X), {}^V(\nabla_Z X)) \\ &= g(Y, Z) + g(\nabla_Y X, \varphi \nabla_Z X) \end{aligned}$$

Hence, X is an isometric immersion if and only if

$$g^\varphi(dX(Y), dX(Z)) = g(Y, Z),$$

for any Y and Z two vector fields on M.
Therefore, X is an isometric immersion if and only if $\nabla X = 0$. □

As a direct consequence of Proposition 4.1, we obtain the following corollaries.

Corollary 4.3. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and (TM, g^{φ}) its tangent bundle equipped with the φ-Sasaki metric. Then any isometric vector field on M is harmonic.*

Corollary 4.4. *Let (M^{2m}, φ, g) be a compact oriented para-Kähler-Norden manifold and (TM, g^{φ}) its tangent bundle equipped with the φ-Sasaki metric. Then any harmonic vector field on M is isometric.*

4.2. Harmonicity of vector fields along smooth maps.

Lemma 4.5. [26] *Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds and $Y \in \Im^1_0(N)$ be a vector field on N. Let σ be a smooth map defined by $\sigma := Y \circ \phi$. Then*

$$d\sigma(X) = {}^{H}(d\phi(X)) + {}^{V}(\nabla^{\phi}_{X}\sigma), \tag{4.19}$$

for any vector field $X \in \Im^1_0(M)$.

Proposition 4.2. *Let (M^m, g) be a Riemannian manifold, $\big(N^{2n}, h, \varphi\big)$ be a para-Kähler-Norden manifold and (TN, h^{φ}) its tangent bundle equipped with the φ-Sasaki metric. Let $\phi : M \to N$ be a smooth map and $Y \in \Im^1_0(N)$ be a vector field on N. Then the tension field of the map $\sigma := Y \circ \phi$ is given by*

$$\tau(\sigma) \;=\; {}^{H}\big(\tau(\phi) + Tr_g\, R^N(\varphi\sigma, \nabla^{\phi}_{*}\sigma)d\phi(*)\big) - {}^{V}\Delta^{\phi}\sigma, \tag{4.20}$$

where $\Delta^{\phi}\sigma := -Tr_g(\nabla^{\phi})^2\sigma = -Tr_g(\nabla^{\phi}_{*}\nabla^{\phi}_{*} - \nabla^{\phi}_{\nabla_{*}*})\sigma$ denotes the rough Laplacian of σ on the pull-back bundle $\phi^{-1}TN$.

Proof. Let $x \in M$, $v \in T_{\phi(x)}N$ and $\{e_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M at x and $\sigma(x) = (\phi(x), Y_{\phi(x)}), Y_{\phi(x)} = v \in T_{\phi(x)}N$. Using (4.19) we have

$$\begin{aligned}
\tau(\sigma)_x &= \sum_{i=1}^{m} \big(\nabla^{\sigma}_{e_i} d\sigma(e_i) - d\sigma(\nabla_{e_i}e_i)\big)_x \\
&= \sum_{i=1}^{m} \nabla^{TN}_{d\sigma(e_i)} d\sigma(e_i) - {}^{H}(d\phi(\nabla_{e_i}e_i)) - {}^{V}(\nabla^{\phi}_{\nabla_{e_i}e_i}\sigma))_{(\phi(x),v)} \\
&= \sum_{i=1}^{m} \big(\nabla^{TN}_{({}^{H}(d\phi(e_i)) + {}^{V}(\nabla^{\phi}_{e_i}\sigma))}({}^{H}(d\phi(e_i)) + {}^{V}(\nabla^{\phi}_{e_i}\sigma)) - {}^{H}(d\phi(\nabla_{e_i}e_i)) \\
&\quad - {}^{V}(\nabla^{\phi}_{\nabla_{e_i}e_i}\sigma)\big)_{(\phi(x),v)} \\
&= \sum_{i=1}^{m} \big(\nabla^{TN}_{{}^{H}(d\phi(e_i))}{}^{H}(d\phi(e_i)) + \nabla^{TN}_{{}^{H}(d\phi(e_i))}{}^{V}(\nabla^{\phi}_{e_i}\sigma) + \nabla^{TN}_{{}^{V}(\nabla^{\phi}_{e_i}\sigma)}{}^{H}(d\phi(e_i)) \\
&\quad + \nabla^{TN}_{{}^{V}(\nabla^{\phi}_{e_i}\sigma)}{}^{V}(\nabla^{\phi}_{e_i}\sigma) - {}^{H}(d\phi(\nabla_{e_i}e_i)) - {}^{V}(\nabla^{\phi}_{\nabla_{e_i}e_i}\sigma)\big)_{(\phi(x),v)}.
\end{aligned}$$

From Theorem 3.1, we obtain:

$$\begin{aligned}
\tau(\sigma) &= \sum_{i=1}^{m}\big({}^{H}(\nabla^{N}_{d\phi(e_i)}d\phi(e_i)) - \frac{1}{2}{}^{V}(R^{N}(d\phi(e_i),d\phi(e_i))\sigma) \\
&\quad +\frac{1}{2}{}^{H}(R^{N}(\varphi\sigma,\nabla^{\phi}_{e_i}\sigma)d\phi(e_i)) + {}^{V}(\nabla^{N}_{d\phi(e_i)}\nabla^{\phi}_{e_i}\sigma) \\
&\quad +\frac{1}{2}{}^{H}(R^{N}(\varphi\sigma,\nabla^{\phi}_{e_i}\sigma)d\phi(e_i)) - {}^{H}(d\phi(\nabla_{e_i}e_i)) - {}^{V}(\nabla^{\phi}_{\nabla_{e_i}e_i}\sigma)\big) \\
&= \sum_{i=1}^{m}\big({}^{H}(\nabla^{\phi}_{e_i}d\phi(e_i)) - {}^{H}(d\phi(\nabla_{e_i}e_i)) + {}^{H}(R^{N}(\varphi\sigma,\nabla^{\phi}_{e_i}\sigma)d\phi(e_i)) \\
&\quad +{}^{V}(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\sigma) - {}^{V}(\nabla^{\phi}_{\nabla_{e_i}e_i}\sigma)\big) \\
&= {}^{H}\big(\tau(\phi) + Tr_g\, R^{N}(\varphi\sigma,\nabla^{\phi}_{*}\sigma)d\phi(*)\big) + {}^{V}\big(Tr_g(\nabla^{\phi})^2\sigma\big). \\
&= {}^{H}\big(\tau(\phi) + Tr_g\, R^{N}(\varphi\sigma,\nabla^{\phi}_{*}\sigma)d\phi(*)\big) - {}^{V}\Delta^{\phi}\sigma.
\end{aligned}$$

□

From Proposition 4.2 we obtain

Theorem 4.6. *Let (M^m, g) be a Riemannian manifold, $\big(N^{2n}, h, \varphi\big)$ be a para-Kähler-Norden manifold and (TN, h^{φ}) its tangent bundle equipped with the φ-Sasaki metric. Let $\phi : M \to N$ be a smooth map and $Y \in \Im^1_0(N)$ be a vector field on N. Then the map $\sigma := Y \circ \phi$ is harmonic if and only if the following conditions are verified*

$$\tau(\phi) = -Tr_g\, R^{N}(\varphi\sigma,\nabla^{\phi}_{*}\sigma)d\phi(*)\big) \quad \textit{and} \quad \Delta^{\phi}\sigma = 0.$$

4.3. Harmonicity of a composition of the projection map of the tangent bundle with a smooth map.

Lemma 4.6. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and (TM, g^{φ}) its tangent bundle equipped with the φ-Sasaki metric. Then the canonical projection $\pi : (TM, g^{\varphi}) \to (M^{2m}, \varphi, g)$ is harmonic i.e. $\tau(\pi) = 0$.*

Proof. Let $(e_i)_{i=\overline{1,2m}}$ (resp.$(f_i)_{i=\overline{1,2m}}$) be local orthonormal frame on (M^{2m}, φ, g) (resp. (M^{2m}, φ, G)), where G is the twin Norden metric of g. Then $\{{}^{H}e_i, {}^{V}f_i\}_{i=\overline{1,2m}}$ is a local orthonormal frame on (TM, g^{φ}).

$$\begin{aligned}
\tau(\pi) &= Tr_{g^{\varphi}}\nabla d\pi \\
&= \sum_{i=1}^{2m}\big(\nabla^{\pi}_{{}^{H}e_i}d\pi({}^{H}e_i) - d\pi(\nabla^{TM}_{{}^{H}e_i}{}^{H}e_i) + \nabla^{\pi}_{{}^{V}f_i}d\pi({}^{V}f_i) - d\pi(\nabla^{TM}_{{}^{V}f_i}{}^{V}f_i)\big) \\
&= \sum_{i=1}^{2m}\big(\nabla^{M}_{d\pi({}^{H}e_i)}d\pi({}^{H}e_i) - d\pi(\nabla^{TM}_{{}^{H}e_i}{}^{H}e_i) + \nabla^{M}_{d\pi({}^{V}f_i)}d\pi({}^{V}f_i) - d\pi(\nabla^{TM}_{{}^{V}f_i}{}^{V}f_i)\big)
\end{aligned}$$

Since $d\pi({}^{V}X) = 0$ and $d\pi({}^{H}X) = X \circ \pi$, for any vector field X on M, then we find

$$\begin{aligned}
\tau(\pi) &= \sum_{i=1}^{2m}\big((\nabla^{M}_{e_i}e_i)\circ\pi - d\pi(\nabla^{M}_{e_i}e_i)^{H}\big) \\
&= 0.
\end{aligned}$$

□

Theorem 4.7. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (N^n, h) be a Riemannian manifold, (TM, g^{φ}) its tangent bundle equipped with the φ-Sasaki metric and $\phi : \big(M^{2m}, g, \varphi\big) \to (N^n, h)$ a smooth map. The tension field of the map $\phi \circ \pi$ is given by:*

$$\tau(\phi \circ \pi) \;=\; \tau(\phi) \circ \pi. \tag{4.21}$$

Proof. Let $\{{}^He_i, {}^Vf_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on (TM, g^{φ}) as above. Then the tension field of the composition $\phi \circ \pi$ is given by [7, 9]

$$\tau(\phi \circ \pi) \;=\; d\phi(\tau(\pi)) + Tr_{g^{\varphi}} \nabla d\phi(d\pi, d\pi).$$

$$\begin{aligned}
Tr_{g^{\varphi}} \nabla d\phi(d\pi, d\pi) &= \sum_{i=1}^{2m} \big(\nabla^{\phi}_{d\pi({}^He_i)} d\phi(d\pi({}^He_i)) - d\phi(\nabla^M_{d\pi({}^He_i)} d\pi({}^He_i))\big) \\
&\quad + \sum_{i=1}^{2m} \big(\nabla^{\phi}_{d\pi({}^Vf_i)} d\phi(d\pi({}^Vf_i)) - d\phi(\nabla^M_{d\pi({}^Vf_i)} d\pi({}^Vf_i))\big) \\
&= \sum_{i=1}^{2m} \big(\nabla^{\phi}_{(e_i \circ \pi)} d\phi(e_i \circ \pi) - d\phi(\nabla^M_{e_i \circ \pi}(e_i \circ \pi))\big) \\
&= \sum_{i=1}^{2m} \big(\nabla^{\phi}_{e_i} d\phi(e_i) - d\phi(\nabla^M_{e_i} e_i)\big) \circ \pi \\
&= \tau(\phi) \circ \pi,
\end{aligned}$$

Using Lemma 4.6, we obtain:

$$\tau(\phi \circ \pi) \;=\; \tau(\phi) \circ \pi.$$

□

Theorem 4.8. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g^{φ}) be its tangent bundle equipped with the φ-Sasaki metric, (N^n, h) be a Riemannian manifold and $\phi : (M^{2m}, \varphi, g) \to (N^n, h)$ be a smooth map. Then the map $\phi \circ \pi$ is harmonic if and only if ϕ is harmonic.*

4.4. Harmonicity of the tangent map.

Lemma 4.7. [6] *Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. The map ϕ induces the tangent map*

$$\begin{aligned}
d\phi : TM &\longrightarrow TN \\
(x, u) &\longmapsto (\phi(x), d\phi(u))
\end{aligned}$$

and we have

$$\begin{aligned}
d(d\phi)({}^VX) &= {}^V(d\phi(X)), \\
d(d\phi)({}^HX) &= {}^H(d\phi(X)) + {}^V(\nabla d\phi(u, X)).
\end{aligned}$$

for any vector field $X \in \Im^1_0(M)$.

Theorem 4.9. *Let $\phi : (M^{2m}, \varphi, g) \to (N^{2n}, \psi, h)$ be a smooth map between para-Kähler-Norden manifolds, then the tension field associated to the tangent map $d\phi : (TM, g^{\varphi}) \longrightarrow (TN, h^{\psi})$ is given by:*

$$\begin{aligned}\tau(d\phi) &= {}^{H}\big(\tau(\phi) + Tr_g R^N(\psi d\phi(u), \nabla d\phi(u,*))d\phi(*)\big) \\ &\quad + {}^{V}\big(Tr_g \nabla^{\phi}(\nabla d\phi(u,*))\big). \end{aligned} \tag{4.22}$$

Proof. Let $\{{}^{H}e_i, {}^{V}f_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on (TM, g^{φ}) as above, then

$$\begin{aligned}\tau(d\phi)_{(x,u)} &= Tr_{g^{\varphi}}(\nabla d(d\phi))_{(x,u)} \\ &= \sum_{i=1}^{2m}\big(\nabla^{TN}_{d(d\phi)({}^{H}e_i)} d(d\phi)({}^{H}e_i) - d(d\phi)(\nabla^{TM}_{{}^{H}e_i}{}^{H}e_i) + \nabla^{TN}_{d(d\phi)({}^{V}f_i)} d(d\phi)({}^{V}f_i) \\ &\quad - d(d\phi)(\nabla^{TM}_{{}^{V}f_i}{}^{V}f_i)\big)_{(\phi(x), d\phi(u))}.\end{aligned}$$

Using Theorem 3.1 and Lemma 4.7, we obtain:

$$\begin{aligned}\tau(d\phi) &= \sum_{i=1}^{m}\big(\nabla^{TN}_{{}^{H}(d\phi(e_i))}{}^{H}(d\phi(e_i)) + \nabla^{TN}_{{}^{H}(d\phi(e_i))}{}^{V}(\nabla d\phi(u,e_i)) \\ &\quad + \nabla^{TN}_{{}^{V}(\nabla d\phi(u,e_i))}{}^{H}(d\phi(e_i))\big) \\ &= \sum_{i=1}^{m}\big({}^{H}(\nabla^{N}_{d\phi(e_i)} d\phi(e_i)) + {}^{V}(\nabla^{N}_{d\phi(e_i)}\nabla d\phi(u,e_i)) \\ &\quad + \frac{1}{2}{}^{H}(R^N(\psi d\phi(u), \nabla d\phi(u,e_i))d\phi(e_i)) \\ &\quad + \frac{1}{2}{}^{H}(R^N(\psi d\phi(u), \nabla d\phi(u,e_i))d\phi(e_i))\big) \\ &= \sum_{i=1}^{m}\big({}^{H}\big(\nabla^{\phi}_{e_i} d\phi(e_i) + R^N(\psi d\phi(u), \nabla d\phi(u,e_i))d\phi(e_i)\big) + {}^{V}(\nabla^{\phi}_{e_i}\nabla d\phi(u,e_i))\big) \\ &= {}^{H}\big(\tau(\phi) + Tr_g R^N(\psi d\phi(u), \nabla d\phi(u,*))d\phi(*)\big) + {}^{V}\big(Tr_g \nabla^{\phi}_{*}(\nabla d\phi(u,*))\big).\end{aligned}$$

□

From Theorem 4.9, we obtain the following theorem and corollary

Theorem 4.10. *Let $\phi : (M^{2m}, \varphi, g) \to (N^{2n}, \psi, h)$ be a smooth map between para-Kähler-Norden manifolds, then the tangent map $d\phi : (TM, g^{\varphi}) \longrightarrow (TN, h^{\psi})$ is harmonic if and only if*

$$\tau(\phi) = -Tr_g R^N(\psi d\phi(u), \nabla d\phi(u,*))d\phi(*)\big) \ \text{ and } \ Tr_g \nabla^{\phi}_{*}(\nabla d\phi(u,*)) = 0.$$

Corollary 4.5. *Let $\phi : (M^{2m}, \varphi, g) \to (N^{2n}, \psi, h)$ be a smooth map between para-Kähler-Norden manifolds. If ϕ is totally geodesic, then the tangent map $d\phi : (TM, g^{\varphi}) \longrightarrow (TN, h^{\psi})$ of ϕ is harmonic.*

4.5. Harmonicity of the identity map $I : (TM, g^{\varphi}) \longrightarrow (TN, h^{\psi})$.

Let $(M^{2m}, \varphi, \psi, g)$ be a bi-para-Kähler-Norden manifold and (TM, g^{φ}) (resp. (TM, g^{ψ})) its tangent bundle equipped with the φ-Sasaki metric g^{φ} (resp. ψ-Sasaki metric g^{ψ}).

Theorem 4.11. *The identity map* $I : (TM, g^{\varphi}) \longrightarrow (TN, h^{\psi})$ *is harmonic*

Proof. Let $\{{}^He_i, {}^Vf_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on (TM, g^{φ}) as above. If $\widetilde{\nabla}$ (resp. $\overline{\nabla}$) denote the Levi-Civita connection of (TM, g^{φ}) (resp. (TM, g^{ψ})), then, we have

$$\begin{aligned}
\tau(I) &= trace_{g^{\varphi}}(\nabla dI) \\
&= \sum_{i=1}^{m} \big(\nabla^I_{{}^He_i} dI({}^He_i) - dI(\widetilde{\nabla}_{{}^He_i}{}^He_i) + \nabla^I_{{}^Vf_i} dI({}^Vf_i) - dI(\widetilde{\nabla}_{{}^Vf_i}{}^Vf_i)\big) \\
&= \sum_{i=1}^{m} \big(\overline{\nabla}_{dI({}^He_i)} dI({}^He_i) - dI(\widetilde{\nabla}_{{}^He_i}{}^He_i) + \overline{\nabla}_{dI({}^Vf_i)} dI({}^Vf_i) - dI(\widetilde{\nabla}_{{}^Vf_i}{}^Vf_i)\big) \\
&= \sum_{i=1}^{m} \big(\overline{\nabla}_{{}^He_i}{}^He_i - \widetilde{\nabla}_{{}^He_i}{}^He_i + \overline{\nabla}_{{}^Vf_i}{}^Vf_i - \widetilde{\nabla}_{{}^Vf_i}{}^Vf_i\big) \\
&= 0.
\end{aligned}$$

□

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