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RESTORATION OF POLYNOMIAL COEFFICIENT IN THE DIFFERENTIAL EQUATION OF THE THIRD ORDER

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Abstract. The problem of identifying a polynomial coefficient in an ordinary differential equation of the third order by eigenvalues is considered. It is shown that the case of a differential equation of odd order differs from the case of a differential equation of even order. It is shown below that, in contrast to even-order differential equations, for the uniqueness of the restoration of the polynomial potential in the differential equation of the third order there is no need to use the eigenvalues of the two spectral problems. It suffices to use the eigenvalues of one boundary-value problem. A method for solving the problem based on finding unknown polynomial coefficients is presented. A method is also developed that allows one to prove the uniqueness of the restored polynomial coefficient by a finite number of eigenvalues. The latter method is based on the method of variation of an arbitrary constant. The uniqueness theorems for the solution of the inverse problem and examples of its solution are given.

1. Introduction

The paper considers the inverse problem for equation

$$ly = -y''' + q(x)\,y = \lambda\,y = s^3\,y. \tag{1.1}$$

Direct problems for even order differential operator have been fairly well studied (see, for example, [14], [28], [29]. The case odd order differential operator has been studied little. We note that differential equations of odd order with unbounded operator coefficients acting in the abstract Hilbert space H are considered in [4], [5] [6], [7], [8] on the half-axis $[0, +\infty)$. In these papers, for such equations in a broad aspects, questions of the well-posed and unique solvability of H-valued functions in Sobolev spaces under different boundary conditions at zero are studied. It should be pointed out that the differential equations considered in this paper are covered by the abstract equations of [4], [5] [6], [7], [8].

The study of inverse eigenvalue problems began with the work [9]. In this work, it was shown that if $\lambda_n = n^2$, $(n = 0, 1, \dots)$, then $q(x) \equiv 0$ for a boundary

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value problem with a differential equation

$$ly = -y'' + q(x)\, y = \lambda\, y = s^2\, y,$$

and boundary conditions

$$y'(0) = y'(\pi) = 0,$$

where $q(x)$ is a real continuous function. In other words, it was shown that if the spectrum of the equation

$$y'' + \lambda\, y = 0$$

for those but the boundary conditions were preserved, then there could not be any disturbance. In 1946, [10] G. Borg showed (see also [16]) that one spectrum, generally speaking, does not determine the equation, the Ambartsumian case is an exception. G. Borg considered methods for constructing an equation from two spectra. These results are conditional in nature, since the existence of a differential equation for which these two sequences are spectra is assumed. Later in the 50's and 60's of the 20 century in the works of V.A. Marchenko, B.M. Levitan, N. Levinson and M.G. Gasymov it was shown that the Sturm-Liouville operator with a discrete spectrum is uniquely determined by two spectra of boundary value problems with different boundary conditions at the zero point and one and the same condition at the other end [16], [17], [19], [23].

After the publication of classical monographs V.A. Marchenko [22] and B.M. Levitan [18] where the potential $q(x)$ was either a continuous or summable function, the main efforts of scientists are aimed at generalizing the results obtained both in the direction of restoring more general potentials and differential equations [13], [30], [32], and in the direction of using more general boundary conditions [12], [15], [21], [24], [25], [26], [27]. In all these works, at least two infinite sets of eigenvalues are required to restore a continuous function or a more general function $q(x)$. However, for applied problems such an approach is not very effective, since in reality with the help of frequency meters it is possible to determine only finite sets of natural frequencies. In addition, as a rule, there is some additional information about the identified object, which allows us to specify the class of required functions. Therefore, the problem arises of identifying a potential of a special type by a finite number of eigenfrequencies. Nevertheless, no effective methods of solving this problem have been proposed. Previously, problems were solved for identifying species and parameters of boundary conditions with respect to a finite number of eigenfrequencies [1], [2].

In [31], the uniqueness of the reconstruction of the two-term equations $y^{(2n)} + q(x)\, y = \lambda\, y$ by the spectrum of the problem

$$y^{(2n)} + q(x)\, y = \lambda\, y, \qquad y(0) = y'(0) = \cdots = y^{(2n-2)}(0) = y(1) = 0 \tag{1.2}$$

and the spectrum of the problem

$$y^{(2n)} + q(x)\, y = \lambda\, y, \qquad y(0) = y'(0) = \cdots = y^{(2n-2)}(0) = y'(1) = 0 \tag{1.3}$$

is showed.

In [3] it is shown that it is not enough eigenvalues for one of the problems (1.2) or (1.3) to prove the uniqueness of the restoration of the linear potential in the Sturm-Liouville problem. The corresponding counterexamples are given in the paper. It is also proved that the use of two eigenvalues, one of which is the eigenvalue of problem (1.2), and the other is the eigenvalue of problem (1.3), allows uniquely recover the linear potential $q(x)$.

In this paper it is shown that the case of a differential equation of odd order differs radically from the case of a differential equation of even order. It is shown below that for the uniqueness of the restoration of the polynomial potential in the differential equation of the third order there is no need to use the eigenvalues of the two problems. It suffices to use the eigenvalues of one boundary-value problem.

2. Polynomial Restoration. Examples

We denote by L_1, L_2, L_1^0 and L_2^0, respectively, the following boundary-value problems for the third-order differential equation:

Problem L_1:

$$ly = -y''' + q(x)\,y = \lambda\,y = s^3\,y, \quad y(0) = y'(0) = y(1) = 0;$$

Problem L_2:

$$ly = -y''' + q(x)\,y = \lambda\,y = s^3\,y, \quad y(0) = y'(0) = y'(1) = 0;$$

Problem L_1^0:

$$lz = z''' + s^3\,z = 0, \quad z(0) = z'(0) = z(1) = 0;$$

Problem L_2^0:

$$ly = z''' + s^3\,z = 0, \quad z(0) = z'(0) = z'(1) = 0.$$

Example 2.1. Let the numbers

$$s_1^3 = 4.2852^3, \quad s_2^3 = 7.8756^3, \quad s_3^3 = 11.495^3$$

be the eigenvalues of problem L_1 and $q(x) \equiv q_0 + q_1\,x + q_2\,x^2$. It is required to reconstruct the quadratic function $q(x) \equiv q_0 + q_1\,x + q_2\,x^2$ by these three eigenvalues λ_i, $i = 1, 2, 3$ of problem L_1.

The solutions $y_1(x,\lambda)$, $y_2(x,\lambda)$ and $y_3(x,\lambda)$ can be found as a Taylor series with the help of a package of analytical computations (for analytical computations we used Maple). These solutions will contain unknown coefficients q_0, q_1 and q_2. Substituting the numbers

$$s_1^3 = 4.2852^3, \quad s_2^3 = 7.8756^3, \quad s_3^3 = 11.495^3$$

and the main part of the Taylor series (the first 200 terms of the series) in $\Delta_1(\lambda) = y_3(x,\lambda) = 0$ we obtain a system of equations with respect to q_0, q_1, and q_2. Solving this system of equations q_0, q_1, and q_2 in the package of analytic computations, we find:

$$q_0 = 1.0000, \quad q_1 = 2.0000, \quad q_2 = 3.0000.$$

Whence

$$q(x) = 1 + 2x + 3\,x^2.$$

Example 2.2. Let $q(x)$ be the function $q(x) \equiv q_0 + q_1\,x + q_2\,x^2$ and

$$s_1^3 = 3.1379^3, \quad s_2^3 = 6.6739^3, \quad s_3^3 = 10.288^3$$

be eigenvalues of problem L_2. It is required to reconstruct the quadratic function

$$q(x) \equiv q_0 + q_1\,x + q_2\,x^2$$

by these three eigenvalues λ_i, $i = 1, 2, 3$ of problem L_2.

The solutions

$$y_1(x,\lambda), \quad y_2(x,\lambda), \quad y_3(x,\lambda)$$

can be found as a Taylor series with the help of a package of analytical computations. These solutions will contain unknown coefficients q_0, q_1 and q_2. Substituting the numbers

$$s_1^3 = 3.1379^3, \quad s_2^3 = 6.6739^3, \quad s_3^3 = 10.288^3$$

and the main part of the Taylor series (the first 200 terms of the series) in

$$\Delta_2(\lambda) = y_3'(x,\lambda) = 0$$

we obtain a system of equations with respect to q_0, q_1 and q_2. Solving this system of equations q_0, q_1, q_2 in the package of analytic computations, we find:

$$q_0 = 1.0000, \quad q_1 = 2.0000, \quad q_2 = 3.0000.$$

Whence

$$q(x) = 1 + 2x + 3\,x^2.$$

Example 2.3. Let the numbers

$$s_1^3 = 4.2690^3, \quad s_2^3 = 7.8708^3, \quad s_3^3 = 11.493^3, \quad s_4^3 = 15.118^3$$

be eigenvalues of problem L_1 and $q(x) \equiv q_0 + q_1\,x + q_2\,x^2 + q_3\,x^3$. For these four eigenvalues λ_i, $i = 1, 2, 3, 4$ of problem L_1, the function $q(x) \equiv q_0 + q_1\,x + q_2\,x^2 + q_3\,x^3$.

The solutions $y_1(x,\lambda)$, $y_2(x,\lambda)$ and $y_3(x,\lambda)$ can be found as a Taylor series with the help of a package of analytical computations. These solutions will contain unknown coefficients q_0, q_1, q_2 and q_3. Substituting the numbers

$$s_1^3 = 4.2690^3, \quad s_2^3 = 7.8708^3, \quad s_3^3 = 11.493^3, \quad s_4^3 = 15.118^3$$

and the main part of the Taylor series (the first 200 terms series) in $\Delta_2(\lambda) = y_3(x,\lambda) = 0$ we obtain a system of equations with respect to q_0, q_1, q_2 and q_3. Solving this system of equations q_0, q_1, q_2 in the package of analytic computations, we find:

$$q_0 = 1.0000, \quad q_1 = 1.0000, \quad q_2 = 1.0000, \quad q_3 = 1.0000.$$

Hence

$$q(x) = 1 + x + x^2 + x^3.$$

Are the solutions are unique? Yes, they are. This can be proved by the lemma, which the following section is posed.

3. The Main Lemma

We denote by $y_j(x,\lambda)$ and $z_j(x,\lambda)$ $(j=1,2,3)$ the solutions of equations (1.1) and

$$z''' + \lambda\, z = z''' + s^3\, z = 0 \tag{3.1}$$

respectively, and satisfying the following conditions

$$y_j^{(i-1)}(x,\lambda) = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i\neq j \end{cases}, \quad i,j=1,2,3, \quad z_j^{(i-1)}(x,\lambda) = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i\neq j \end{cases}. \tag{3.2}$$

Lemma 3.1. *An arbitrary solution $y(x)=y(x,\lambda)$ of equation (1.1) can be written in the form*

$$\begin{gathered} y(x) = y(0)\, z_1(x,\lambda) + y'(0)\, z_2(x,\lambda) + y''(0)\, z_3(x,\lambda) + \\ + q_0\, Z_0 + q_1\, Z_1 + \cdots + q_{n-1}\, Z_{n-1}, \end{gathered} \tag{3.3}$$

where

$$\begin{gathered} Z_m = z_1(x,\lambda) \int_0^x \xi^m\, y(\xi) \left(z_2(\xi,\lambda)\, z_3'(\xi,\lambda) - z_2'(\xi,\lambda)\, z_3(\xi,\lambda) \right) d\xi - \\ - z_2(x,\lambda) \int_0^x \xi^m\, y(\xi) \left(z_1(\xi,\lambda)\, z_3'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_3(\xi,\lambda) \right) d\xi + \\ + z_3(x,\lambda) \int_0^x \xi^m\, y(\xi) \left(z_1(\xi,\lambda)\, z_2'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_2(\xi,\lambda) \right) d\xi, \\ m = 0,1,\ldots,n-1. \end{gathered}$$

Proof. Equation (1.1) can be rewritten as

$$y''' + s^3\, y = q(x)\, y. \tag{3.4}$$

Equation (3.1) has a fundamental system of solutions

$$z_1(x,\lambda), \quad z_2(x,\lambda), \quad z_3(x,\lambda).$$

Therefore, consider (3.4) as a nonhomogeneous equation with the right-hand side $q(x)\, y$ and apply the method of variation of arbitrary constants. We have the system of equations

$$\begin{cases} C_1'\, z_1 + C_2'\, z_2 + C_3'\, z_3 = 0, \\ C_1'\, z_1' + C_2'\, z_2' + C_3'\, z_3' = 0, \\ C_1'\, z_1'' + C_2'\, z_2'' + C_3'\, z_3'' = q(x)\, y. \end{cases}$$

From here

$$C_1' = \frac{D_1}{D}, \quad C_2' = \frac{D_2}{D}, \quad C_3' = \frac{D_3}{D},$$

$$D = \begin{vmatrix} z_1 & z_2 & z_3 \\ z_1' & z_2' & z_3' \\ z_1'' & z_2'' & z_3'' \end{vmatrix}, \quad D_1 = \begin{vmatrix} 0 & z_2 & z_3 \\ 0 & z_2' & z_3' \\ q(x)\, y & z_2'' & z_3'' \end{vmatrix},$$

$$D_2 = \begin{vmatrix} z_1 & 0 & z_3 \\ z_1' & 0 & z_3' \\ z_1'' & q(x)\, y & z_3'' \end{vmatrix}, \quad D_3 = \begin{vmatrix} z_1 & z_2 & 0 \\ z_1' & z_2' & 0 \\ z_1'' & z_2'' & q(x)\, y \end{vmatrix}.$$

From the Liouville formula [20, p. 349] and (3.2) it follows that determinant D (Wronskian) is equal to one. Hence we obtain

$$\begin{aligned}
&y(x) = C_1\, z_1(x,\lambda) + C_2\, z_2(x,\lambda) + C_3\, z_3(x,\lambda) + \\
&+ z_1(x,\lambda) \int_0^x q(\xi)\, y(\xi) \big(z_2(\xi,\lambda)\, z_3'(\xi,\lambda) - z_2'(\xi,\lambda)\, z_3(\xi,\lambda)\big)\, d\xi - \\
&- z_2(x,\lambda) \int_0^x q(\xi)\, y(\xi) \big(z_1(\xi,\lambda)\, z_3'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_3(\xi,\lambda)\big)\, d\xi + \\
&+ z_3(x,\lambda) \int_0^x q(\xi)\, y(\xi) \big(z_1(\xi,\lambda)\, z_2'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_2(\xi,\lambda)\big)\, d\xi.
\end{aligned}$$

(Here C_1, C_2, C_3 are already other constants, they are denoted by the previous symbols to avoid an abundance of notation.)

If $x = 0$, then from the last equation we obtain $y(0) = C_1 \cdot 1 + 0$. Whence we have

$$\begin{aligned}
&y(x) = y(0)\, z_1(x,\lambda) + C_2\, z_2(x,\lambda) + C_3\, z_3(x,\lambda) + \\
&+ z_1(x,\lambda) \int_0^x q(\xi)\, y(\xi) \big(z_2(\xi,\lambda)\, z_3'(\xi,\lambda) - z_2'(\xi,\lambda)\, z_3(\xi,\lambda)\big)\, d\xi - \\
&- z_2(x,\lambda) \int_0^x q(\xi)\, y(\xi) \big(z_1(\xi,\lambda)\, z_3'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_3(\xi,\lambda)\big)\, d\xi + \\
&+ z_3(x,\lambda) \int_0^x q(\xi)\, y(\xi) \big(z_1(\xi,\lambda)\, z_2'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_2(\xi,\lambda)\big)\, d\xi.
\end{aligned} \tag{3.5}$$

Similarly we obtain

$$\begin{aligned}
&y'(0) = y(0)\, z_1'(0,\lambda) + C_2\, z_2'(0,\lambda) + C_3\, z_3'(0,\lambda) + \\
&+ z_1'(0,\lambda) \int_0^0 q(\xi)\, y(\xi) \big(z_2(\xi,\lambda)\, z_3'(\xi,\lambda) - z_2'(\xi,\lambda)\, z_3(\xi,\lambda)\big)\, d\xi + \\
&+ z_1(0,\lambda)\, q(0)\, y(0) \big(z_2(0,\lambda)\, z_3'(0,\lambda) - z_2'(0,\lambda)\, z_3(0,\lambda)\big) - \\
&- z_2'(0,\lambda) \int_0^0 q(\xi)\, y(\xi) \big(z_1(\xi,\lambda)\, z_3'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_3(\xi,\lambda)\big)\, d\xi - \\
&- z_2(0,\lambda)\, q(0)\, y(0) \big(z_1(0,\lambda)\, z_3'(0,\lambda) - z_1'(0,\lambda)\, z_3(0,\lambda)\big)\, d\xi + \\
&+ z_3'(0,\lambda) \int_0^0 q(\xi)\, y(\xi) \big(z_1(\xi,\lambda)\, z_2'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_2(\xi,\lambda)\big)\, d\xi + \\
&+ z_3(0,\lambda)\, q(0)\, y(0) \big(z_1(0,\lambda)\, z_2'(0,\lambda) - z_1'(0,\lambda)\, z_2(0,\lambda)\big)\, d\xi.
\end{aligned}$$

From this and (3.1) we have $y'(0) = C_2$.

By analogy, we also obtain $y''(0) = C_3$.

Using (3.5) we obtain equation (3.3). □

4. Uniqueness Theorems

Characteristic determinants $\Delta_1(\lambda)$, $\Delta_2(\lambda)$ of problems L_1 and L_2 respectively have the form

$$\Delta_1(\lambda) = \begin{vmatrix} y_1(0) & y_2(0) & y_3(0) \\ y_1'(0) & y_2'(0) & y_3'(0) \\ y_1(1) & y_2(1) & y_3(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y_1(1) & y_2(1) & y_3(1) \end{vmatrix} = y_3(1),$$

$$\Delta_2(\lambda) = \begin{vmatrix} y_1(0) & y_2(0) & y_3(0) \\ y_1'(0) & y_2'(0) & y_3'(0) \\ y_1'(1) & y_2'(1) & y_3'(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y_1'(1) & y_2'(1) & y_3'(1) \end{vmatrix} = y_3'(1).$$

Taking into account the lemma, the characteristic determinants $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ of problems L_1 and L_2 respectively, can be written in the following form

$$\Delta_1(\lambda) = y_3(1,\lambda) = \\ = \Big[z_3(x,\lambda) + q_0\, Z_0(1,\lambda) + q_1\, Z_1(1,\lambda) + \cdots + q_{n-1}\, Z_{n-1}(1,\lambda) \Big]_{x=1},$$

$$\Delta_2(\lambda) = y_3'(1,\lambda) = \\ = \Big[z_3'(x,\lambda) + q_0\, Z_0'(1,\lambda) + q_1\, Z_1'(1,\lambda) + \cdots + q_{n-1}\, Z_{n-1}'(1,\lambda) \Big]_{x=1} = \\ = z_3'(1,\lambda) + q_0\, Q_0(1,\lambda) + q_1\, Q_1(1,\lambda) + \cdots + q_{n-1}\, Q_{n-1}(1,\lambda),$$

where

$$\begin{aligned} Q_m(1,\lambda) = {} & z_1'(1,\lambda) \int_0^1 \xi^m\, y_3(\xi) \big(z_2(\xi,\lambda)\, z_3'(\xi,\lambda) - z_2'(\xi,\lambda)\, z_3(\xi,\lambda) \big)\, d\xi - \\ & - z_2'(1,\lambda) \int_0^1 \xi^m\, y_3(\xi) \big(z_1(\xi,\lambda)\, z_3'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_3(\xi,\lambda) \big)\, d\xi + \\ & + z_3'(1,\lambda) \int_0^1 \xi^m\, y_3(\xi) \big(z_1(\xi,\lambda)\, z_2'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_2(\xi,\lambda) \big)\, d\xi + \\ & + z_1(1,\lambda)\, \xi^m\, y_3(\xi) \big(z_2(\xi,\lambda)\, z_3'(\xi,\lambda) - z_2'(\xi,\lambda)\, z_3(\xi,\lambda) \big) - \\ & - z_2(1,\lambda)\, \xi^m\, y_3(\xi) \big(z_1(\xi,\lambda)\, z_3'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_3(\xi,\lambda) \big) + \\ & + z_3(1,\lambda)\, \xi^m\, y(\xi) \big(z_1(\xi,\lambda)\, z_2'(\xi,\lambda) - z_1'(\xi,\lambda)\, z_2(\xi,\lambda) \big), \\ & m = 0, 1, \ldots, n-1. \end{aligned}$$

Let λ_i $(i = 1, 2, \ldots, n)$ be eigenvalues of problem L_1. If the system of equations

$$z_3(1,\lambda_i) + q_0\, Z_0(1,\lambda_i) + q_1\, Z_1(1,\lambda_i) + \cdots + q_{n-1}\, Z_{n-1}(1,\lambda_i) = 0, \quad i = \overline{1,n}, \tag{4.1}$$

has a unique solution, this means that the polynomial $q(x)$ is uniquely determined. Hence we have proved

Theorem 4.1. *If eigenvalues λ_i of problem L_1 are such that the system of equations (4.1) has a unique solution, then the problem of finding the polynomial $q(x)$ has a unique solution.*

Similarly, if λ_i $(i = 1, 2, \ldots, n)$ are eigenvalues of problem L_2 and the system of equations

$$z_3'(1,\lambda_i) + q_0\, Q_0(1,\lambda_i) + q_1\, Q_1(1,\lambda_i) + \cdots + q_{n-1}\, Q_{n-1}(1,\lambda_i) = 0, \quad i = \overline{1,n}, \tag{4.2}$$

has a unique solution, then this means that the polynomial $q(x)$ is uniquely determined. Hence we have proved

Theorem 4.2. *If the eigenvalues λ_i of problem L_2 are such that the system of equations (4.2) has a unique solution, then the problem of finding the polynomial $q(x)$ has a unique solution.*

5. Proof of the Uniqueness of Finding Polynomials. Examples

Are the polynomials obtained in Examples 2.1 and 2.2 the unique solutions? Below it is shown that this is so.

Example 5.1. We substitute the found polynomial $q(x) = 1 + 2x + 3\,x^2$ from Example 2.1 and the eigenvalues

$$s_1^3 = 4.2852^3, \quad s_2^3 = 7.8756^3, \quad s_3^3 = 11.495^3$$

of problem L_1 in (4.1). As a result, we obtain the system of equations

$$0.0049136\,q_0 + 0.0024643\,q_1 + 0.0013660\,q_2 - 0.013940 = 0,$$
$$-0.002574\,q_0 - 0.0012864\,q_1 - 0.00080851\,q_2 + 0.0075717 = 0,$$
$$0.0034587\,q_0 + 0.0017286\,q_1 + 0.0011199\,q_2 - 0.010276 = 0.$$

This system has the unique solution

$$q_0 = 1.0000, \quad q_1 = 2.0000, \quad q_2 = 3.0000.$$

Where does it follow that

$$q(x) = 1 + 2x + 3\,x^2$$

is the only polynomial that can be recovered by eigenvalues

$$s_1^3 = 4.2852^3, \quad s_2^3 = 7.8756^3, \quad s_3^3 = 11.495^3$$

of problem L_1.

Example 5.2. We substitute the found polynomial $q(x) = 1 + 2x + 3\,x^2$ from Example 2.2 and the eigenvalues

$$s_1^3 = 3.1379^3, \quad s_2^3 = 6.6739^3, \quad s_3^3 = 10.288^3$$

of problem L_2 in (4.2). As a result, we obtain the system of equations

$$0.030569\,q_0 + 0.018827\,q_1 + 0.012590\,q_2 - 0.10599 = 0,$$
$$-0.018296\,q_0 - 0.0095634\,q_1 - 0.0063691\,q_2 + 0.056530 = 0,$$
$$0.030341\,q_0 + 0.015451\,q_1 + 0.010294\,q_2 - 0.092126 = 0.$$

This system has the unique solution

$$q_0 = 1.0000, \quad q_1 = 2.0000, \quad q_2 = 3.0000.$$

Where does it follow that

$$q(x) = 1 + 2x + 3\,x^2$$

is the unique polynomial that can be recovered by eigenvalues

$$s_1^3 = 3.1379^3, \quad s_2^3 = 6.6739^3, \quad s_3^3 = 10.288^3$$

of problem L_2.

Example 5.3. We substitute the found polynomial $q(x) = 1 + x + x^2 + x^3$ from Example 2.3 and the eigenvalues

$$s_1^3 = 4.2690^3, \quad s_2^3 = 7.8708^3, \quad s_3^3 = 11.493^3, \quad s_4^3 = 15.118^3$$

of problem L_1 in (4.1). As a result, we obtain the system of equations

$$0.0049375\,q_0 + 0.0024741\,q_1 + 0.0013709\,q_2 + 0.00081836\,q_3 - 0.0096009 = 0,$$
$$-0.0025713\,q_0 - 0.0012854\,q_1 - 0.00080804\,q_2 - 0.00056948\,q_3 + 0.0052341 = 0,$$
$$0.0034560\,q_0 + 0.0017275\,q_1 + 0.0011194\,q_2 + 0.00081544\,q_3 - 0.0071183 = 0,$$
$$-0.0070691\,q_0 - 0.0035337\,q_1 - 0.0023167\,q_2 - 0.0017084\,q_3 + 0.014628 = 0.$$

This system has the unique solution

$$q_0 = 1.0000, \quad q_1 = 1.0000, \quad q_2 = 1.0000, \quad q_3 = 1.0000.$$

Where does it follow that

$$q(x) = 1 + x + x^2 + x^3$$

is the only polynomial that can be recovered by eigenvalues

$$s_1^3 = 4.2690^3, \quad s_2^3 = 7.8708^3, \quad s_3^3 = 11.493^3, \quad s_4^3 = 15.118^3$$

of problem L_1.

6. About the Methods Used in the Work

The method of finding the eigenvalues by means of an expansion in a power series was also applied in the book [11, p. 25.6], but the eigenvalues were not by breaking the series and solving the corresponding algebraic equation, but by obtaining recurrence formulas.

A method based on the method of variation of an arbitrary constant and used by us to prove the uniqueness of the found polynomial potential was used earlier (see, for example, [23, chapter II, p. 4.3]) for another purpose as obtaining asymptotic relations of a fundamental system of solutions for large $|\lambda|$.

In solving the direct and inverse problems in the package of analytical computations, the first 200 terms of the series were used as the main part of the expansion series of the Taylor series in x and λ, and 50 significant digits were used in the calculations. Numerical experiments show that quite satisfactory results are obtained when using the first 60 terms of the series and 15 significant figures in the calculations. This is due to the fact that the linearly independent solutions of the equation and the characteristic determinant turn out to be alternating series and, therefore, according to the Leibnitz test, the remainder of the series can be estimated by its first term, which turns out to be a very small quantity. Therefore, the errors in the calculation of the eigenvalues are small.

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