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ON THE COMPLETENESS OF EIGEN AND ASSOCIATED VECTORS OF A CLASS OF THIRD ORDER QUASI-ELLIPTIC OPERATOR PENCILS

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Abstract. In the paper we find sufficient conditions on the coefficients of a class of third order quasi-elliptic operator pencils providing the completeness of the part of eigen and associated vectors in a separable Hilbert space. The completeness of the system of decreasing elementary solutions of the corresponding homogeneous equation in the space of solutions of this equation is proved. All these conditions are expressed by the coefficients of the operator pencil.

1. Introduction

In separable Hilbert space H we consider the operator pencil

$$P(\lambda) = (\lambda E - A)^2(\lambda E + A) + \lambda^2 A_1 + \lambda A_2, \tag{1.1}$$

where λ is a spectral parameter, E is a unit operator in H, the operator coefficients of the operator pencil (1.1) satisfy the following conditions:

(1) A is a self-adjoint positive-definite operator in H;
(2) The linear operators $B_1 = A_1 A^{-1}$, $B_2 = A_2 A^{-2}$ are bounded operators in H.

Note that in the sequel we will denote $P_0(\lambda) = (\lambda E - A)^2(\lambda E + A), P_1(\lambda) = = \lambda^2 A_1 + \lambda A_2$, $P(\lambda) = P_0(\lambda) + P_1(\lambda)$ and $P^*(\lambda) = (\lambda E - A)^2(\lambda E + A) + \lambda A_1^* + \lambda A_2^*$ i.e. $P^*(\lambda) = P^*(\overline{\lambda})$.

Note that

$$P(\lambda) = \lambda^3 E - \lambda^2 A - \lambda A^2 + A^3 + \lambda^2 A_1 + \lambda A_2 = (E + L(\lambda))A^3,$$

where

$$L(\lambda) = \lambda(B_2 - E)A^{-1} + \lambda^2(B_1 - E)A^{-2} + \lambda^3 A^{-3}.$$

If A^{-1} is a completely continuous operator in H, then $L(\lambda)$ will be an operator-valued function with completely continuous coefficients. Considering that $L(0) = 0$, $E + L(\lambda)$ is invertible at the point $\lambda = 0$. Then from the M.V. Keldysh lemma [9] the operator-function $E + L(\lambda)$ has a discrete spectrum with a unique limit point at infinity.

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Note that at the points $E+L(\lambda_0)$ is not invertible, λ_0 will be an eigen-value of the operator pencil $E+L(\lambda)$. Since $P(\lambda)=(E+L(\lambda))A^3$, then the pencil $P(\lambda)$ also possesses this property.

Note that if at the point λ_0 the operator pencil is not invertible, then in the vicinity of the point λ_0 the resolvent $P^{-1}(\lambda)$ is of the form

$$P^{-1}(\lambda)=\frac{Q_0}{(\lambda-\lambda_0)^{m+1}}+\frac{Q_1}{(\lambda-\lambda_0)^{m_0}}+...+\frac{Q_m}{\lambda-\lambda_0}+Q(\lambda), \qquad (1.2)$$

where $Q(\lambda)$ is a holomorphic operator-function in the vicinity of the point λ_0 where $Q_0, Q_1, ..., Q_m$ are finite dimensional operators.

Denote by $H_\gamma (\gamma \geq 0)$ Hilbert space whose domain of definition is $D(A^\gamma)$ with scalar product $(x,y)_\gamma = (A^\gamma, A^\gamma y), x, y \in D(A^\gamma)$. For $\gamma = 0$ we will consider $(x,y)_0 = (x,y)$ and $H_0 = H$.

Definition 1.1. If $x_0 \in H_3$ is a nonzero vector and satisfies the equation $P(\lambda_0)x_0 = 0$, then λ_0 is said to be an eigen-value of the operator pencil $P(\lambda)$.

If the vectors $x_0, x_1, ..., x_m$ satisfy the equation $(x_0 \neq 0)$

$$\begin{array}{l} P(\lambda_0)x_0 = 0 \\ P(\lambda_0)x_1 + \frac{d}{d\lambda}P(\lambda)\big|_{\lambda=\lambda_0} x_0 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P(\lambda_0)x_m + \frac{1}{1!}\left.\frac{dP(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} x_{m-1} + \frac{1}{2!}\left.\frac{d^2P(\lambda)}{d\lambda^2}\right|_{\lambda=\lambda_0} x_{m-2} + \frac{1}{3!}\left.\frac{d^3P(\lambda)}{d\lambda^3}\right|_{\lambda=\lambda_0} x_{m-3} = 0 \end{array} \qquad (1.3)$$

then $x_0, x_1, ..., x_m$ is said to be the system of eigen and associated vectors of the pencil $P(\lambda)$ responding to the eigen value λ_0.

Note that the eigen-value λ_0 can respond to several finite number systems of eigen and associated vectors.

Expanding $P(\lambda)$ in the vicinity of the point λ_0 in powers $(\lambda-\lambda_0)$ we see that for any $x \neq 0$ the system $Q_0x, Q_1x, ..., Q_mx$ are eigen and associated vectors $P(\lambda)$ responding to the eigen-value λ_0.

Denote by $K(\Pi_-)$ all the systems of eigen and associated vectors responding to eigen-values from the left hand side of the halfplane

$$\Pi_- = \{\lambda : Re\lambda < 0\}.$$

Note that various theorems on the completeness of all systems and their parts have been studied for example in [2],[5-7],[9], [11-14].

Third order operator pencils with a multiple characteristics of the principle part of third order

$$\tilde{P}(\lambda) = (\lambda E + A)^2(\lambda E - A) + \lambda^2 A_1 + \lambda A_2 \qquad (1.4)$$

and appropriate boundary value problem have been studied in [1-4].

Denote by $L_2(R_+; H)$ a Hilbert space of vector-functions determined in $R_+ = (0,\infty)$ always everywhere with the values in H with the norm

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^\infty \|f(t)\|^2\, dt\right)^{1/2} < +\infty$$

Following [10] we denote

$$W_2^3(R_+;H)=\left\{u:u^{(3)}\in L_2(R_+;H),A^3u\in L_2(R_+;H)\right\} \tag{1.5}$$

with the norm

$$\|u\|_{W_2^3(R_+;H)}=\left(\left\|u^{(3)}\right\|^2_{L_2(R_+;H)}+\left\|A^3u\right\|^2_{L_2(R_+;H)}\right)^{1/2}$$

It follows from the trace theorem [10] that

$$W_2^3(R_+;H;0)=\left\{u:u\in W_2^3(R_+;H),u(0)=0\right\} \tag{1.6}$$

and

$$\overset{0}{W}{}_2^3(R_+;H)=\left\{u:u\in W_2^3(R_+;H),u(0)=u'(0)=u''(0)=0\right\} \tag{1.7}$$

are complete subspaces of space $W_2^3(R_+;H)$.

Note that for $\varphi\in H_{5/2}$ and $e^{-tA}\varphi\in W_2^3(R_+;H)$ we have the inequalities

$$\left\|A^3e^{-tA}\varphi\right\|_{L_2(R_+;H)}\le\frac{1}{\sqrt{2}}\|\varphi\|_{5/2} \tag{1.8}$$

and

$$\left\|e^{-tA}\varphi\right\|_{W_2^3(R_+;H)}\le\|\varphi\|_{5/2} \tag{1.9}$$

Indeed, from the spectral expansion, assuming $y=A^{5/2}\varphi\in H$

$$\left\|A^3e^{-tA}\varphi\right\|^2_{L_2(R_+;H)}=\int_0^\infty\left(A^3e^{-tA}\varphi,A^3e^{-tA}\varphi\right)dt=\int_0^\infty\left(A(e^{-tA}y,e^{-tA}y\right)dt\le$$

$$\le\int_0^\infty\left(\int_{\mu_0}^\infty\mu e^{-2t\mu}dt\right)d\left(E_\mu y,y\right)=\int_{\mu_0}^\infty\mu d(E_\mu y,y)\left(\int_0^\infty e^{-2t\mu}dt\right)=$$

$$=\frac{1}{2}\int_{\mu_0}^\infty(dE_\mu y,y)=\frac{1}{2}\|y\|^2=\frac{1}{2}\|\varphi\|^2_{5/2}\,.$$

Hence it follows that $\left\|e^{-tA}\varphi\right\|^2_{W_2^3(R_+;H)}=2\left\|A^3e^{-tA}\varphi\right\|^2_{L_2(R_+;H)}\le\|\varphi\|^2_{5/2}\,.$

Note that for $R=(-\infty,\infty)$ the spaces $L_2(R_+;H)$ and $W_2^3(R_+;H)$ are determined in the same way.

We will use some facts from the book [10].

1^0. Theorem on intermediate derivatives.

If $u\in W_2^3(R_+;H)$, then $A^{3-j}u^{(j)}\in L_2(R_+;H)$ and

$$\left\|A^{3-j}u^{(j)}\right\|_{L_2(R_+;H)}\le const\,\|u\|_{W_2^3(R_+;H)}\,,\ j=1,2 \tag{1.10}$$

2^0. The trace theorem

If $u\in W_2^3(R_+;H)$, then $u^{(j)}(0)\in H_{3-j-1/2}$ and

$$\left\|u^{(j)}(0)\right\|_{H_{3-j-1/2}}\le const\,\|u\|_{W_2^3(R_+;H)}\,,j=1,2 \tag{1.11}$$

3^0. The class $D(R;H)$ of infinitely differentiable functions with values in H with compact supports in R is everywhere dense in $W_2^3(R;H)$.

Note that the principal part of the operator pencil $P(\lambda)$

$$P_0(\lambda) = (\lambda E - A)^2(\lambda E + A) \tag{1.12}$$

has spectra in the right and left semi-axes, but the power is an odd number, therefore, the pencil (1.1) is said to be a third order quasi-elliptic operator.

Since the characteristic polynomial has one root from the left half plane, we will study boundary value problems in one condition at the point 0.

Definition 1.2. Let $x_0, x_1, ..., x_m$ be the system of eigen and associated vectors responding to the eigen-value λ_0, then the vector-functions

$$u_h(t) = e^{\lambda_0 t}\left(\frac{t^h}{h!}x_0 + \frac{t^{h-1}}{(h-1)!}x_1 + ... + x_h\right), h = \overline{0, m} \tag{1.13}$$

are called elementary solutions of $P(d/dt)u(t) = 0$.

For $Re\lambda_0 < 0$ they are said to be decreasing elementary solutions.

The equalities $P(d/dt)u_h(t) = 0$ are verified by using the definition of eigen and associate vectors. Thus, for $Re\lambda_0 < 0$ the elementary solutions $u_h(t) \in W_2^3(R_+; H)$. Let us associate the operator pencil $P(\lambda)$ with the boundary value problem

$$P(d/dt)u(t) = \left(\frac{d}{dt} - A\right)^2\left(\frac{d}{dt} + A\right)u(t) + A_1\frac{d^2u(t)}{dt^2} +$$

$$+A_2\frac{du(t)}{dt} = f(t), t \in R_+ \tag{1.14}$$

$$u(0) = 0. \tag{1.15}$$

Definition 1.3. If for $f(t) \in L_2(R_+; H)$ there exists a vector-function $u(t) \in$ $\in W_2^3(R_+; H)$ satisfying the equation (1.14) almost everywhere in $R_+ = (0, \infty)$, then $u(t)$ it is said to be a regular solution of equation (1.14)

Definition 1.4. If for any $f(t) \in L_2(R_+; H)$ there exists regular solution $u(t)$ of the equation (1.14), boundary conditions (1.15) in the sense of convergence

$$\lim_{t\to+0} \|u(t)\|_{5/2} = 0$$

and we have the estimations

$$\|u\|_{W_2^3(R_+;H)} \le const\,\|f\|_{L_2(R_+;H)}$$

then problem (1.14), (1.15) is said to be regularly solvable.

2. On the norms of intermediate derivatives and regular solvability conditions of a boundary value problem

Denote

$$\mathcal{P}_0 u = P_0(d/dt)u = (d/dt - A)^2\,(d/dt + A)\,u(t), \quad u \in W_2^3(R_+; H; 0)$$

$$\mathcal{P}_1 u = P_1(d/dt)u = A_1\frac{d^2u}{dt^2} + A_2\frac{du}{dt} \quad \text{and } \mathcal{P}u = \mathcal{P}_0 u + \mathcal{P}_1 u,\ u \in W_2^3(R_+; H; 0)$$

acting from $W_2^3(R_+H; 0)$ to the space $L_2(R_+; H)$. The boundedness of the operators $\mathcal{P}_0, \mathcal{P}_1$ and $\mathcal{P}$ follows from conditions 1), 2) and the theorem on intermediate derivatives [10].

The following theorem is valid.

Theorem 2.1. *The operator $\mathcal{P}_0$ isomorphically maps the space $W_2^3(R_+H;0)$ onto the space $L_2(R_+;H)$*

Proof. Let us show that the equation $\mathcal{P}_0 u = f$ has a solution for all $f \in L_2(R_+;H)$.

Introduce the function $f_1(t) \in L_2(R;H)$ in the following way: $f_1(t) = f(t)$, for $t \in R_+$, $f_1(t) = 0$, for $t \in R/R_+$. Obviously, $\|f_1\|_{L_2(R,H)} = \|f\|_{L_2(R_+,H)}$.

Consider in R the equation

$$P_0(d/dt)v(t) = f_1(t), \quad t \in R = (-\infty, \infty).$$

After Fourier transformation we obtain

$$\hat{v}(\xi) = P_0^{-1}(i\xi)\hat{f}_1(\xi),\ \xi \in R$$

where $\hat{v}(\xi)$ and $\hat{f}_1(\xi)$ are Fourier transformations of the functions $v(t)$ and $f_1(t)$, respectively. Obviously,

$$\left\|A^2\hat{v}(\xi)\right\|^2_{L_2(R,H)} = \left\|A^2P_0^{-1}\hat{f}_1(\xi)\right\|^2_{L_2(R,H)} \le \sup_{\xi\in R}\left\|A^2P_0^{-1}(i\xi)\right\|^2\left\|\hat{f}_1(\xi)\right\|^2_{L_2(R,H)}$$

From the spectral theory of self-adjoint operators we obtain

$$\left\|A^2P_0^{-1}(\xi)\right\| = \sup_{\sigma\in\sigma(A)}\left|\sigma^3(i\xi-\sigma)^{-2}(i\xi+\sigma)^{-1}\right| = \sup_{\sigma\in\sigma(A)}\left|\sigma^3(\xi^2+\sigma^2)^{-3/2}\right| \le 1$$

i.e. $A^2\hat{v}(\xi) \in L_2(R,H)$. It is similarly proved that $(i\xi)^3\hat{v}_1(\xi) \in L_2(R,H)$. Then $v(t) \in W_2^3(R;H)$ and $\|v(t)\|_{W_2^3(R;H)} \le const\,\|f_1(t)\|_{L_2(R;H)} \le const\,\|f\|_{L_2(R;H)}$.

Denote by $u_0(t)$ contraction of the function $v(t)$ on $[0,\infty)$.

Then $u_0(t) \in W_2^3(R_+;H)$ is a regular solution of (1.14) and we will look for the solution in the form

$$u(t) = u_0(t) + e^{-tA}\varphi,$$

where $\varphi \in H_{5/2}$ is an unknown vector. Then it follows from $u(0) = 0$ condition that $\varphi = -u_0(0)$. Since $v_0(0) = u_0(0)$ and by the theorem on traces $\|v_0(0)\|_{5/2} \le const\,\|v\|_{W_2^3(R;H)} \le const\,\|f_1\|_{L_2(R;H)} = const\,\|f\|_{L_2(R_+;H)}$, then $\varphi = -u_0(0) \in H_{5/2}$. On the other hand,

$$\|u\|_{W_2^2(R_+;H)} \le \|u_0\|_{W_2^2(R_+;H)} + \left\|e^{-tA}u_0(0)\right\|_{W_2(R_+;H)} \le const\,\|f\|_{L_2(R_+;H)}$$

Then, by the Banach bounded inverse theorem the inverse operator $\mathcal{P}_0^{-1}$ exists and is bounded. The theorem is proved.

□

If follows from this theorem that the norms $\|u\|_{W_2^3(R;H)}$ and $\|\mathcal{P}_0 u\|_{L_2(R;H)}$ are equivalent in the space $W_2^3(R_+;H,0)$. Then the following norms are finite

$$N_j(R_+;0) = \sup_{0\ne u\in W_2^3(R_+;H,0)}\left\|A^{3-j}u^{(j)}\right\|_{L_2(R_+;H)}\|\mathcal{P}_0u\|^{-1}_{L_2(R_+;H)},\ j=1,2, \quad (2.1)$$

and

$$\overset{\circ}{N}_j(R_+) = \sup_{0\ne u\in \overset{\circ}{W}{}_2^3(R_+;H)}\left\|A^{3-j}u^{(j)}\right\|_{L_2(R_+;H)}\|\mathcal{P}_0u\|^{-1}_{L_2(R_+;H)},\ j=1,2,$$

These norms $N_j(R_+;0)$ are very important for finding the conditions of regular solvability of problem (1.14), (1.15) .

Let $d_j=\left(\frac{j}{3}\right)^{j/3}\left(\frac{3-j}{3}\right)^{\frac{3-j}{3}}$, $j=1,2$ and the parameter $\beta\in(0,d_j^{-3})$. Let us determine the following operator pencils

$$P_j(\lambda;\beta;A)=P_0(\lambda;A)P_0(-\lambda,A)-\beta(i\lambda)^{2j}A^{6-2j},\ j=1,2 \qquad (2.2)$$

We have

Theorem 2.2. *For $\beta\in(0,d_j^{-3})$ the operator pencil $P_j(\lambda;\beta;A)$ has no spectrum on the imaginary axis, and $P_j(\lambda;\beta;A)$ is represented in the form:*

$$P_j(\lambda;\beta;A)=F_j(\lambda;\beta;A)P_j(-\lambda;\beta;A) \qquad (2.3)$$

where

$$F_j(\lambda;\beta;A)=(\lambda E-w_1(\beta)A)\,(\lambda E-w_2(\beta)A)\,(\lambda E-w_3(\beta)A)= \\ =\lambda^3+a_{2,j}(\beta)\lambda^2A+a_{1,j}\lambda A^2+1 \qquad (2.4)$$

where $Rew_k(\beta)<0, k=1,2,3$ and $a_{2,j}(\beta)>0,\ a_{1,j}(\beta)>0,\ j=1,2.$

Proof. Let $\sigma\in\sigma(A)$. Then for $\lambda=i\xi,\ \xi\in R$ we obtain

$$P_j(i\xi,\beta;\sigma)=P_0(i\xi,\sigma)P_0(-i\xi,\beta)-\beta\xi^{2j}\sigma^{6-2j}=(\xi^2+\sigma^2)^3-\beta\xi^{2j}\sigma^{6-2j}=$$

$$=(\xi^2+\sigma^2)^3-\beta\xi^{2j}\sigma^{6-2j}=(\xi^2+\sigma^2)^3\left(1-\beta\xi^{2j}\sigma^{6-2j}(\xi^2+\sigma^2)^{-3}\right)>$$

$$>(\xi^2+\sigma^2)^3(1-\beta\sup_{\mu\geq 0}\mu^j(1+\mu)^{-3})=$$

$$=(\xi^2+\sigma^2)^3(1-\beta d_j^3)>0,\quad j=1,2.$$

Hence it follows that $P_j(i\xi,\beta;\sigma)$ has no root on the imaginary axis.

Since $P_j(\lambda,\beta;\sigma)=P_j(-\lambda,\beta;\sigma)$ and $P_j(\lambda,\beta;\sigma)$ is a polynomial with real coefficients, and if λ is a root of the polynomial $P_j(\lambda,\beta;\sigma)$, then $-\lambda$ and $\overline{\lambda}$ also are roots of $P_j(\lambda,\beta;\sigma)$. Therefore, three of the roots lie on the left half-plane and three of them lie on the right half-plane. Then assuming

$$F_j(\lambda,\beta;\sigma)=(\lambda E-w_{1,j}(\beta)A)(\lambda E-w_{2,j}(\beta)A)(\lambda E-w_{3,j}(\beta)A)$$

where $Rew_{k,j}(\beta)<0$ $(k=1,2,3)$, we obtain

$$F_j(\lambda,\beta;\sigma)=a_{3,j}(\beta)\lambda^3+a_{2,j}(\beta)\lambda^2\sigma+a_{1,j}(\beta)\lambda\sigma^2+a_{0,j}(\beta)$$

Note that $a_{3,j}(\beta)=1$, and from the Wiett theorem it follows that

$$a_{1,j}(\beta)=-(w_1(\beta)+w_2(\beta)+w_3(\beta)>0,$$

$$a_{2,j}(\beta)=w_{1,j}(\beta)w_{2,j}(\beta)+w_{1,j}(\beta)w_{3,j}(\beta)+w_{2,j}(\beta)w_{3,j}(\beta)>0,$$

$$a_0(\beta)=-w_{1,j}(\beta)w_{2,j}(\beta)w_{3,j}(\beta)>0,$$

since if $w_k(\beta)<0,(k=1,2,3)$ this is obvious, if one the roots is complex, then its complex conjugate is also a root of the polynomial and again $Re\overline{w_k(\beta)}<0$.

On the other and, from the equality

$$P_j(\lambda,\beta;\sigma)=F_j(\lambda,\beta;\sigma)F_j(\lambda,\beta;\sigma) \qquad (2.5)$$

comparing the coefficients for the powers λ we get $a_{0j}^2(\beta)=1$, since $a_{0,j}(\beta)>0$, $a_{0,j}(\beta)=1, j=1,2$. For $j=1$ we have

$$a_{0,1}(\beta)=a_{3,1}(\beta)=1,\ a_{2,1}^2(\beta)-2a_{1,1}(\beta)=3,\ a_{1,1}^2(\beta)-2a_{1,1}(\beta)=3-\beta \quad (2.6)$$

For $j=2$ we have

$$a_{0,2}(\beta)=a_{3,2}(\beta)=1,\ a_{2,2}^2(\beta)-2a_{1,2}(\beta)=3-\beta,\ a_{1,2}^2(\beta)-2a_{2,2}(\beta)=3 \quad (2.7)$$

In what follows, using spectral expansion of the operator A from equality (2.5) , we obtain the validity of formula (2.3) The theorem has been proved. □

Theorem 2.3. *For any R we have the equality*

$$\begin{aligned}&\|\mathcal{P}_0u\|^2_{L_2(R_+;H)}-\beta\left\|A^{3-j}u^{(j)}\right\|^2_{L_2(R_+;H)}=\\&=\|F(d/dt;\beta A)u\|^2_{L_2(R_+;H)}+(R_j(\beta),\tilde\varphi,\tilde\varphi)_{H^2},\ j=1,2\end{aligned} \quad (2.8)$$

where $\tilde\varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$, $\varphi_j=A^{-3-j-1/2}u^{(j)}(0),\ j=1,2,\quad H^2=H\times H,$

$$R_j(\beta)=\begin{pmatrix}a_{1,j}(\beta)a_{2,j}(\beta)-1 & a_{1,j}(\beta)+1\\ a_{1,j}(\beta)+1 & a_{2,j}(\beta)+1\end{pmatrix},\ j=1,2$$

Proof. Let $u\in W_2^3(R_+;H;0)(u(0)=0)$. Then

$$\begin{aligned}&\|F_j(d/dt;\beta;A)u\|^2_{L_2(R_+;H)}=\left\|u^{(3)}+a_{2,j}(\beta)Au''+a_{1,j}(\beta)A^2u'+A^3u\right\|^2_{L_2(R_+;H)}=\\&=\left\|u^{(3)}\right\|^2_{L_2(R_+;H)}+a_{2,j}^2(\beta)\left\|Au''\right\|^2_{L_2(R_+;H)}+a_{1,j}^2(\beta)\left\|A^2u\right\|^2_{L_2(R_+;H)}+\\&+\left\|A^3u\right\|^2_{L_2(R_+;H)}+a_{2,j}(\beta)2Re\left(u^{(3)},Au''\right)_{L_2(R_+;H)}+\\&+a_{1,j}(\beta)2Re\left(u^{(3)},A^2u'\right)_{L_2(R_+;H)}+2Re(u^{(3)},A^3u)_{L_2(R_+;H)}\\&+a_{2,j}(\beta)a_{1,j}(\beta)2Re(Au'',A^2u')_{L_2(R_+;H)}+\\&+a_{2,j}(\beta)2Re(Au'',A^3u)_{L_2(R_+;H)}+a_{1,j}(\beta)2Re(A^2u',u^{(3)})_{L_2(R_+;H)}\end{aligned} \quad (2.9)$$

After integrating by parts we obtain

$$2Re\left(u^{(3)},A^2u''\right)_{L_2(R_+;H)}=-\left\|\varphi_2\right\|^2_H,\ 2Re\left(u^{(3)},A^2u'\right)=$$

$$=-2Re(\varphi_2,\varphi_1)-2\left\|Au''\right\|^2_{L_2(R_+;H)},2Re\left(u^{(3)},A^3u\right)_{L_2(R_+;H)}=$$

$$=\left\|\varphi_1\right\|^2_H,\ 2Re\left(Au'',A^2u'\right)_{L_2(R_+;H)}=-\left\|\varphi_1\right\|^2$$

$$2Re\left(Au'',A^3u\right)_{L_2(R_+;H}=-2\left\|A^2u'\right\|^2_{L_2(R_+;H)}.$$

Considering this equality in (2.9), we obtain

$$\begin{aligned}&\|F_j(d/dt;\beta;A)u\|^2_{L_2(R_+;H)}=\left\|u^{(3)}\right\|^2_{L_2(R_+;H)}+a_{2,j}^2(\beta)\left\|Au''\right\|^2_{L_2(R_+;H)}+\\&+a_{1,j}^2(\beta)\left\|A^3u'\right\|^2_{L_2(R_+;H)}-a_{2,j}(\beta)\left\|\varphi_2\right\|^2-2a_{1,j}(\beta)Re(\varphi_2,\varphi_1)-\\&-2a_{i,j}^{(\beta)}\left\|Au''\right\|^2_{L_2(R_+;H)}+\left\|\varphi_1\right\|^2-a_{1,j}(\beta)a_{2,j}(\beta)\left\|\varphi_1\right\|^2-\\&-2a_{2,j}(\beta)\left\|A^2u'\right\|^2_{L_2(R_+;H)}=\left\|u^{(3)}\right\|^2_{L_2(R_+;H)}+\end{aligned}$$

$$+\left\|A^3u\right\|^2_{L_2(R_+;H)}+(a^2_{2,j}(\beta)-2a_{1,j}(\beta))\left\|Au''\right\|^2_{L_2(R_+;H)}+ \\ +(a^2_{1,j}(\beta)-2a_{1,j}(\beta))\left\|Au'\right\|^2_{L_2(R_+;H)}-a_{1,j}(\beta)a_{2,j}(\beta)\left\|\varphi_1\right\|^2- \\ -a_{1,j}(\beta)\left\|\varphi_2\right\|^2-2a_{1,j}(\beta)Re(\varphi_2,\varphi_1). \tag{2.10}$$

It follows from equality (2.10) that

$$\|\varphi_1\|^2=\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix}\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix}\right)_{H^2},$$

$$2Re(\varphi_2,\varphi_1)=\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix}\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix}\right)_{H^2}$$

$$\|F_j(d/dt;\beta;A)u\|^2_{L_2(R_+;H)}=\left\|u^{(3)}\right\|^2_{L_2(R_+;H)}+\left\|A^3u\right\|^2_{L_2(R_+;H)}+$$

$$+(a^2_{2,j}(\beta)-2a_{1,j}(\beta))\left\|Au''\right\|^2_{L_2(R_+;H)}+(a^2_{1,j}(\beta)-2a_{2,j}(\beta))\left\|Au'\right\|^2_{L_2(R_+;H)}$$

$$-\left(\begin{pmatrix}a_{1,j}(\beta)\cdot a_{2,j}(\beta)-1 & a_{1,j}(\beta)\\ a_{1,j}(\beta) & a_{2,j}(\beta)\end{pmatrix}\tilde{\varphi},\tilde{\varphi}\right)_{H^2}.$$

In the same way we obtain:

$$\|P_0u\|^2_{L_2(R_+;H)}=\left\|u^{(3)}-A^2u''-A^2u'+A^3u\right\|^2_{L_2(R_+;H)}=\left\|u^{(3)}\right\|^2_{L_2(R_+;H)}+$$

$$+3\left\|Au''\right\|^2_{L_2(R_+;H)}+3\left\|A^2u'\right\|^2_{L_2(R_+;H)}+\left(\begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix}\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix}\right)_{H^2}$$

Hence we obtain

$$\|F_j(d/dt;\beta;A)u\|^2_{L_2(R_+;H)}+(R_j(\varphi)\tilde{\varphi},\tilde{\varphi})_{H^2}=$$

$$=\|P_0u\|^2_{L_2(R_+;H)}+(a^2_{2,j}(\beta)-2a_{1,j}(\beta)-3)\times$$

$$\times\left\|Au'''\right\|^2_{L_2(R_+;H)}+(a^2_{1,j}(\beta)-2a_{2,j}(\beta)-3)\left\|Au'\right\|^2_{L_2(R_+;H)},\ j=1,2$$

Taking into account equalities (2.6) and (2.7) in the last equality, we complete the proof of the theorem. □

In [8] it is proved that

$$N_j(R)=\sup_{0\neq u\in W^3_2(R;H)}\left\|A^{3-j}u^{(j)}\right\|_{L_2(R;H)}\|P_0(d/dt)u\|^{-1}_{L_2(R;H)}=d_j^{3/2},\ j=1,2$$

Then from the density of $D(R;H)$ in $W^3_2(R;H)$ it follows that there exists the vector-function $v_n(t)\in D(R;H)$ with such a support in the interval $[-n,n]$ that $v_n(t)\to v(t)$ in the norm of the space $W^3_2(R;H)$. Then the vector-function $u_n(t)=v(t-n)\in \overset{\circ}{W}{}^3_2(R_+;H)$ and $\|u_n(t)\|_{W^3_2(R_+;H)}=\|v_n(t)\|_{W^3_2(R;H)}$. Then for any $\varepsilon>0$ there exists n_0 such that

$$\left\|A^{3-j}u^{(j)}_{n_0}\right\|_{L_2(R_+;H)}\left\|\mathcal{P}_0u_{n_0}\right\|^{-1}_{L_2(R_+;H)}>d_j^{3/2}-\varepsilon$$

On the other hand, it follows from Theorem 2.3 that for all $u \in \overset{0}{\underset{2}{W}}(R;H)(u^{(j)}(0) = 0, j = 1,2,3)$

$$\|\mathcal{P}_0 u\|^2_{W_2(R_+;H)} - \beta \left\| A^{3-j} u^{(j)} \right\|_{L_2(R_+;H)} \geq 0$$

Passing to the limit as $\beta \to d_j^{-3}$ we obtain, consequently $\overset{\circ}{N}_j(R_+) \leq d_j^{3/2}$, it follows from $\overset{\circ}{N}_j(R_+) = d_j^{3/2}, j = 1,2$ that $\overset{\circ}{W}{}^3_2(R_+;H) \subset W_2(R_+;H;0)$ consequently,

$$N_j(R_+;0) \geq d_j^{3/2} (j = 1,2) \tag{2.11}$$

Now we show that when $N_j(R_+;0) = d_j^{3/2} (j = 1,2)$.

Theorem 2.4. *For $N_j(R_+;0) = d_j^{3/2} (j = 1,2)$ it is necessary and sufficient that for all $\beta \in (0, d_j^{-3})$ the matrix $R_j(\beta) > 0$, where the matrix $R_j(\beta)$ was determined from Theorem 2.3.*

Proof. Let $N_j(R_+;0) = d_j^{3/2}$. Then for any $\beta \in (0, d_j^{-3})$ for any $\tilde{\varphi} \in H^2$, the Cauchy problem

$$F_j(d/dt; \beta : A)u(t) = 0, u(0) = 0, \; u'(0) = A^{-3/2}\varphi_1, \; u''(0) = A^{-1/2}\varphi_2$$

has the solution $u(t)$ from $W_2^3(R_+;H)$. This follows from the form of $F_j(\lambda;\beta;A)$. Then for any $\tilde{\varphi} \in H^2$ we have

$$\begin{aligned}
&(R(\beta)\tilde{\varphi}, \tilde{\varphi})_{H^2} = \|\mathcal{P}_0 u\|^2_{L_2(R_+;H)} - \beta \left\| A^{3-j} u^{(j)} \right\|_{L_2(R_+;H)} = \\
&= \|\mathcal{P}_0 u\|^2_{L_2(R_+;H)} \left(1 - \beta \left\| A^{3-j} u^{(j)} \right\|^2_{L_2(R_+;H)} \|\mathcal{P}_0 u\|^{-2}_{L_2(R_+;H)} \right) \geq \\
&\geq \|\mathcal{P}_0 u\|^2_{L_2(R_+;H)} (1 - \beta N_j^2(R_+;0)) = \|\mathcal{P}_0 u\|^2_{L_2(R_+;H)} (1 - \beta d_j^3) > 0.
\end{aligned}$$

Consequently, $R_j(\beta) > 0$. On the other hand, if $R_j(\beta) > 0$, then if follows from Theorem 2.3 that for all $\beta \in (0, d_j^{-3})$ and $W_2^3(R_+;H;0)$

$$\|\mathcal{P}_0 u\|^2_{L_2(R_+;H)} - \beta \left\| A^{3-j} u^{(j)} \right\|^2_{L_2(R_+;H)} > 0.$$

For $\beta \to d_j^{-3}$ we have

$$\left\| A^{3-j} u^{(j)} \right\|_{L_2(R_+;H)} \cdot \|\mathcal{P}_0 u\|^{-1}_{L_2(R_+;H)} \leq d_j^{3/2}$$

i.e. $N_j(R_+; 0 \leq d_j^{3/2}$. On the other hand, it follows from inequality (2.11) that $N_j(R_+;0) \geq d_j^{3/2}$. Then $N_j(R_+;0) = d_j^{3/2}$. The theorem is proved. □

For $N_j(R_+;0) > d_j^{3/2}$ finding $N_j(R_+;0)$ we prove the following theorem.

Theorem 2.5. *Let $N_j(R_+;H) > d_j^{3/2}$, then $N_j^{-2}(R_+;0)$ is the least root of the equation* $\det R_j(\beta) = 0$ *from the interval* $(0, d_j^{-3})$.

Proof. Obviously, $N_j(R_+;0) \in (0, d_j^{-3})$. Then for $\beta \in (0, N_j^{-2}(R_+;0))$, for any $\tilde{\varphi} \in H^2$, the Cauchy problem

$$F(d/dt; \beta; A)u = 0, \; u(0) = 0, \; u'(0) = A^{-3/2}\varphi_1, \; u''(0) = A^{-1/2}\varphi_2$$

has a unique solution $u_0(t) \in W_2^3(R_+; H; 0)$. This follows from the representation of $F(\lambda; \beta; A)$ in (2.4).

Then for any $\tilde{\varphi} \in H^2$, it follows from the inequality that

$$(R(\beta)\varphi, \varphi) \geq \|\mathcal{P}_0 u\|^2_{L_2(R_+;H)} ((1 - \beta \cdot N_j^2(R_+; 0)) > 0, \beta \in (0, N_j^{-2}(R_+; 0)).$$

Thus, the first eigenvalue $\lambda_1(\beta) > 0$, for $\beta \in (0, N_j^{-2}; (R_+; 0))$.

From the definition of $N_j(R_+; 0)$ it follows that for $\beta > N_j^{-2}(R_+; 0)$ there exists such vector-function $u_\beta(t) \in W_2^3(R_+; H; 0)$ that

$$\|\mathcal{P}_0 u_\beta\|^2_{L_2(R_+;H)} - \beta \left\| A^{3-j} u_\beta^{(j)} \right\|^2_{L_2(R_+;H)} < 0.$$

Then it follows from Theorem 2.3 that $(R(\beta)\tilde{\varphi}_\beta, \tilde{\varphi}_\beta) < 0$, where $\tilde{\varphi}_\beta = \begin{pmatrix} \varphi_{1,\beta} \\ \varphi_{2,\beta} \end{pmatrix}$, $\varphi_{j,\beta} = A^{3-j1/2} u_\beta^{(j)}(0), j = 1, 2$. Thus, we obtain that the first eigenvalue of the matrix $R_j(\beta)$ for $\beta > N_j^{-2}(R_+; 0)$ is negative, i.e. $\lambda_1(\beta) < 0$ for $\beta \in (N_j^{-2}(R_+; 0), d_j^{-3})$. Then it follows from the continuity of $\lambda_1(\beta)$ with respect to β that $\lambda_1(N_j^{-2}(R_+; 0)) = 0$.

Consequently, $\det R(N_j^{-2}(R_+; 0)) = 0$. Since $\lambda_1(\beta) > 0$ for all $\beta \in (N_j^{-2}(R_+; 0), d_j^{-3})$, we have $\beta_0 = N_j^{-2}(R_+; 0)$ is the least root of the equation $\det R_j(\beta) = 0$ from the interval $(0, d_j^{-3}) (j = 1, 2)$. The theorem is proved. □

Using Theorems 2.4 and 2.5, we can find $N_1(R_+; 0)$ and $N_2(R_+; 0)$. We have

Theorem 2.6. *The norm* $N_1(R_+; 0) = \left(\frac{\sqrt{5}-1}{8}\right)^{1/2}$.

Proof. To find $N_1(R_+; 0)$, we will solve the system of equations:

$$\begin{cases} (a_1(\beta)a_{21}(\beta) - 1)(a_{21}(\beta) + 1) = (a_{1,j}(\beta) + 1)^2 \\ a_{1,1}^2(\beta) - 2a_{2,1}(\beta) = 3 - \beta \\ a_{2,1}^2(\beta) - 2a_{1,1}(\beta) = 3 \end{cases}$$

For simplicity we omit the argument β and the second index

$$a_1 a_2^2 - a_2 + a_1 a_2 - 1 = a_1^2 + 2a_1 + 1 \Rightarrow a_1(2a_1 + 3) - a_2 + a_1 a_2 - 1 =$$
$$= a_1^2 + 2a_1 + 1 \Rightarrow a_1^2 + a_1 - a_2 + a_1 a_2 - 2 = 0 \Rightarrow (a_1 - 1)(a_1 + a_2 + 2) = 0.$$

Since $a_1 + a_2 + 2 > 0$, $a_1 = 1$. Then $a_2 = \sqrt{5}$. Consequently from the equation $1 - 2\sqrt{5} = 3 - \beta$ it follows that $\beta = 2(\sqrt{5} + 1) \in \left(0, \frac{27}{4}\right)$ i.e.$\beta_0 = 2\left(\sqrt{5} + 1\right)$ and $N_1(R_+; 0) = \beta_0^{-1/2} = \left(\frac{\sqrt{5}-1}{8}\right)^{1/2}$. □

Theorem 2.7. *The norm* $N_2(R_+; 0) = \beta_0^{-1/2}$, *where* β_0 *is a positive root of the equation*

$$4\beta^3 - 11\beta - 20\beta - 16 = 0.$$

Proof. We must solve the system

$$\begin{cases} (a_1 a_2 - 1)(a_2 + 1) = (a_1 + 1)^2 \\ a_1^2 - 2a_2 = 3 \\ a_2^2 - 2a_1 = 3 - \beta \end{cases}$$

From the equality $(a_1a_2-1)(a_2+1)=(a_1+1)^2 \Rightarrow (a_1a_2^2-a_2+a_1a_2-1)=$

$$=(a_1+1)^2 \Rightarrow a_1(2a_1+3-\beta)-a_2-a_1a_2-1=a_1^2+2a_1+1 \Rightarrow$$
$$\Rightarrow a_1^2+a_1-\beta a_1-a_2+a_1a_2=0 \Rightarrow 1+a_1+a_2+a_1a_2-\beta a_1=0$$

i.e.

$$1+a_1+a_2+a_1a_2=\beta a_1 \tag{2.12}$$

Multiplying the equality by the number a_1, we obtain

$$\begin{aligned}&a_1+a_1^2+a_1a_2+a_1^2a_2=\beta a_1^2 \Rightarrow a_1+2a_2+3+a_1a_2+a_2(2a_2+3)=\\&=\beta(2a_2+3) \Rightarrow (1+a_1+a_2+a_1a_2)+(a_2+2+2a_2^2+3a_2)=\\&=2\beta a_2+3\beta \Rightarrow \beta a_1+4a_2+4a_1+8-2\beta=2\beta a_2+3\beta\end{aligned}$$

and finally

$$(4+\beta)a_1+(4-2\beta)a_2=5\beta-8 \tag{2.13}$$

Then, considering $a_1^2-1=2(a_2+1)$, we obtain the equality

$$\begin{aligned}&(a_1a_2-1)(a_1-1)=2(a_1+1) \Rightarrow a_1^2a_2-a_1+1-a_1a_2=2a_1+2 \Rightarrow\\&\Rightarrow (2a_2+3)a_2-a_1-a_1a_2-1=2a_1 \Rightarrow 2a_2^2+4a_2-\\&-(a_2+a_1+a_1a_2+1)=2a_1 \Rightarrow\\&\Rightarrow 4a_1+6-2\beta+4a_2-\beta a_1=2a_1\end{aligned}$$

i.e.

$$(2-\beta)a_1+4a_2=2\beta-6 \tag{2.14}$$

From the equations (2.13), and (2.14) we obtain

$$a_1=\frac{2\beta^2-4}{4+6\beta-\beta^2} \tag{2.15}$$

It follows from equation (2.14) that

$$2a_1^2+(2-\beta)a_1=2\beta$$

Considering in this equation the value a_1 from (2.15) with respect to β we obtain an equation with respect to β. After simple calculations we obtain the equation

$$4\beta^3-11\beta^2-20\beta-16=0$$

By the Descartes theorem this equation has a unique real root β_0 from the interval $\left(0,\frac{27}{4}\right)$ and $N_2(R_+;0)=\beta_0^{-1/2}$. The theorem is proved. □

We now prove a theorem on regular solvability of problem (1.14),(1.15)

Theorem 2.8. *Let conditions 1), 2) be fulfilled, and the condition*

$$q=N_2(R_+;0)\,\|B_1\|+N_1(R_+;0)\,\|B_2\|<1$$

be fulfilled, where the numbers $N_1(R_+;0)$ *and* $N_2(R_+;0)$ *were determined from Theorem 2.6 and 2.7. Then the problem* (1.14),(1.15) *is regularly solvable.*

Proof. We write problem (1.14),(1.15) in the operator form

$$\mathcal{P}u\equiv\mathcal{P}_0u+\mathcal{P}_1u=f,$$

where $f \in L_2(R_+;0)$, $f \in W_2^3(R_+;0)$. After replacing $\mathcal{P}_0 u = \omega$ we obtain the equation $(E + \mathcal{P}_1\mathcal{P}_0^{-1})\omega = f$. We now estimate the norm of the operator $P_1 P_0^{-1}$. We can write that

$$\left\|\mathcal{P}_1\mathcal{P}_0^{-1}u\right\|_{L_2(R_+;H)} = \|\mathcal{P}_1 u\|_{L_2(R_+;H)} \le \|B_1\| \, \|Au''\|_{L_2(R_+;H)} + \\ + \|B_2\| \, \|Au'\|_{L_2(R_+;H)} \le (N_2 \|B_1\| \,| + N_1 \|B_2\|) \, \|\mathcal{P}_0 u\|_{L_2(R_+;H)} = \\ = (N_2 \|B_1\| \,| + N_1 \|B_2\|) \, \|\omega\|_{L_2(R_+;H)} = q \, \|\omega\|_{L_2(R_+;H)}, \ (q < 1).$$

Consequently, subject to the theorem, the operator $E + \mathcal{P}_1\mathcal{P}_0^{-1}$ is invertible in $L_2(R_+;0)$. Hence we obtain

$$\|u\|_{W_2^2(R_+;H)} \le const \, \|f\|_{L_2(R_+;H)}$$

The theorem is proved □

3. Theorem on the completeness of the system $K(\Pi_-)$ and the system of decreasing elementary solutions of homogeneous equation

In [6], M.G. Gasymov shows that for studying the completeness of the system $K(\Pi_-)$ it suffices to prove the existence of regular solutions of the problem

$$P(d/dt)u(t) = 0 \quad t \in R_+ = (0, \infty) \tag{3.1}$$

$$u(0) = \varphi, \quad \varphi \in H_{5/2} \tag{3.2}$$

and some analytic properties of the resolvent $P^{-1}(\lambda)$.

Definition 3.1. If for any $\varphi \in H_{5/2}$ there exists a regular solution of equation (3.1), satisfying the boundary condition in the sense

$$\lim_{t \to +0} \|u(t) - \varphi\|_{5/2} = 0$$

and

$$\|u\|_{W_2^2(R_+;H)} \le const \, \|\varphi\|_{5/2},$$

then the problem (3.1) and (3.2) is said to be regularly solvable.

We have

Theorem 3.1. *Let all the conditions of Theorem 2.8 be fulfilled. Then the problem* (3.1), *(3,2) is regularly solvable.*

Proof. We make substitution $u(t) = v(t) + e^{-tA}\varphi$, where $v(t) \in W_2^3(R_+;H;0)$ and $\varphi \in H_{5/2}$. Then the problem (3.1),(3.2) can be written in the following form:

$$P(d/dt)v(t) = g(t) \quad t \in R_+ \tag{3.3}$$

$$v(0) = 0 \tag{3.4}$$

where $g(t) \in L_2(R_+;H)$. Indeed $g(t) = -A_1 A^2 e^{-tA}\varphi + A_2 A e^{-tA}\varphi$ and considering (1.8) we have:

$$\|g(t)\|_{L_2(R_+;H)} = \left\|A_1 A^2 e^{-tA}\varphi\right\|_{L_2(R_+;H)} + \left\|A_2 A e^{-tA}\varphi\right\|_{L_2(R_+;H)} \le$$

$$\le \|B_1\| \left\|A^3 e^{-tA}\varphi\right\|_{L_2(R_+;H)} + \|B_2\| \left\|A^3 e^{-tA}\varphi\right\|_{L_2(R_+;H)} \le const \, \|\varphi\|_{5/2}$$

Then problem (3.3), (3.4) subject to the conditions of the theorem has a regular solution, and

$$\lim_{t\to+0}\|v(t)\|_{3/2}=\lim_{t\to+0}\|u(t)-\varphi\|_{5/2}=0$$

the following estimation holds

$$\|v(t)\|_{W_2^2(R_+;H)}\le const\,\|g(t)\|_{L_2(R_+;H)}\le const\,\|\varphi\|_{5/2}$$

Using the inequality (1.9) we obtain:

$$\|u\|_{W_2^2(R_+;H)}\le\|v\|_{W_2^2(R_+;H)}+\left\|e^{-tA}\varphi\right\|_{W_2^2(R_+;H)}\le const\,\|\varphi\|_{5/2}$$

The theorem is proved □

We now study some properties of the resolvent $P^{-1}(\lambda)$.

Let T be some completely continuous operator in H. The eigenvalue of the operator $|T|=(T^*T)^{1/2}$ is called $s-$ numbers of the operator T. We will consider

$$s_1\ge s_2\ge...\ge s_n\ge....$$

If for $\rho>0$,

$$\sum_{k=1}^{\infty}s_k^{\rho}<\infty,$$

then we say that $T\in\sigma_\rho(H)$. For $T\in\sigma_\rho(H)$ the operators BT and $TB\in\sigma_\rho(H)$, for any bounded operator B. If $A=A^*\ge\mu_0E,\mu_0>0$, and A^{-1} is a completely continuous operator, and $\lambda_1\le\lambda_2\le...\le\lambda_k\le...$ are eigenvalues of the operator A, then we say that

$$\sum_{k=1}^{\infty}\lambda_k^{-\rho}<\infty$$

Note that if $u_h(t)$ is an elementary solution of the homogeneous equation, then $u_h(0)=x_h$.

We have

Lemma 3.1. *Let conditions 1), 2) be fulfilled. Then for any $\alpha\in(0,\pi/4]$ subject to the inequality*

$$\|B_1\|+\|B_2\|<\sqrt{2}d^{-3/2}\sin\frac{\alpha}{2},$$

where $d=d_1=d_2=\left(\frac{4}{27}\right)^{1/2}$, on the rays $\Gamma_{\pi\pm\alpha}=\left\{\lambda:\lambda=re^{i(\pi\pm\alpha)},r>0\right\}$ there exists a resolvent $P^{-1}(\lambda)$ and for large $|\lambda|$ we have the estimations

$$\left\|P^{-1}(\lambda)\right\|\le const\,|\lambda|^{-3}.$$

Proof. Let $\lambda\in\Gamma_{\pi\pm\alpha}$. Then from the representation $P(\lambda)=(E+P_1(\lambda)P_0^{-1}(\lambda))P_0(\lambda)$ for the invertibility of $P(\lambda)$, we estimate the norms $\left\|P_1(\lambda)P_0^{-1}(\lambda)\right\|$ on the rays $\Gamma_{\pi\pm\alpha}$. Obviously,

$$\left\|P_1(\lambda)P_0^{-1}(\lambda)\right\|\le\|B_1\|\left\|\lambda^2AP_0^{-1}(\lambda)\right\|+\|B_2\|\left\|\lambda^2A^2P_0^{-1}(\lambda)\right\|$$

On the rays $\Gamma_{\pi\pm\alpha}$ from the spectral expansion of A we obtain:

$$\begin{aligned} \left\|\lambda^2 A P_0^{-1}(\lambda)\right\| &= \sup_{\sigma\in\sigma(A)} \left|r^2\sigma(re^{i(\pi\pm\alpha)}-\sigma)^{-2}(re^{i(\pi\pm\alpha)}-\sigma)^{-1}\right| = \\ &= \sup_{\sigma\in\sigma(A)} \left|r^2\sigma(r^2+\sigma^2+2r\sigma\cos\alpha)^{-1}(r^2+\sigma^2-2r\sigma\cos\alpha)^{-1/2}\right| \leq \\ &\leq \sup_{\sigma\in\sigma(A)} \left|r^2\sigma(r^2+\sigma^2)^{-3/2}2^{-1/2}\sin^{-1}\tfrac{\alpha}{2}\right| \leq d^{3/2}2^{-1/2}\sin^{-1}\tfrac{\alpha}{2}. \end{aligned}$$

In a similar way we obtain

$$\left\|\lambda A^2 P_0^{-1}(\lambda)\right\| \leq d^{3/2}2^{-1/2}\sin^{-1}\frac{\alpha}{2}.$$

Then for $\lambda\in\Gamma_{\pi\pm\alpha}$ and subject to the condition of the theorem

$$\left\|P_1(\lambda)P_0^{-1}(\lambda)\right\| \leq d^{3/2}2^{-1/2}\sin^{-1}\frac{\alpha}{2}\left(\|B_1\|+\|B_2\|\right) < 1.$$

Subject to the condition of the theorem, we obtain $\left\|P_1(\lambda)P_0^{-1}(\lambda)\right\| < 1$ for $\lambda\in\Gamma_{\pi\pm\alpha}$ therefore $P^{-1}(\lambda)$ exists and for large $|\lambda|$ we have the estimations

$$\left\|P^{-1}(\lambda)\right\| \leq const\left\|P_0^{-1}(\lambda)\right\| \leq const\,|\lambda|^{-3}.$$

□

The following corollary holds.

Corollary 3.1. *If conditions 1), 2) are fulfilled and the inequality*

$$N_2(R_+;0)\,\|B_1\| + N_1(R_+;0)\,\|B_2\| < \sqrt{2}\sin\frac{\alpha}{2}$$

holds then $P^{-1}(\lambda)$exists on all the rays and for large $\lambda\in\Gamma_{\pi\pm\alpha}$ we have the estimations $\left\|P^{-1}(\lambda)\right\| \leq const\,|\lambda|^{-3}$.

Here the numbers $N_1(R_+;0)$ and $N_2(R_+;0)$ were determined from Theorems 2.6 and 2.7.

Proof. Since, $N_2(R_+;0) > d^{3/2}$ and $N_1(R_+;0) > d^{3/2}$, then $N_2(R_+;0)\cdot d^{-3/2} > 1$ $N_1(R_+;0)\cdot d^{-3/2} > 1$. From the condition of the corollary we obtain

$$\|B_1\|+\|B_2\| < d^{-3/2}(N_2(R_+;0)\,\|B_1\| + N_1(R_+;0)\,\|B_2\|) < d^{-3/2}\sqrt{2}\sin\frac{\alpha}{2}.$$

Therefore, by Lemma 3.1 $P^{-1}(\lambda)$ exists, and $\left\|P^{-1}(\lambda)\right\| \leq const\,|\lambda|^{-3}$. The corollary is proved. □

We now prove a theorem on the completeness of the system $K(\Pi_-)$ in space $H_{5/2}$

Theorem 3.2. *Let conditions 1), 2) be fulfilled, $A^{-1}\in\sigma_\rho(H)$ $(0<\rho<\infty)$ and the following inequality hold:*

$$N_2(R_+;0)\,\|B_1\| + N_1(R_+;0)\,\|B_2\| < \begin{cases} 1; & \rho\in(0,1] \\ \sqrt{2}\sin\frac{\pi}{4\rho}; & \rho\in[1,\infty) \end{cases}$$

Then the system $K(\Pi_-)$ is complete in $H_{5/2}$

Proof. Let $\rho \in (0,1]$ an opening between the angles $(-i\infty,0)$ and $(0,i\infty)$ be equal to π; on these rays subject to the inequality

$$N_2(R_+;0)\,\|B_1\| + N_1(R_+;0)\,\|B_2\| < 1$$

there exists a regular solution of problem (3.1) and (3.2), and $P^{-1}(\lambda)$ is invertible on the imaginary axis (see Corollary 3.1 for $\alpha = \frac{\pi}{2}$) . For $\rho \in [1,\infty)$ on the rays $\Gamma_{\pi\pm\pi/2\rho}$ subject to the condition

$$N_2(R_+;0)\,\|B_1\| + N_1(R_+;0)\,\|B_2\| < \sqrt{2}\sin\frac{\alpha}{4\rho},$$

the solution of the problem (3.1), (3.2) $u(t)$ exists and a resolvent also exists. Since

$$P(\lambda) = \left(\lambda^3 A^{-3} + \lambda^2(B_1 - E)A^{-2} + \lambda(B_2 - E)A^{-1} + E\right) A^3$$

and $A^{-1} \in \sigma_\rho(H)$, $A^{-2} \in \sigma_{2\rho}(H)$,$A^{-3} \in \sigma_{3\rho}(H)$, by M. G. Gasymov's lemma [5] $P^{-1}(\lambda)$ is represented with respect to two entire functions at most order ρ, and for the order ρ these functions are of minimum type. On the other hand, after the Laplace transformation this solution $u(t)$ can be represented in the form

$$u(t) = \frac{1}{2\pi t}\int_{-i\infty}^{i\infty} P^{-1}(\lambda)\sum_{\nu=0}^{2} Q_\nu u^{(\nu)}(0)e^{\lambda t}d\lambda \tag{3.5}$$

where the polynominal $Q_\nu(\lambda)(\nu = \overline{0,3})$ has at most two orders. Using the behavior of the resolvent $P^{-1}(\lambda)$ on the imaginary axis, we can represent the integral (3.5) in the form:

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_\theta} P^{-1}(\lambda)\sum_{\nu=0}^{2} Q_\nu(\lambda)u^{(\nu)}(0)e^{\lambda t}d\lambda \tag{3.6}$$

where Γ_θ for sufficiently small θ coincides with the rays $\Gamma_{\pi-\theta}$ and $\Gamma_{\pi+\theta}$. Note that $u(t) \in W_2^3(R_+;H)$, and therefore its Laplace transform is an analytic function in the right half plane, is bounded and has finite values on the imaginary axis.

From (3.6) it follows that for $t > 0$

$$(u(t),\varphi)_{5/2} = \frac{1}{2\pi i}\int_{\Gamma_\theta}\left(A^{5/2}P^{-1}(\lambda)\sum_{\nu=0}^{2} Q_\nu^{(\lambda)}u^{(j)}(0), A^{5/2}\varphi\right)e^{\lambda t}d\lambda =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\theta}\left(\sum_{\nu=0}^{2} Q_\nu^{(\lambda)}u^{(j)}(0), (A^{5/2}P^{-1}(\overline{\lambda}))^* A^{5/2}\varphi\right)e^{\lambda t}d\lambda.$$

It follows from M.G. Gasymov's [5,6] theorem that the system $K(\Pi_-)$ is complete in $H_{5/2}$if and only if the function $v(\lambda) = \left(A^{5/2}P^{-1}(\overline{\lambda})\right)^* A^{5/2}\varphi$ is analytic in the left hand plane Π_-, it follows that $\varphi = 0$. Assume $\varphi \neq 0$ and $v(\lambda)$ is an analytic function in the left half plane Π_-. Then for $t > 0$ from the representation of the solution $u(t)$ we obtain:

$$(u(t),\varphi)_{5/2} = \frac{1}{2\pi i}\int_{\Gamma_\theta}\left(\sum_{\nu=0}^{2} Qu^{(\nu)}(0), v(\overline{\lambda})\right)e^{\lambda t}d\lambda$$

and the function $\eta(\lambda) = \left(\sum\limits_{\nu=0}^{2} Q_\nu^{(\lambda)} u^{(\nu)}(0), v(\overline{\lambda}) \right)$ will be an analytic function in the Π_-, $\eta(\lambda)$ grows at infinity no rapidly than $|\lambda|^3$, since $\left\| \sum\limits_{\nu=0}^{2} Q_\nu u^{(\nu)}(0) \right\| \leq$ $\leq const\, |\lambda|^2$, and

$\|v(\lambda)\| \leq \left\| A^{5/2} P^{-1}(\lambda) \right\| \left\| A^{5/2}\varphi \right\| \leq const \left\| A^{5/2} P_0^{-1}(\lambda) \right\| \left\| A^{5/2}\varphi \right\| \leq const\, |\lambda|^{-1/2}$.

Thus, applying the Phragmen-Lindelof theorem for $\rho \in (0,1]$ we obtain that $\eta(\lambda)$ is a polynomial of degree not more than three. For $\rho \in (1,\infty)$ the function $\eta(\lambda)$, by the Phragmen-Lindelof theorem, is an analytic function in the sector $\left\{ \lambda : |\arg\lambda - \pi| < \frac{\pi}{2\rho} \right\}$. It follows from Lemma 3.1 that subject to the conditions of the theorem in the sector where the sides are the rays $\Gamma_{\pi/2\rho}$ and $\Gamma_{3\pi/2\rho}$, by the Phragmen-Lindelof theorem, $\eta(\lambda)$ is an analytic function. Continuing this process, we obtain that subject to the condition of the theorem, $\eta(\lambda)$ is an entire function. Since $\eta(\lambda)$ is an entire function and grows no rapidly than $|\lambda|^2$, we have

$$\eta(\lambda) = \sum_{k=0}^{2} h_k \lambda^k, h_k \in \mathbb{C}; k = 0,1,2.$$

For $t > 0$

$$(u(t),\varphi)_{5/2} = \frac{1}{2\pi i} \int_{\Gamma_\theta} \left(\sum_{k=0}^{2} h_k \lambda^k \right) e^{\lambda t} d\lambda = 0,$$

since

$$\int_{\Gamma_\theta} \lambda^k e^{\lambda t} d\lambda = 0, k = 0,1,2.$$

Thus for $t > 0$ we have $(u(t),\varphi)_{5/2} = 0$.

Passing to the limit as $t \to 0$ we obtain

$$\lim_{t\to+0} (u(t),\varphi)_5 = (\varphi,\varphi)_{5/2} = \|\varphi\|_{5/2}^2 = 0$$

i.e. $\varphi = 0$. We get a contradiction, which proves the theorem. □

From the completeness of the system $K(\Pi_-)$ in $H_{5/2}$ and regular solvability of problem (3.1), (3.2) we obtain a theorem on the completeness of the system of decreasing elementary solutions in the space of regular solutions of the homogeneous equation.

Theorem 3.3. *Let all the conditions of theorem 3.2 be fulfilled . Then the system of decreasing elementary solutions of the homogeneous equations is complete in the space of all regular solutions of the homogeneous equation.*

Proof. Let $\varepsilon_1 > 0$ be any number. Then there exist a number $N > 0$, numbers $c_{i,j,h}(\varepsilon)$ and a system of eigen and associated vectors λ_i responding to the eigen values $Re\lambda_i < 0$, (the index shows that the number λ_i can respond to several systems of eigen and associated vectors) such that

$$\left\| \varphi - \sum_{i=1}^{N} \sum_{(j,h)} c_{i,j,h}(\varepsilon) x_{i,j,h} \right\| < \varepsilon_1.$$

Since $x_{i,j,h}(t) = u_{i,j,h}(0)$ and $\varphi = u(0)$, the regular solvability of the problem (3.1), (3.2) yields

$$\left\| u(t) - \sum_{i=1}^{N} \sum_{(j,h)} c_{i,j,h} u_{i,j,h}(t) \right\|_{W_2^3(R_+;H)} \leq$$

$$\leq const \left\| \varphi - \sum_{i=1}^{N} \sum_{(j,h)} c_{i,j,h}(\varepsilon) x_{i,j,h} \right\| < \varepsilon_1, const = \varepsilon$$

where ε is any positive number. The theorem has been proved. □

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