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CONTROL OF LOADED POINTS OF A PARABOLIC EQUATION

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Abstract. We consider a problem of optimal control of loading points and the corresponding reaction functions described by a loaded parabolic equation. Optimality conditions for control actions are obtained. The objective functional gradient formulas contained in these conditions are used in the algorithm for numerically solving the problem of optimization of loading points and reaction functions based on first-order optimization methods. The results of numerical experiments are provided.

1. Introduction

It is known that the states of many objects and processes are described by loaded differential equations with ordinary or partial derivatives [1, 2, 15, 16, 20, 21]. The state of such an object at the loading points affects the state of the entire object as a whole, so the choice of optimal locations for the loading points and the corresponding response functions is important for the functioning of the object.

Similar optimization and optimal control problems arise when designing the placement of wells in water, oil and gas fields, enterprises, taking into account the ecology of the region and others. The same mathematical statements also arise at the stage of exploitation of the above objects for parametric identification of the corresponding mathematical models in order to control these objects [7, 13, 17, 18, 22, 23]. Loaded initial-boundary value problems also arise in the feedback control of objects with distributed parameters, in which the loading points are the places where the measurement points of the current state of the object are set [3, 4, 6, 8, 9, 14].

In this regard, in recent years, interest has increased in optimal control problems for objects with distributed parameters described by various types of loaded partial differential equations and types of initial boundary conditions [16, 21]. Loaded differential equations of various types have been studied by many authors both from the point of view of the existence and uniqueness issues of their solutions [11, 12, 21], the development of numerical methods for solving [5, 10, 24], and the optimal control of processes described by the corresponding initial boundary value problems [2, 3, 4, 8, 9, 15]. In all the above studies, the positions of

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loading points were specified [3, 4, 5, 8, 9, 10, 11, 12, 15, 16, 21, 25], or optimized [1], but the case of changing their location in time, and even more so, the control of the change was not considered by anyone. In this paper, regarding a loaded initial-boundary value problem of a parabolic type, the case is considered when the coordinates of loading points change in time under the influence of control actions and are described by systems of ordinary differential equations.

The necessary optimality conditions of control actions on the movement of loading points and the corresponding reaction functions are obtained. Using the formulas for the gradient of the objective functional contained in these conditions, numerical optimization methods of the first order are applied for the numerical solution of the problem. Considering that the loading points move, a scheme of a finite-difference integro-interpolation approximation of a loaded initial-boundary value problem is proposed.

The paper presents the results of a numerical solution of a model problem for optimizing the parameters of a loaded system.

2. Problem statement

The optimal control problem of loading points and the corresponding reaction functions is considered, which is described by the following two-dimensional parabolic equation:

$$\frac{\partial u(x,t)}{\partial t} = Lu(x,t) + \sum_{s=1}^{n} q_s(t) N(\xi_s(t))\, u(x,t) +$$

$$+F(x,t), \quad \Omega = \{(x,t):\ 0 < x < l,\ \ t_0 < t \le t_f\}, \tag{2.1}$$

with initial and boundary conditions:

$$u(x,t_0) = \varphi_0(x), \quad x \in [0,l], \tag{2.2}$$

$$u(0,t) = \chi_1(t), \quad \frac{\partial u(l,t)}{\partial x} = \chi_2(t), \quad t \in [t_0,t_f]. \tag{2.3}$$

Here $u(x,t)$ – function determining the phase state of the process under study; t_0, t_f – start and end time of the process; $\varphi_0 \in L_2(0,l)$, χ_1, $\chi_2 \in L_2(t_0,t_f)$ functions are given. Optimized functions $q_s(t) \in L_2(t_0,t_f)$, $s = 1,2,...,n$ determining the reactions at loading points must satisfy the conditions:

$$\underline{q}_s \le q_s(t) \le \overline{q}_s, \quad t \in [t_0,t_f], \quad s = 1,2,\dots,n, \tag{2.4}$$

n– the number of loading points is given.

Let us assume that the initial and the left-end boundary conditions satisfy the matching condition:

$$\chi_1(t_0) = \varphi_0(0)\ . \tag{2.5}$$

The operator L and the loading operator $N(\cdot)$ are defined as follows:

$$Lu(x,t) = a_0^2 \frac{\partial^2 u(x,t)}{\partial x^2} + a_1 \frac{\partial u(x,t)}{\partial x} - a_2 u(x,t)\,,\ (x,t) \in \Omega, \tag{2.6}$$

$$N(\xi)\, u(x,t) = u(\xi,t)\,. \tag{2.7}$$

Here, a_0, a_1, $a_2 > 0$ are given numbers.

The coordinates of the loading points $\xi(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_n(t))$ are controllable and are described by the differential equations:

$$\frac{d\xi_s(t)}{dt} = f_s(\xi_s(t), \vartheta_s(t), t), \quad t \in (t_0, t_f], \quad s = 1, 2, \ldots, n, \tag{2.8}$$

with the initial conditions:

$$\xi_s(t_0) = \xi_s^0 \in (0, l), \quad s = 1, 2, \ldots, n. \tag{2.9}$$

Here the given functions $f_s(\cdot, \cdot, t)$, $s = 1, 2, \ldots, n$ are continuously differentiable with respect to the first two arguments and piecewise continuous with respect to t and determine the law of motion of the loaded points, functions $\vartheta_s(t) \in L_2(t_0, t_f)$, $s = 1, 2, ..., n$ are the control influences on the movement of the loading points $\xi_s(t)$, $s = 1, 2, \ldots, n$. The loading points must satisfy the natural conditions

$$0 < a_s \leq \xi_s(t) \leq b_s < l, \quad t \in (t_0, t_f], \quad s = 1, 2, \ldots, n,$$

which, as it is not difficult to show, can be brought to the form:

$$g_s(t) = \frac{b_s - a_s}{2} - \left|\xi_s(t) - \frac{b_s + a_s}{2}\right| \geq 0, \quad t \in (t_0, t_f], \quad s = 1, 2, \ldots, n.$$

Let there be positional constraints on controls:

$$\underline{\vartheta}_s \leq \vartheta_s(t) \leq \overline{\vartheta}_s, \quad t \in (t_0, t_f], \quad s = 1, 2, \ldots, n. \tag{2.10}$$

The problem under consideration consists in finding the optimal control actions $q = q(t) = (q_1(t), q_2(t), \ldots, q_n(t))$ and $\vartheta = \vartheta(t) = (\vartheta_1(t), \vartheta_2(t), \ldots, \vartheta_n(t))$, which, together with the corresponding phase state $u(x, t)$ deliver the minimum value to the functional:

$$J(q, \vartheta) = \int_0^l [u(x, t_f; q, \vartheta) - U(x)]^2 \, dx +$$

$$+\mu_1 \|q(t)\|^2_{L_2^n(t_0, t_f)} + \mu_2 \|\vartheta(t)\|^2_{L_2^n(t_0, t_f)}. \tag{2.11}$$

Here the function $U(x) \in L_2(0, l)$, determining the desired final state of the process; μ_1, μ_2– the parameters are given.

The main specificity of the problem (2.1)–(2.11) consists, firstly, in the loading(ness) of the differential equation (2.1) [1, 3], and secondly, in the fact that the coordinates of the loading points change in time under the influence of control actions and are the solution of a system of differential equations (2.8) with the initial conditions (2.9). Another specificity of the problem is, in general, the no convexity of the problem functional, as it is easy to see, by the second term of the right-hand side of the differential equation (2.1).

For each fixed control $(q(t),\ \vartheta(t))$, the corresponding solution $u(x, t) = u(x, t; q, \vartheta)$ is uniquely determined from the boundary value problem (2.1)–(2.3). Since the control can have an infinite number of discontinuities, a classical solution to problem (2.1)–(2.3) may not exist. Therefore, the solution to this boundary value problem will be understood in a generalized sense.

Definition 2.1. By the solution of the initial-boundary value problem (2.1)–(2.3), corresponding to the control $(q(t),\ \vartheta(t))$, we understand a generalized solution, a function $u = u(x,t) = u(x,t;q,\vartheta)$ from $V_2^{1,0}(\Omega)$, satisfying the integral identity

$$\int_{t_0}^{t_f}\int_0^l \left[-u(x,t)\frac{\partial\eta(x,t)}{\partial t} + a_0^2\frac{\partial u(x,t)}{\partial x}\frac{\partial\eta(x,t)}{\partial x} + a_1 u(x,t)\frac{\partial\eta(x,t)}{\partial x} - \right.$$

$$\left. + a_2 u(x,t)\eta(x,t) - \sum_{s=1}^{n} q_s(t)u(\xi_s(t),t)\eta(x,t) - F(x,t)\eta(x,t)\right] dxdt +$$

$$+\int_{t_0}^{t_f}\left[a_0^2 u(0,t)\eta(0,t) - a_1 u(l,t)\eta(l,t)\right]dt =$$

$$= \int_{t_0}^{t_f}\varphi(x)\eta(x,0)dx + \int_{t_0}^{t_f}\left[a_0^2\chi_2(t)\eta(l,t) - a_1\chi_1(t)\eta(0,t)\right]dt$$

at $\forall\eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ and $\eta(x,t_f) = 0$.

Using the methods of [19, 26], it is possible to show that for each given control of the initial-boundary value problem (2.1)–(2.3) there is a unique generalized solution from $V_2^{1,0}(\Omega)$. The goal of this work is to numerically solve the optimal control problem (2.1)–(2.11). In this paper, we will assume that the existence, uniqueness, and stability conditions for the solution of the initial-boundary value problem (2.1)–(2.3) and the Cauchy problem (2.8), (2.9) are satisfied for all admissible reaction functions $q_s(t)$, the trajectories $\xi_s(t)$ and controls $\vartheta_s(t)$, $s = 1, 2, \ldots, n$.

As is known, the complexity of optimal control problems both for theoretical studies and for their numerical solution is due to the presence of phase constraints. To study and solve the problem under consideration, we use the method of penalty functions. For minimization, instead of the functional (2.11), we use the external penalty functional:

$$\hat{J}(q,\vartheta;r) = J(q,\vartheta) + rG(t;\xi(t)),\ \ G(t;\xi(t)) = \sum_{s=1}^{n}\int_{t_0}^{t_f}\left[\min(0, g_s(t))\right]^2 dt, \quad (2.12)$$

where $r > 0$ is a positive penalty coefficient tending to $+\infty$.

3. Necessary optimality conditions in the problem (2.1)–(2.12)

First of all, let us prove the differentiability of the objective functional (2.11) for each given value of the penalty coefficient. To do this, we use the method of estimating the increment of the functional due to the increment of the optimized parameters. We will use the following notation for functions:

$$f(x^-) = f(x-0),\ \ f(x^+) = f(x+0).$$

Regarding problem (2.1)–(2.3), (2.6), (2.7), (2.11) consider the following auxiliary initial-boundary value problem:

$$\frac{\partial \psi\,(x,t)}{\partial t} = -a_0^2 \frac{\partial^2 \psi(x,t)}{\partial x^2} + a_1 \frac{\partial \psi(x,t)}{\partial x} + a_2 \psi(x,t), \quad (x,t) \in \Omega,, \tag{3.1}$$

$$\psi\,(x,t_f) = -2[u(x,t_f) - U(x)],\; x \in [0,l], \tag{3.2}$$

$$\psi\,(0,t) = 0, \quad a_0^2 \frac{\partial \psi\,(l,t)}{\partial x} = a_1 \psi(l,t), \quad t \in [t_0,t_f], \tag{3.3}$$

at points $\xi_s,\ s = 1,\ 2\,,\ ...,\ n$, at $t \in [t_0, t_f)$ must meet the conditions

$$\psi_x(\xi_s^-,t) = \psi_x(\xi_s^+,t) - \frac{q_s(t)}{a_0^2} \int_0^l \psi(x,t)dx\,, \quad s = 1,\ 2,\ ...,n \tag{3.4}$$

which we will call the adjoint problem. According to its solution, a function $\psi(x,t)$ from the class $V_2^{1,0}(\Omega)$ is understood that satisfies the following equality

$$\int_{t_0}^{t_f}\int_0^l \left[\psi(x,t)\frac{\partial \eta_1(x,t)}{\partial t} - a_0^2 \frac{\partial \psi(x,t)}{\partial x}\frac{\partial \eta_1(x,t)}{\partial x} + a_1 \psi(x,t)\frac{\partial \eta_1(x,t)}{\partial x} + \right.$$

$$\left. -a_2\psi(x,t)\eta_1(x,t) + \sum_{s=1}^{n} q_s(t)\int_0^l \psi(\gamma,t)d\gamma\,\delta(x-\xi_s(t))\eta_1(x,t)\right] dxdt =$$

$$= \int_0^l 2\,(u(x,t_f) - U(x))\,\eta_1(x,t_f)dx + a_0^2 \int_{t_0}^{t_f} \frac{\partial \psi(0,t)}{\partial x}\eta_1(0,t)\,dt,$$

for an arbitrary function $\eta_1(x,t) \in W_2^{1,1}(\Omega)$ such that $\eta_1(x,0) = 0,\ x \in (0,l)$.

Theorem 3.1. *Under the conditions imposed on the functions involved in problem (2.1)–(2.3), the penalty functional of problem (2.12) for a given penalty coefficient r, is Frechet differentiable in the space $H = L_2(t_0,t_f) \times L_2(t_0,t_f)$ and the components of its gradient are determined by the formulas:*

$$\frac{\partial \hat{J}\,(q,\vartheta;r)}{\partial q_s} = -u(\xi_s(t),t)\int_0^l \psi(x,t)dx + 2\mu_1 q_s(t),\; t \in (t_0,t_f], \tag{3.5}$$

$$\frac{\partial \hat{J}\,(q,\vartheta;r)}{\partial \vartheta_s} = -\varphi_s(t)\frac{\partial f_s\,(\xi_s(t),\vartheta_s(t),t)}{\partial \vartheta_s} + 2\mu_2\vartheta_s(t),\; t \in (t_0,t_f]. \tag{3.6}$$

Here $s = 1,2,\ldots,n$, function $\psi = \psi(x,t)$ solution to the adjoint initial-boundary value problems (3.1)–(3.4), functions $\varphi_s\,(t)\,, s = 1,2,\ldots,n$ are almost everywhere continuously differentiable and are solutions to the following adjoint Cauchy problems

$$\frac{d\varphi_s\,(t)}{dt} = -\varphi_s\,(t)\,\frac{\partial f_s\,(\xi_s\,(t)\,,\vartheta_s\,(t)\,,t)}{\partial \xi_s} - q_s\,(t)\,\left.\frac{\partial u\,(x,t)}{\partial x}\right|_{x=\xi_s(t)} \int_0^l \psi(\gamma,t)d\gamma +$$

$$+r\min\left(0,-2sgn\left(\xi_s\left(t\right)-\frac{b_s+a_s}{2}\right)\left(\frac{b_s-a_s}{2}-\left|\xi_s\left(t\right)-\frac{b_s+a_s}{2}\right|\right)\right),$$

$$t\in(t_0,t_f],\quad s=1,2,\ldots,n, \tag{3.7}$$

$$\varphi_s(t_f)=0,\quad s=1,2,\ldots,n. \tag{3.8}$$

Proof. Let, for given $r>0$ the function $u(x,t)=u(x,t;q,\xi)$ and the vector-function $\xi(t;\vartheta)=(\xi_1(t;\vartheta_1),\xi_2(t;\vartheta_2),\ldots,\xi_n(t;\vartheta_n))$ be the solutions of the initial-boundary value problem (2.1)–(2.3) and the Cauchy problem (2.8), (2.9), respectively, for the given admissible values $q(t)$, $\vartheta(t)$, and the function $\tilde{u}(x,t)=\tilde{u}(x,t;\tilde{q},\tilde{\xi})$, $\tilde{\xi}(t;\tilde{\vartheta})=(\tilde{\xi}_1(t;\tilde{\vartheta}_1),\ldots,\tilde{\xi}_n(t;\tilde{\vartheta}_n))$ – be the solution of problems (2.1)–(2.3) and (2.8), (2.9) with incremented admissible values of the functions $\tilde{q}(t)$, $\tilde{\vartheta}(t)$. We will use the designations:

$$\tilde{q}(t)=q(t)+\Delta q(t),\quad \tilde{\vartheta}(t)=\vartheta(t)+\Delta\vartheta(t),$$

$$\tilde{u}(x,t)=u(x,t)+\Delta u(x,t),\quad \tilde{\xi}(t)=\xi(t)+\Delta\xi(t).$$

It is easy to show that $\Delta u(x,t)$ and $\Delta\xi(t)$ with the accuracy of the first order terms of smallness are solutions of the following, respectively, initial-boundary value problem

$$\frac{\partial\Delta u\left(x,t\right)}{\partial t}=L\Delta u\left(x,t\right)+\sum_{s=1}^{n}q_s(t)N\left(\xi_s\left(t\right)\right)\left(\Delta u\left(x,t\right)+\frac{\partial u\left(x,t\right)}{\partial x}\Delta\xi_s\left(t\right)\right)+$$

$$+\sum_{s=1}^{n}\Delta q_s(t)N\left(\xi_s\left(t\right)\right)u\left(x,t\right),\quad (x,t)\in\Omega, \tag{3.9}$$

$$\Delta u\left(x,t_0\right)=0,\quad x\in[0,l], \tag{3.10}$$

$$\Delta u\left(0,t\right)=0,\quad \frac{\partial\Delta u\left(l,t\right)}{\partial x}=0,\quad t\in[t_0,t_f], \tag{3.11}$$

and Cauchy problem:

$$\frac{d\Delta\xi_s\left(t\right)}{dt}=\frac{\partial f_s\left(\xi_s\left(t\right),\vartheta_s\left(t\right),t\right)}{\partial\xi_s}\Delta\xi_i\left(t\right)+\frac{\partial f_s\left(\xi_s\left(t\right),\vartheta_s\left(t\right),t\right)}{\partial\vartheta_s}\Delta\vartheta_s\left(t\right),$$

$$t\in(t_0,t_f],\quad s=1,2,\ldots,n, \tag{3.12}$$

$$\Delta\xi_s\left(t_0\right)=0,\quad s=1,2,\ldots,n. \tag{3.13}$$

In (3.9), the Lagrange formula is used with the accuracy of first order terms of smallness

$$\Delta u(\xi_s(t),t)=\tilde{u}(\xi_s(t)+\Delta\xi_s(t),t)-$$

$$-u(\xi_s(t),t)-\left.\frac{\partial u\left(x,t\right)}{\partial x}\right|_{x=\xi_s(t)}\Delta\xi_s\left(t\right)+O\left(\left(\Delta\xi_s(t)\right)^2\right).$$

Considering the stability properties of boundary value problems described by parabolic equations and Cauchy problems of relative to the right-hand side differential equations, the terms of the second order of smallness in (3.9) and (3.12) can be neglected [19]:

$$\left\|\Delta u(x,t)\right\|^2_{V_2^{1,0}(\Omega)}\le M_1\left\|\Delta q(t)\right\|^2_{L_2^n(t_0,t_f)}+M_2\left\|\Delta\xi(t)\right\|^2_{L_2^n(t_0,t_f)}, \tag{3.14}$$

$$\|\Delta\xi(t)\|^2_{L_2^n(t_0,t_f)} \le M_3 \|\Delta\vartheta(t)\|^2_{L_2^n(t_0,t_f)}. \tag{3.15}$$

Let us estimate the increment of the objective functional obtained by incrementing the optimized functions:

$$\Delta\hat{J}(q,\vartheta;r) = \Delta J(q,\vartheta;r) + r\Delta G(t;\xi(t)) =$$

$$= J(q+\Delta q, \vartheta+\Delta\vartheta) - J(q,\vartheta) + r\left(G(t;\xi(t)+\Delta\xi(t)) - G(t;\xi(t))\right). \tag{3.16}$$

$$\Delta J(q,\vartheta) = 2\int_0^l [u(x,t_f) - U(x)]\,\Delta u(x,t_f)dx +$$

$$+2\mu_1\int_{t_0}^{t_f} q(t)\Delta q(t)dt + 2\mu_2\int_{t_0}^{t_f} \vartheta(t)\Delta\vartheta(t)dt +$$

$$+O\left(\|\Delta u(x,t_f)\|^2_{W_2^1(0,l)}\right) + O\left(\|\Delta q(t)\|^2_{L_2^n(t_0,t_f)}\right) + O\left(\|\Delta\vartheta(t)\|^2_{L_2^n(t_0,t_f)}\right).$$

$$\Delta G(t;\xi(t)) = G(t;\xi(t)+\Delta\xi(t)) - G(t;\xi(t)) =$$

$$= \sum_{s=1}^n \int_{t_0}^{t_f}\left[\min\left(0, \frac{b_s-a_s}{2} - \left|\xi_s(t) + \Delta\xi_s(t) - \frac{b_s+a_s}{2}\right|\right)\right]^2 dt -$$

$$-\sum_{s=1}^n \int_{t_0}^{t_f}\left[\min\left(0, \frac{b_s-a_s}{2} - \left|\xi_s(t) - \frac{b_s+a_s}{2}\right|\right)\right]^2 dt =$$

$$= \sum_{s=1}^n \int_{t_0}^{t_f} \min\left[0, -2sgn\left(\xi_s(t) - \frac{b_s+a_s}{2}\right)\times\right.$$

$$\left.\times\left(\frac{b_s-a_s}{2} - \left|\xi_s(t) - \frac{b_s+a_s}{2}\right|\right)\right]\Delta\xi_s(t)\,dt + O\left(\|\Delta\xi(t)\|^2_{L_2^n(t_0,t_f)}\right).$$

Moving all the terms of equation (3.9) the left, we multiply both sides of the resulting equality by a function $\psi(x,t)$.

Similarly, shifting all the terms of equations (3.12) to the left-hand side, we multiply both parts of the obtained equalities by the yet arbitrary functions $\varphi_i(t)$, $i = 1, 2, \dots, n$ from the class of functions continuously differentiable with respect to $t \in [t_0, t_f]$. Integrating the left-hand sides of the obtained equalities, which are equal to zero, over $x \in [0, l]$ and $t \in [t_0, t_f]$ and adding the obtained relations with (3.16), we get:

$$\Delta\hat{J}(q,\vartheta;r) = 2\int_0^l [u(x,t_f) - U(x)]\,\Delta u(x,t_f)dx +$$

$$+2\mu_1\int_{t_0}^{t_f} q(t)\Delta q(t)dt + 2\mu_2\int_{t_0}^{t_f} \vartheta(t)\Delta\vartheta(t)dt +$$

$$+r\sum_{s=1}^{n}\int_{t_0}^{t_f}\min\left[0,-2sgn\left(\xi_s\left(t\right)-\frac{b_s+a_s}{2}\right)\times\right.$$

$$\left.\times\left(\frac{b_s-a_s}{2}-\left|\xi_s\left(t\right)-\frac{b_s+a_s}{2}\right|\right)\right]\Delta\xi_s\left(t\right)dt+$$

$$+\int_{t_0}^{t_f}\int_{0}^{l}\psi(x,t)\left[\frac{\partial\Delta u\left(x,t\right)}{\partial t}-L\Delta u\left(x,t\right)-\right.$$

$$\left.-\sum_{s=1}^{n}q_s(t)N\left(\xi_s\left(t\right)\right)\left(\Delta u\left(x,t\right)+\frac{\partial u\left(x,t\right)}{\partial x}\Delta\xi_s\left(t\right)\right)-\sum_{s=1}^{n}\Delta q_s(t)u\left(\xi_s,t\right)\right]dxdt+$$

$$+\sum_{i=1}^{n}\int_{t_0}^{t_f}\varphi_i(t)\left[\frac{d\Delta\xi_i\left(t\right)}{dt}-\frac{\partial f_i\left(\xi_i\left(t\right),\vartheta_i\left(t\right),t\right)}{\partial\xi_i}\Delta\xi_i\left(t\right)-\right.$$

$$\left.-\frac{\partial f_i\left(\xi_i\left(t\right),\vartheta_i\left(t\right),t\right)}{\partial\vartheta_i}\Delta\vartheta_i\left(t\right)\right]dt+O\left(\left\|\Delta u(x,t_f)\right\|^2_{W_2^1(0,l)}\right)+$$

$$+O\left(\left\|\Delta q(t)\right\|^2_{L_2^n\left(t_0,t_f\right)}\right)+O\left(\left\|\Delta\vartheta(t)\right\|^2_{L_2^n\left(t_0,t_f\right)}\right)+O\left(\left\|\Delta\xi(t)\right\|^2_{L_2^n\left(t_0,t_f\right)}\right).$$

After carrying out simple calculations (integration by parts, grouping), we obtain:

$$\Delta\hat{J}\left(q,\vartheta;r\right)=2\mu_1\sum_{s=1}^{n}\int_{t_0}^{t_f}q_s(t)\Delta q_s(t)dt+2\mu_2\sum_{s=1}^{n}\int_{t_0}^{t_f}\vartheta_s(t)\Delta\vartheta_s(t)dt+$$

$$+\int_{0}^{l}\left(\psi(x,t_f)+2\left[u(x,t_f)-U(x)\right]\right)\Delta u(x,t_f)dx+$$

$$+r\sum_{s=1}^{n}\int_{t_0}^{t_f}\min\left[0,-2sgn\left(\xi_s\left(t\right)-\frac{b_s+a_s}{2}\right)\times\right.$$

$$\left.\times\left(\frac{b_s-a_s}{2}-\left|\xi_s\left(t\right)-\frac{b_s+a_s}{2}\right|\right)\right]\Delta\xi_s\left(t\right)dt+$$

$$+\int_{t_0}^{t_f}\left(a_0^2\psi_x(l,t)-a_1\psi(l,t)\right)\Delta u(l,t)dt+a_0^2\int_{t_0}^{t_f}\psi(0,t)\Delta u_x(0,t)dt+$$

$$+\int_{t_0}^{t_f}\int_{0}^{l}\left(-\frac{\partial\psi(x,t)}{\partial t}-a_0^2\frac{\partial^2\psi(x,t)}{\partial x^2}+a_1\frac{\partial\psi(x,t)}{\partial x}+a_2\psi(x,t)\right)\Delta u(x,t)dxdt+$$

$$-a_0^2\sum_{s=1}^{n}\int_{t_0}^{t_f}\left[\psi_x(\xi_s^-,t)-\psi_x(\xi_s^+,t)+\frac{q_s(t)}{a_0^2}\int_{0}^{l}\psi(x,t)dx\right]\Delta u(\xi_i,t)\,dt-$$

$$-a_0^2\sum_{s=1}^{n}\int_{t_0}^{t_f}\left[\left(\psi(\xi_i^+,t)-\psi(\xi_i^-,t)\right)\right]\Delta u_x(\xi_i,t)dt-$$

$$-\sum_{s=1}^{n}\int_0^l\int_{t_0}^{t_f}\psi(x,t)\Delta q_s(t)u(\xi_s(t),t)dtdx-$$

$$-\sum_{s=1}^{n}\int_0^l\int_{t_0}^{t_f}\psi(x,t)q_s(t)\frac{\partial u(\xi_s(t),t)}{\partial x}\Delta\xi_s(t)dtdx+\sum_{s=1}^{n}\varphi_s(t_f)\xi_s(t_f)-$$

$$+\sum_{s=1}^{n}\int_{t_0}^{t_f}\left[-\frac{d\varphi_s(t)}{dt}\Delta\xi_s(t)-\varphi_s(t)\frac{\partial f_s\left(\xi_s(t),\vartheta_s(t),t\right)}{\partial\xi_s}\Delta\xi_s(t)-\right.$$

$$\left.-\varphi_s(t)\frac{\partial f_s\left(\xi_s(t),\vartheta_s(t),t\right)}{\partial\vartheta_s}\Delta\vartheta_s(t)\right]dt+O\left(\|\Delta u(x,t_f)\|^2_{W_2^1(0,l)}\right)+$$

$$+O\left(\|\Delta q(t)\|^2_{L_2^n(t_0,t_f)}\right)+O\left(\|\Delta\vartheta(t)\|^2_{L_2^n(t_0,t_f)}\right)+O\left(\|\Delta\xi(t)\|^2_{L_2^n(t_0,t_f)}\right),\quad(3.17)$$

where $O\left(\|\Delta\xi(t)\|^2_{L_2^n(t_0,t_f)}\right)$, $O\left(\|\Delta q(t)\|^2_{L_2^n(t_0,t_f)}\right)$, $O\left(\|\Delta\vartheta(t)\|^2_{L_2^n(t_0,t_f)}\right)$ and $O\left(\|\Delta u(x,t_f)\|^2_{W_2^1(0,l)}\right)$ are small of the second order of the relative to $\Delta\xi(t)$, $\Delta q(t)$, $\Delta\vartheta(t)$, $t\in(t_0,t_f]$ and $\Delta u(x,t_f)$, $x\in[0,l]$.

Since the functions $\psi(x,t)$, $\varphi_s(t)$, $s=1,2,\dots,n$ are arbitrary, they are required to be solutions of the initial-boundary value problems (3.1)–(3.4) and the Cauchy problems (3.7)–(3.8).

Considering the assessments (3.14), (3.15), it follows from (3.17) that the functional is differentiable, and from the conditions (3.10), (3.11), (3.14), for the components of the gradient vector of the objective functional, which are the linear parts of the functional increment with respect to the corresponding optimized parameters, we obtain formulas (3.5), (3.6).

Now it is not difficult to formulate the necessary optimality conditions for the functions $q(t)$ and $\vartheta(t)$ relative to the penalty functional in the following variational form.

Theorem 3.2. *Let $q^*(t),\vartheta^*(t)$ satisfy the conditions of the problem (2.1)-(2.3), (2.8) and (2.9) for the given values r deliver a minimum to the functional (2.12). Then, for arbitrary admissible parameters $q(t),\vartheta(t)$, satisfying the conditions (2.4) and (2.10), the following inequality is true:*

$$\sum_{s=1}^{n}\int_{t_0}^{t_f}\left(\frac{\partial\hat{J}\left(q^*\left(t\right),\vartheta^*\left(t\right);r\right)}{\partial q_s},\left(q(t)-q_s^*(t)\right)\right)dt+$$

$$+\sum_{s=1}^{n}\int_{t_0}^{t_f}\left(\frac{\partial\hat{J}\left(q^*\left(t\right),\vartheta^*\left(t\right);r\right)}{\partial\vartheta_s},\left(\vartheta(t)-\vartheta_s^*(t)\right)\right)dt=$$

$$= \sum_{s=1}^{n} \int_{t_0}^{t_f} \left(\left(-u(\xi_s(t), t; \vartheta_s^*(t)) \int_0^l \psi(x, t; q_s^*(t)) dx + 2\mu_1 q_s^*(t) \right) , \right.$$

$$\left. (q_s(t) - q_s^*(t)) \right) dt$$

$$+ \sum_{s=1}^{n} \int_{t_0}^{t_f} \left(\left(-\varphi_s(t; \vartheta_s^*(t)) \left. \frac{\partial f_s(\xi_s(t), \vartheta_s(t), t)}{\partial \vartheta_s} \right|_{\vartheta_s(t) = \vartheta_s^*(t)} + 2\mu_2 \vartheta_s^*(t) \right) , \right.$$

$$\left. (\vartheta_s(t) - \vartheta_s^*(t)) \right) dt \geq 0 \, .$$

4. Numerical scheme for solving the problem (2.1)–(2.12)

For the numerical solution of the considered control problem of loading points, it is proposed to use the method of projection of the penalty functional gradient on the constraints (2.4), (2.10) imposed on the optimized functions $q(t)$, $\vartheta(t)$. The iterative procedure is constructed as follow [28]:

$$\begin{pmatrix} q(t) \\ \vartheta(t) \end{pmatrix}^{k+1} = P_{(2.4),(2.10)} \left[\begin{pmatrix} q(t) \\ \vartheta(t) \end{pmatrix}^{k} - \alpha_k \text{grad} \hat{J} \left(q^k(t), \vartheta^k(t), r \right) \right] . \quad (4.1)$$

Here k = 0,1,…, $P_{(2.4),(2.10)} [\cdot]$ is the projection operator on the admissible range of values of the functions being optimized. Taking into account that this area is a segment, the projection operator on it has a constructive character [28]. The step $\alpha_k \geq 0$ in the antigradient direction can be determined, for example, by any method of one-dimensional minimization from the condition:

$$\alpha_k = \arg\min_{\alpha \geq 0} \hat{J} \left(P_{(2.4),(2.10)} \left[\begin{pmatrix} q(t) \\ \vartheta(t) \end{pmatrix}^{k} - \alpha \text{grad} \hat{J} \left(q^k(t), \vartheta^k(t), r \right) \right] \right) .$$

Arbitrary functions can be assigned as the initial approximation $(q(t), \vartheta(t))^0$, in particular, with values satisfying the constraints (2.4), (2.10). The penalty functional gradient is determined by the formulas given in Theorem 3.1.

For the numerical solution of the direct initial-boundary value problem with respect to the loaded differential equation (2.1)–(2.3) and the adjoint problem (3.1)–(3.4), the integro-interpolation scheme of the grid method (finite-difference approximation) will be used [24, 27].

Let us introduce on segments $[0, l]$, $[t_0, t_f]$ nodal points $x_i = ih_x$, $i = 0, 1, \ldots, N_x$, $h_x = l/N_x$, $t_j = jh_t$, $j = 0, 1, \ldots, N_t$, $h_t = (t_f - t_0)/N_t$, $h_x \leq l - b_s$, $s = 1, 2, \ldots, n$ and designations $\omega = \{x_0, x_1, \ldots, x_{N_x}\}$, $u_i^j = u(x_i, t_j)$, $u_i = u(x_i, t)$, $F_i^j = F(x_i, t_j)$, $i = 1, 2, \ldots, N_x - 1$, $j = 1, 2, \ldots, N_t$, $\varphi_{0,i} = u(x_i, t_0)$, $i = 0, 1, \ldots, N_x$, $\chi_1^j = \chi_1(t_j)$, $\chi_2^j = \chi_2(t_j)$, $j = 1, 2, \ldots, N_t$.

To approximate the initial-boundary value problem at the nodal points of the grid area, we use the following schemes [24]:

$$\left. \frac{\partial u(x,t)}{\partial t} \right|_{(x_i, t_j)} = \frac{u_i^j - u_i^{j-1}}{h_t} + O(h_t),$$

$$\left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{(x_i,t_j)} = \frac{u^j_{i+1} - 2u^j_i + u^j_{i-1}}{h_x^2} + O(h_x^2). \tag{4.2}$$

To approximate the operator $N(\xi)$ we use the following scheme

$$\xi = \xi(t) \in (x_{\gamma_\xi}, x_{\gamma_\xi+1}]. \tag{4.3}$$

$$\hat{N}(\xi)\, u(x,t) = \frac{(\xi - x_{\gamma_\xi+1})(\xi - x_{\gamma_\xi+2})}{2h_x^2} u_{\gamma_\xi}(t) + \frac{(\xi - x_{\gamma_\xi})(x_{\gamma_\xi+2} - \xi)}{h_x^2} u_{\gamma_\xi+1}(t) +$$

$$+ \frac{(\xi - x_{\gamma_\xi})(\xi - x_{\gamma_\xi+1})}{2h_x^2} u_{\gamma_\xi+2}(t), \quad t \in (t_0, t_f], \quad \xi \in (x_{\gamma_\xi}, x_{\gamma_\xi+1}], \tag{4.4}$$

which, as can be easily shown, has the order of accuracy $O\left(h_x^2\right)$, those

$$\left|\hat{N}(\xi)\, u(x,t) - N(\xi)\, u(x,t)\right| \le M h_x^2, \quad t \in (t_0, t_f]. \tag{4.5}$$

To solve the initial-boundary value problem (2.1)–(2.3) for each given vectors $q(t)$, $\xi(t)$ we use a finite difference approximation based on the implicit scheme for approximating the (2.1) , and for the loading operator $N(\xi)\, u(x,t)$ the approximation (4.4) ([5, 10, 24, 25]):

$$u_i^0 = \varphi_{0,i}, \quad 0 \le i \le N_x, \quad u_0^j = \chi_1^j, \quad j > 0,$$

$$u^j_{N_x} = E u^j_{N_x-1} + \sum_{s=1}^{n} q_s^j \left(D^0_{\gamma_{\xi_s}} u^j_{\gamma_{\xi_s}} + D^1_{\gamma_{\xi_s}} u^j_{\gamma_{\xi_s}+1} + D^2_{\gamma_{\xi_s}} u^j_{\gamma_{\xi_s}+2} \right) +$$

$$+ C u^{j-1}_{N_x} + h_t C F^{j-1}_{N_x} + H \chi_2^j, \quad j \ge 1,$$

$$E = \left(\frac{2a_0^2 h_t}{h_x^2} \right) \left(1 + \frac{2a_0^2 h_t}{h_x^2} + a_2 h_t \right)^{-1},$$

$$H = \left(a_1^2 h_t - \frac{2a_0^2 h_t}{h_x} \right) \left(1 + \frac{2a_0^2 h_t}{h_x^2} + a_2 h_t \right)^{-1},$$

$$u_i^j = A u^j_{i-1} + B u^j_{i+1} + \sum_{s=1}^{n} q_s^j \left(D^0_{\gamma_{\xi_s}} u^j_{\gamma_{\xi_s}} + D^1_{\gamma_{\xi_s}} u^j_{\gamma_{\xi_s}+1} + D^2_{\gamma_{\xi_s}} u^j_{\gamma_{\xi_s}+2} \right) +$$

$$+ C u_i^{j-1} + h_t C F_i^{j-1},$$

$$A = \left(\frac{a_0^2 h_t}{h_x^2} - \frac{a_1 h_t}{2h_x} \right) \left(1 + \frac{2a_0^2 h_t}{h_x^2} + a_2 h_t \right)^{-1},$$

$$B = \left(\frac{a_0^2 h_t}{h_x^2} + \frac{a_1 h_t}{2h_x} \right) \left(1 + \frac{2a_0^2 h_t}{h_x^2} + a_2 h_t \right)^{-1},$$

$$C = \left(1 + \frac{2a_0^2 h_t}{h_x^2} + a_2 h_t \right)^{-1},$$

$$D^0_{\gamma_{\xi_s}} = \frac{(\xi_s - x_{\gamma_{\xi_s}+1})(\xi_s - x_{\gamma_{\xi_s}+2})}{2h_x^2} \left(1 + \frac{2a_0^2 h_t}{h_x^2} + a_2 h_t \right)^{-1},$$

$$D^1_{\gamma_{\xi_s}} = \frac{(\xi_s - x_{\gamma_{\xi_s}})(x_{\gamma_{\xi_s}+2} - \xi_s)}{h_x^2}\left(1 + \frac{2a_0^2 h_t}{h_x^2} + a_2 h_t\right)^{-1},$$

$$D^2_{\gamma_{\xi_s}} = \frac{(\xi_s - x_{\gamma_{\xi_s}})(\xi_s - x_{\gamma_{\xi_s}+1})}{2h_x^2}\left(1 + \frac{2a_0^2 h_t}{h_x^2} + a_2 h_t\right)^{-1}, \quad 0 < i < N_x, j \geq 1.$$

Considering [23], the error of such an approximation of the problem (2.1)–(2.3) is $O\left(h_t + h_x^2\right)$. To solve the Cauchy problem (2.8), (2.9) for a given control vector function $\vartheta(t)$ we use the fourth-order Runge-Kutta method.

We use similar finite-difference approximation schemes for the adjoint initial-boundary value problem (3.1)–(3.4) and the Cauchy problem (3.7), (3.8).

To solve the grid initial-boundary direct and adjoint problems for the given functions $q(t)$, $\xi(t)$ and taking into account the specifics of grid difference systems of equations on each time layer $t = t_j$, the special numerical methods proposed in [5, 24, 29], were used.

Let us present the results of numerical experiments obtained by solving the following optimal control problem.

A control problem is considered, which is described by the following initial-boundary value problem with respect to a loaded parabolic differential equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + q_1(t)u(\xi_1(t), t) + q_2(t)u(\xi_2(t), t) +$$
$$+0.1\sin^2(4\pi x)(t + 0.1), x \in (0,1), t \in (0,1],$$
$$u(x,0) = 0.2\sin^2\left(2\pi x^3\right), \quad x \in [0,1],$$
$$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t \in [0,1].$$

There were the following restrictions on the reaction control functions:

$$-2.3 \leq q_i(t) \leq 2.3, \quad t \in [0,1], \quad i = 1,2.$$

The controlled motions of two loading points are described by the following Cauchy problems:

$$\frac{d\xi_1(t)}{dt} = 2\xi_1(t) + \vartheta_1(t), \quad \frac{d\xi_2(t)}{dt} = -3\xi_2(t) + \vartheta_2(t), \quad t \in (0,1],$$
$$\xi_1(0) = 0.36, \quad \xi_2(0) = 0.62,$$

with restrictions:

$$0.05 \leq \xi_i(t) \leq 0.95, \quad t \in (0,1], \quad i = 1,2,$$
$$-1 \leq \vartheta_i(t) \leq 1, \quad t \in [0,1], \quad i = 1,2.$$

The objective functional is in the form:

$$\tilde{J}(q,\vartheta;r) = J(q,\vartheta) + rG(q,\vartheta).$$

$$J(q,\vartheta) = \int_0^1 \left[u(x,t_f;q,\vartheta) - U(x)\right]^2 dx + \mu_1 \|q(t)\|^2_{L_2^n(0,1)} + \mu_2 \|\vartheta(t)\|^2_{L_2^n(0,1)},$$

$$G(q,\vartheta) = \int_0^1 \left[\min(0, g_1(t))\right]^2 dt + \int_0^1 \left[\min(0, g_2(t))\right]^2 dt.$$

Here $\mu_1 = 0.1$, $\mu_2 = 0.1$, $U(x) = 5$, $x \in [0,1]$.

In the results of computer experiments presented below, the following parameters of the finite-difference approximation of the loaded initial-boundary value problem and the Cauchy problem were used:

$$n = 2, \quad l = 1, \quad t_0 = 0, \quad t_f = 1, \quad h_x = 0.01, \quad h_t = 0.005.$$

As an unconditional minimization of the penalty functional, the conjugate gradient method [28] was used, the value of the penalty coefficient was changed three times $R_k = 5R_{k-1}$, $k = 1, 2, 3$, $R_0 = 5$.

Experiment 1. The initial approximations for the control functions $q_1^0(t)$, $q_2^0(t)$,$\vartheta_1^0(t)$), $\vartheta_2^0(t)$) are taken as follows:

$$q_1^0(t) = 1.5\,\sin^2(2\pi\, t), \quad q_2^0(t) = 1.5\,\sin^2(3\pi\, t), \quad t \in [0,1]$$

$$\vartheta_1^0(t) = \text{ - } 0.6 + 0.4\cos(2\pi t), \quad \vartheta_2^0(t) = 0.5 \text{ - } 0.4\cos(4\pi t) \quad t \in [0,1]\,.$$

For these initial values of control functions we have:

$$J\left(q^0, \vartheta^0\right) = 24.91, \quad \int_0^1 [u(x,t_f) - U(x)]^2\, dx = 24.66, \quad G(q^0, \vartheta^0) = 0,$$

$$\left\|q^0(t)\right\|^2_{L_2^n[0,1]} = 1.69, \quad \left\|\vartheta^0(t)\right\|^2_{L_2^n[0,1]} = 0.82.$$

And after performing the iterative procedures, the following optimal values were obtained:

$$J\left(q^*, \vartheta^*\right) = 0.0094, \quad \int_0^1 [u(x,t_f) - U(x)]^2\, dx = 0, \quad G(q^*, \vartheta^*) = 0,$$

$$\|q^*(t)\|^2_{L_2^n[0,1]} = 9.1321, \quad \|\vartheta^*(t)\|^2_{L_2^n[0,1]} = 0.2712.$$

On the fig. 1. the graphs of the reaction functions for the initial approximation are given $q_1^0(t)$, $q_2^0(t)$ and for the obtained optimal functions $q_1^*(t)$, $q_2^*(t)$.

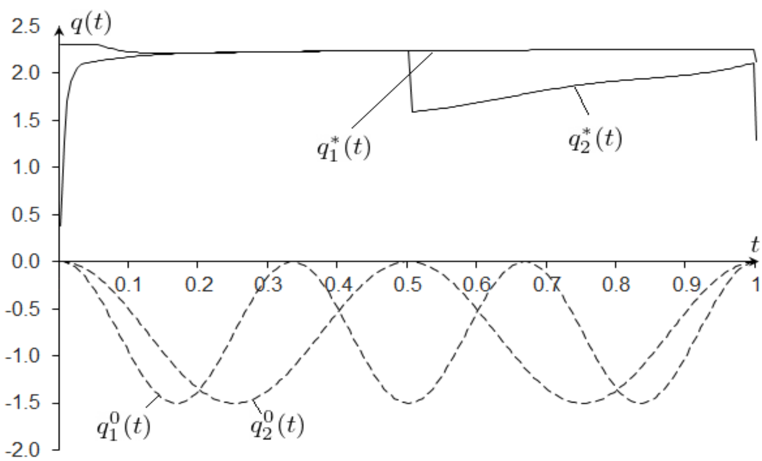

FIGURE 1. *Graphs of initial approximations for reaction functions $q_1^0(t), q_2^0(t)$ (dashed lines) and for the obtained optimal functions $q_1^*(t), q_2^*(t)$ (continuous lines)*

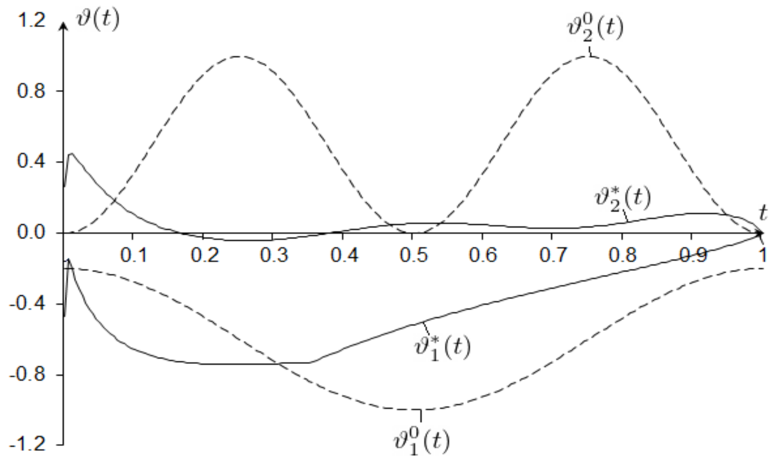

FIGURE 2. *Graphs of the initial control functions* $\vartheta_1^0(t), \vartheta_2^0(t)$ *(dashed lines) and for the obtained optimal functions* $\vartheta_1^*(t), \vartheta_2^*(t)$ *(continuous lines).*

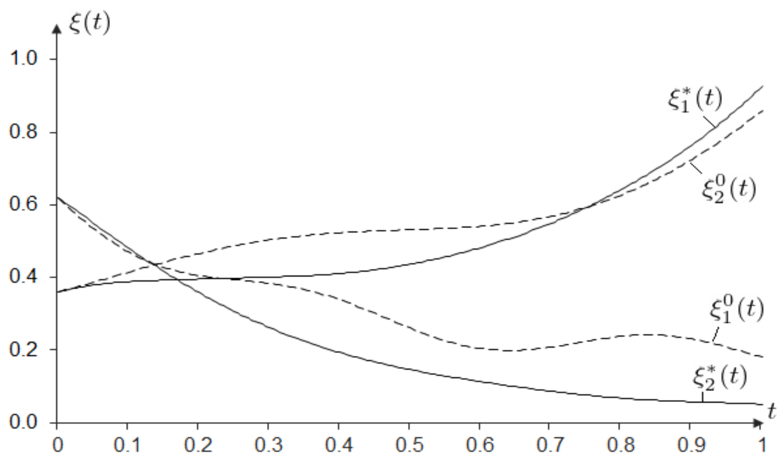

FIGURE 3. *Trajectories of movement of loading points* $\xi_1^0(t), \xi_2^0(t)$ *(dashed lines) at initial controls* $(q^0(t), \vartheta^0(t))$ *and trajectories of movement of loading points* $\xi_1^*(t), \xi_2^*(t)$ *(continuous lines) with optimal control functions* $(q^*(t), \vartheta^*(t))$

.

Experiment 2. The initial approximations for the control functions $q_1^0(t)$, $q_2^0(t)$,$\vartheta_1^0(t))$, $\vartheta_2^0(t))$ are taken as follows:

$$q_1^0(t) = 0.80 + 0.80\sin(2.0\pi t), \quad q_2^0(t) = 0.85 + 1.05\sin(3\pi t), \quad t \in [0,1],$$

$$\vartheta_1^0(t) = \text{ - } 0.58 + 0.18(\cos(8\pi t)\sin(2\pi t) \text{ - } t),$$

$$\vartheta_2^0(t) = 0.2 \text{ - } 0.2(\cos(2\pi t)\sin(8\pi t) \text{ - } t), \quad t \in [0,1].$$

For these initial values of control functions we have:

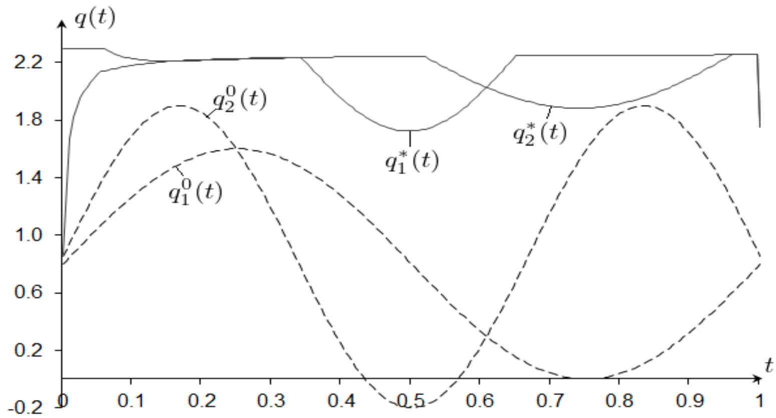

FIGURE 4. *Graphs of Initial approximations for reaction functions $q_1^0(t), q_2^0(t)$ (dashed lines) and for the obtained optimal functions $q_1^*(t), q_2^*(t)$ (continuous lines).*

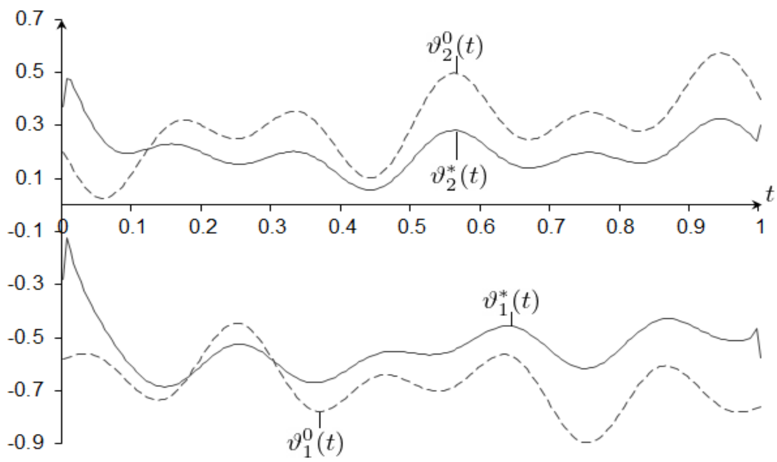

FIGURE 5. *Graphs of the initial control functions $\vartheta_1^0(t), \vartheta_2^0(t)$ (dashed lines) and for the obtained optimal functions $\vartheta_1^*(t), \vartheta_2^*(t)$ (continuous lines).*

$$J\left(q^0, \vartheta^0\right) = 20.69, \quad \int_0^1 [u(x,t_f) - U(x)]^2\, dx = 20.37,$$

$$G(q^0, \vartheta^0) = 0, \quad R = 1,$$

$$\left\|q^0(t)\right\|^2_{L_2^n[0,1]} = 2.61, \quad \left\|\vartheta^0(t)\right\|^2_{L_2^n[0,1]} = 0.57.$$

And after performing the iterative procedures, the following optimal values were obtained:

$$J\left(q^*, \vartheta^*\right) = 10^{-5}, \quad \int_0^1 [u(x,t_f) - U(x)]^2\, dx = 10^{-5}, \quad G(q^*, \vartheta^*) = 10^{-5},$$

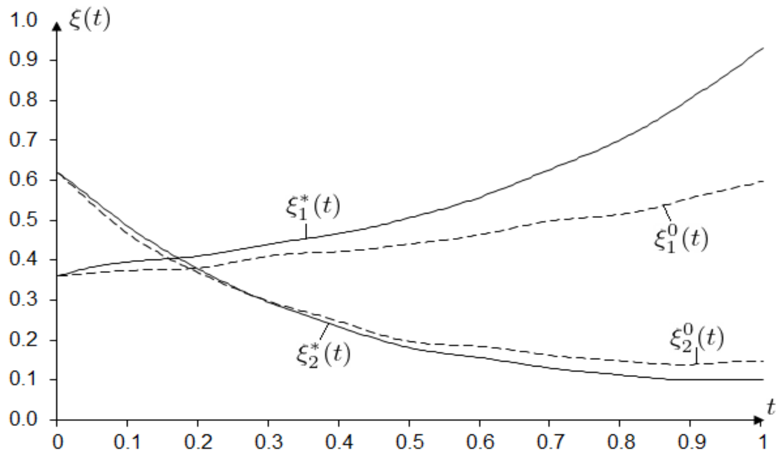

FIGURE 6. *Trajectories of movement of loading points $\xi_1^0(t), \xi_2^0(t)$ (dashed lines) at initial controls $(q^0(t), \vartheta^0(t))$ and trajectories of movement of loading points $\xi_1^*(t), \xi_2^*(t)$ (continuous lines) with optimal control functions $(q^*(t), \vartheta^*(t))$.*

$$\|q^*(t)\|^2_{L_2^n[0,1]} = 9.0937, \quad \|\vartheta^*(t)\|^2_{L_2^n[0,1]} = 0.3461.$$

Numerous computer experiments were carried out both for other values of the parameters involved in the numerical methods of solutions to the boundary value problem and the Cauchy problem, and for other data of the problem itself. The results of the experiments differed little from the results presented above.

5. Conclusion

In the paper, the problem of controlling the change in loading places and the corresponding reaction functions for objects that are described by a loaded partial differential equation of parabolic type is studied. The problem statements under consideration can arise both in the optimization of loaded systems and in the solution of inverse problems on the identification of unknown loading places and the reaction functions to loadings. The specificity and novelty of the problem under study lies in the fact that the coordinates of the loading points are the solution of a system of differential equations and change in time under the influence of control actions.

Necessary optimality conditions are obtained for the optimized parameters involved in the loaded initial-boundary value problem and Cauchy problems describing the motions of loading points. The optimality conditions contain formulas for the gradient of the objective functional, which are used in the application of first-order optimization methods for the numerical solution of the problem of optimizing the loading sites and loading response functions.

The statement of the problem proposed in the paper and the approach to obtaining calculation formulas for its numerical solution can be extended to the cases of controlling many other processes described by other types of loaded partial differential equations and initial-boundary conditions.

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