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A STUDY OF ONE APPROACH TO SOLUTION OF THE FIRST-ORDER NON-LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH MULTIPOINT BOUNDARY CONDITIONS

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Abstract. In this paper the existence and uniqueness of solution for a nonlinear first order ordinary differential system with multipoint boundary conditions and impulse are given. The proof is obtained by defining a suitable Green function, which converts the differential problem into an equivalent integral equation so that the existence and uniqueness can be easily studied on this equivalent problem by using the Banach contraction principle. Existence results are shown by Schaefer's and Krasnoselskii's fixed point theorems. An example is provided to see the applicability of the obtained results.

1. Introduction and Problem Statement

Many problems in modern science, in technology and in economics are described by some differential equations, with a first kind discontinuous solution at a fixed value of the independent variable. Such differential equations are called differential equations with impulse effects [8, 9, 15, 18, 31]. We will study the existence and uniqueness of impulsive equations coupled with multipoint boundary conditions. Multipoint boundary value problems with impulse arise in many natural science disciplines such as physics, mathematics, and biology. For example, the mathematical model of a dynamical system with n degrees of freedom, with n states observed at n different instants of time leads to a multipoint boundary value problem.

Multipoint boundary value problems for ODEs and their systems have been intensively explored in the last few years and still are very attractive because of their importance in the solution of concrete real problems. This is related with their strong relation with a myriad of applications in various fields of physics and mathematics [11, 10]. It is epitomized the fact that the vibrations of a uniform cross-section string composed by N parts of different densities appear in the theory of elastic stability [32]. It is noteworthy that these problems are modelled by the multipoint boundary value problems in mathematical formulations. One of the essential point is that multipoint boundary value problems also arise when discretizing some boundary value problems for partial differential equations.

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Up-to-date, the multipoint boundary value problems for second-order differential equations have been mainly investigated (see [4, 12, 13, 19, 26] and references therein). The initial study of multipoint boundary value problems for linear second order ordinary differential equations was started by Il'in and Moiseev [16]. Since then nonlinear multipoint boundary value problems have been analyzed by several authors using the Leray-Schauder Continuation Theorem, nonlinear alternatives of Leray-Schauder coincidence degree theory and fixed point theorem in cones. However, differential equations of the first order haven't been sufficiently studied. Examples of such works can be found in [1, 3, 20, 21, 25, 27, 28, 29, 33, 34]. Similar problems restricted to only two-point and integral boundary value problems are considered in [2, 5, 6, 7, 12, 14, 22, 23, 24, 30].

In this paper, we study the existence and uniqueness of the solution of the following nonlinear differential system

$$\dot{x} = f(t,x),\ t \in [0,T],\ t \neq \eta \in (0,T),\ \ x \in R^n, \tag{1.1}$$

with multipoint boundary conditions

$$\sum_{i=0}^{m} l_i x\left(t_i\right) = \alpha, \tag{1.2}$$

and the impulsive condition

$$\Delta x\left(\eta\right) = J\left(x\left(\eta\right)\right), \tag{1.3}$$

where l_i, $i = 1, 2, ..., m$ are constant square matrices of order n such that $\det N \neq 0$, $N = \sum\limits_{i=0}^{m} l_i$; η is a some known fixed point; $f : [0,T] \times R^n \to R^n$ and $J : R^n \to R^n$ are given functions; points $t_i,\ i = 1, 2, ..., m$ satisfies the condition $0 = t_0 < t_1 < ... < t_m = T$ and $\eta \in (t_k, t_{k+1})$,

$$\Delta x\left(\eta\right) = x\left(\eta^+\right) - x\left(\eta^-\right)\ ,$$

where

$$x\left(\eta^+\right) = \lim_{h \to +0} x\left(\eta + h\right), x\left(\eta^-\right) = \lim_{h \to +0} x\left(\eta - h\right) = x\left(\eta\right),$$

are the right- and left-hand limits of $x\left(t\right)$ at $t = \eta$, respectively.

In order to show the existence and uniqueness, a suitable Green function is constructed for the multipoint boundary value problem and the considered problem is reduced to an equivalent integral equations. Then the existence and uniqueness of the solutions are studied using the Banach contraction principle. The existence of the solution is also proved by applying Schaefer's and Krasnoselskii's fixed point theorems. The Banach contraction principle, Schaefer's and Krasnoselskii's fixed point theorem are particularly useful for proving the existence and uniqueness results.

The organization of the paper is as follows. In Section 2, we introduce some definitions and lemmas which are the key tools for our main task. Section 3 is devoted to the theorems on the existence and uniqueness of the solution of problem (1.1)-(1.3) established under some sufficient conditions on the nonlinear terms. An example is included.

2. Preliminaries

We denote by $C([0,T];R^n)$ the Banach space of all continuous functions from $[0,T]$ into R^n. We consider the linear space

$$PC([0,T];R^n) = \{x : [0,T] \to R^n;\ x(t) \in C([0,\eta], R^n) \cup C((\eta,T], R^n]),$$

$$x\left(\eta^-\right) \text{ and } x\left(\eta^+\right) \text{ exist and } x\left(\eta^-\right) = x(\eta)\}.$$

$PC([0,T];R^n)$ is a Banach space with the norm

$$\|x\|_{PC} = \max\{\|x\|_{C([0,\eta],R^n)},\ \|x\|_{C((\eta,T],R^n)}\}.$$

We define the solution of problem (1.1)-(1.3) as follows:

Definition 2.1. A function $x \in PC([0,T]\ ;R^n)$ is said to be a solution of problem (1.1)-(1.3) if $\dot{x} = f(t,x)$ for each $t \in [0,T]$, and boundary conditions (1.2) and (1.3) are satisfied.

For simplicity, let us first consider the following problem:

$$\dot{x}(t) = y(t), \qquad t \in [0,T], \tag{2.1a}$$

$$\sum_{i=0}^{m} l_i x(t_i) = \alpha, \tag{2.1b}$$

$$\Delta x(\eta) = J(x(\eta)), \tag{2.1c}$$

where $y(t)$ is continuous function.

Lemma 2.1. *Let $y \in C([0,T];R^n)$. Then the unique solution of the boundary value problem for differential equation (2.1a) with impulsive boundary conditions (2.1b), (2.1c) is given by*

$$x(t) = N^{-1}\alpha + \int_0^T G(t,\tau)y(\tau)d\tau + g(t,\eta)\,a, \tag{2.1}$$

where

$$G(t,\tau) = \begin{cases} G_1(t,\tau),\ t \in [0,t_1], \\ G_2(t,\tau),\ t \in (t_1,t_2], \\ \dots\dots\dots\dots\dots\dots \\ G_m(t,\tau),\ t \in (t_{m-1},T], \end{cases}$$

$$g(t,\eta) = \begin{cases} -N^{-1} \sum\limits_{i=k;t<\eta}^{m} l_i, \\ N^{-1} \sum\limits_{i=1;t\geq\eta}^{k+1} l_i, \end{cases}$$

with

$$G_i(t,\tau)=\begin{cases} N^{-1}l_0, \quad t_0\le\tau\le t_1, \\ N^{-1}\left(\sum\limits_{k=0}^{1} l_k\right), \quad t_1<\tau\le t_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ N^{-1}\left(\sum\limits_{k=0}^{i-1} l_k\right), \quad t_{i-1}<\tau\le t_i, \\ N^{-1}\left(\sum\limits_{k=0}^{i} l_k\right), \quad t_i<\tau\le t, \\ -N^{-1}\left(\sum\limits_{k=i+1}^{m} l_i\right), \quad t<\tau\le t_{i+1}, \\ -N^{-1}\left(\sum\limits_{k=i+2}^{m} l_i\right), \quad t_{i+1}<\tau\le t_{i+2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -N^{-1}l_m, \quad t_m<\tau\le T, \end{cases}$$

where $i=1,2,...,m$.

Proof. If the function $x=x(\cdot)$ is a solution of the differential equation (2.1a) with impulsive boundary conditions (2.1c) then for any $t\in(0,T)$, it is

$$x(t)=x_0+\chi(t-\eta)\,a+\int_0^t y(\tau)d\tau, \tag{2.2}$$

where x_0 is a constant vector and $\chi(t-\eta)=\begin{cases} 0, & if \;\; t<\eta, \\ 1, & if \;\; t\ge\eta. \end{cases}$

Then, in order to fulfill the multipoint boundary conditions (2.1b), we have

$$\sum_{i=0}^{m} l_i[x_0+\chi(t_i-\eta)\,a+\int_0^{t_i} y(s)\,ds]=\alpha.$$

So that the constant vector x_0 must be equal to

$$x_0=N^{-1}\alpha-N^{-1}\left[\sum_{i=1}^{m} l_i\chi(t_i-\eta)\,a+\sum_{i=1}^{m} l_i\int_0^{t_i} y(s)\,ds\right]. \tag{2.3}$$

From Eq. (2.2) if we take into account this value of x_0 we get

$$x(t)=N^{-1}\alpha-N^{-1}\left[\sum_{i=1}^{m} l_i\chi(t_i-\eta)\,a+\sum_{i=1}^{m} l_i\int_0^{t_i} y(s)\,ds\right]+$$
$$+\chi(t-\eta)\,a+\int_0^t y(s)\,ds. \tag{2.4}$$

Now suppose that $t\in[0,t_1]$. Then we can write the equality (2.4) as follows:

$$x(t)=N^{-1}\alpha-N^{-1}\left(l_1\int_0^t y(\tau)d\tau+l_1\int_t^{t_1} y(\tau)d\tau\right)-$$
$$-N^{-1}\left(l_2\int_0^t \mu(\tau)d\tau+l_2\int_t^{t_1} y(\tau)d\tau\right)-N^{-1}l_2\int_{t_1}^{t_2} y(\tau)\,d\tau-N^{-1}\times$$

$$\left(l_3\int_0^t y\left(\tau\right)d\tau+l_3\int_t^{t_1} y\left(\tau\right)d\tau\right)-N^{-1}l_3\left(\sum_{i=1}^{2}\int_t^{t_{i+1}} y\left(\tau\right)d\tau\right)-...-$$

$$-N^{-1}\left(l_m\int_0^t y\left(\tau\right)d\tau+l_m\int_t^{t_1} y\left(\tau\right)d\tau\right)-N^{-1}l_m\left(\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}} y\left(\tau\right)d\tau\right)-$$

$$-N^{-1}\sum_{i=1}^{m}l_i\chi\left(t_i-\eta\right)a+\chi\left(t-\eta\right)a+\int_0^t y\left(\tau\right)d\tau\,.$$

This equality can be rewritten in the following equivalent form:

$$x(t)=N^{-1}\alpha+\int_0^t\left(E-N^{-1}\sum_{i=1}^{m}l_i\right)y(\tau)d\tau-N^{-1}\int_t^{t_1}\left(\sum_{i=1}^{m}l_i\right)y(\tau)d\tau-$$

$$-N^{-1}\left(\sum_{i=2}^{m}l_i\right)\int_{t_1}^{t_2} y\left(\tau\right)d\tau-N^{-1}\left(\sum_{i=3}^{m}l_i\right)\int_{t_2}^{t_3} y\left(\tau\right)d\tau-...-$$

$$-N^{-1}l_m\int_{t_{m-1}}^{T} y\left(\tau\right)d\tau+g\left(t,\eta\right)a, \tag{2.5}$$

where E is an identity matrix.

Since the equality

$$\left(E-N^{-1}\sum_{i=1}^{m}l_i\right)=N^{-1}l_0$$

holds true, we can introduce the following functions:

$$G_1\left(t,\tau\right)=\begin{cases} N^{-1}l_0, \quad t_0\le\tau\le t,\\ -N^{-1}\left(\sum\limits_{i=1}^{m}l_i\right), \quad t<\tau\le t_1,\\ -N^{-1}\left(\sum\limits_{i=2}^{m}l_i\right), \quad t_1<\tau\le t_2,\\ -N^{-1}\left(\sum\limits_{i=3}^{m}l_i\right), \quad t_2<\tau\le t_3,\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ -N^{-1}l_m, \quad t_{m-1}<\tau\le T,\end{cases}$$

and

$$g\left(t,\eta\right)=\begin{cases} -N^{-1}\sum\limits_{i=k;t<\eta}^{m}l_i,\\ N^{-1}\sum\limits_{i=1;t\ge\eta}^{k+1}l_i.\end{cases}$$

By using this function, Eq. (2.5) can be written as the integral equation

$$x(t)=N^{-1}\alpha+\int_0^T G_1(t,\tau)y(\tau)d\tau+g\left(t,\eta\right)a, \quad t\in\left[0,t_1\right].$$

Now, let us assume $t\in(t_k,t_{k+1}]$. Then we can write Eq. (2.4) as

$$x(t)=N^{-1}\alpha-N^{-1}\left(\sum_{i=1}^{m}l_i\right)\int_0^{t_1} y(t)dt-N^{-1}\left(\sum_{i=2}^{m}l_i\right)\int_{t_1}^{t_2} y(\tau)d\tau-...-$$

$$-N^{-1}\left(\sum_{i=k+1}^{m} l_i\right)\left(\int_{t_k}^{t} y(\tau)\,d\tau+\int_{t}^{t_{k+1}} y(\tau)\,d\tau\right)-...-N^{-1}l_m\int_{t_{m-1}}^{T} y(\tau)\,d\tau+$$

$$-N^{-1}\sum_{i=1}^{m} l_i\chi(t_i-\eta)\,a+\chi(t-\eta)\,a+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} y(t)\,dt+\int_{t_k}^{t} y(\tau)\,d\tau.$$

From here we obtain

$$x(t)=N^{-1}\alpha+N^{-1}l_0\int_0^{t_1} y(t)dt+N^{-1}\left(\sum_{i=0}^{1} l_i\right)\left(\int_{t_1}^{t_2} y(\tau)d\tau\right)-...-$$

$$-...-N^{-1}\left(\sum_{i=0}^{k-1} l_i\right)\int_{t_{k-1}}^{t_k} y(\tau)\,d\tau+N^{-1}\left(\sum_{i=0}^{k} l_i\right)\int_{t_k}^{t} y(\tau)\,d\tau-$$

$$-N^{-1}\left(\sum_{i=k+1}^{m} l_i\right)\int_{t}^{t_{k+1}} y(\tau)\,d\tau-...-N^{-1}l_m\int_{t_{m-1}}^{T} y(\tau)\,d\tau-$$

$$-N^{-1}\sum_{i=k+1}^{m} l_i a+\chi(t-\eta)\,a+\sum_{i=1}^{k}\int_{i-1}^{t_i} y(t)\,dt+\int_{t_k}^{t} y(\tau)d\tau.$$

Let's iterate again to define a new function as follows:

$$G_k(t,\tau)=\begin{cases} N^{-1}l_0, \quad t_0\le\tau\le t_1,\\ N^{-1}\left(\sum\limits_{i=0}^{1} l_i\right), \quad t_1<\tau\le t_2,\\ \dots\dots\dots\dots\dots\dots\dots\dots \\ N^{-1}\left(\sum\limits_{i=0}^{k-1} l_i\right), \quad t_{k-1}<\tau\le t_k,\\ N^{-1}\left(\sum\limits_{i=0}^{k} l_i\right), \quad t_k<\tau\le t,\\ -N^{-1}\left(\sum\limits_{i=k+1}^{m} l_k\right), \quad t<\tau\le t_{k+1},\\ -N^{-1}\left(\sum\limits_{i=k+2}^{m} l_i\right), \quad t_{k+1}<\tau\le t_{k+2},\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -N^{-1}l_m, \quad t_{m-1}<\tau\le T, \end{cases}$$

and

$$g(t,\eta)=\begin{cases} -N^{-1}\sum\limits_{i=k;t<\eta}^{m} l_i,\\ N^{-1}\sum\limits_{i=0;\eta\le t}^{k+1} l_i. \end{cases}$$

Thus, we have obtained that if $t\in(t_k,t_{k+1}]$, then the solution of the boundary value problem can be written in the form

$$x(t)=N^{-1}\alpha+\int_0^T G_k(t,\tau)y(\tau)d\tau+g(t,\eta)\,a, \quad t\in(t_k,t_{k+1}].$$

Similarly for every segment $t \in (t_i, t_{i+1}]$, we get

$$G_i\left(t,\tau\right)=\begin{cases} N^{-1}l_0, \quad t_0 \le \tau \le t_1, \\ N^{-1}\left(\sum\limits_{i=0}^{1} l_i\right), \quad t_1 < \tau \le t_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ N^{-1}\left(\sum\limits_{k=0}^{i-1} l_k\right), \quad t_{i-1} < \tau \le t_i, \\ N^{-1}\left(\sum\limits_{k=0}^{i} l_k\right), \quad t_i < \tau \le t, \\ -N^{-1}\left(\sum\limits_{k=i+1}^{m} l_i\right), \quad t < \tau \le t_{i+1}, \\ -N^{-1}\left(\sum\limits_{k=i+2}^{m} l_i\right), \quad t_{i+1} < \tau \le t_{i+2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -N^{-1}l_m, \quad t_{m-1} < \tau \le T. \end{cases}$$

So that the solution of the impulsive boundary value problem (2.1a)-(2.1c) can be written as

$$x(t) = N^{-1}\alpha + \int_0^T G(t,\tau)y(\tau)d\tau + g\left(t,\eta\right)a.$$

The proof is completed. □

This first result, which was obtained for the simple given vector $y(t)$, shows that the problem (2.1a)-(2.1c) is equivalent to an impulsive integral equation. This holds true also for the more general case (1.1)-(1.3) according to the following lemma.

Lemma 2.2. *Assume that* $f \in C\left([0,T]\times R^n; R^n\right)$ *and* $J \in C\left(R^n; R^n\right)$. *Then the function* $x\left(t\right)$ *is a solution of boundary value problem* (1.1)-(1.3) *if and only if* $x\left(t\right)$ *is a solution of the impulsive integral equation*

$$x(t) = N^{-1}\alpha + \int_0^T G(t,\tau)f\left(\tau, x\left(\tau\right)\right)d\tau + g\left(t,\eta\right)J\left(x\left(\eta\right)\right). \tag{2.6}$$

Proof. Clearly, this lemma can be derived by a similar argument used for the proof of Lemma 2.1. By direct verification, we can show that the solution of impulsive integral equation (2.6) satisfies the boundary value problem (1.1) - (1.3). Lemma 2.2 is proved. □

For our purposes, we will use the following classical theorem:

Theorem 2.1. *[17]. Let* M *be a bounded, closed, convex, and nonempty subset of a Banach space* X. *Let* A_1 *and* A_2 *be two operators such that*

(i) $A_1x + A_2y \in M$ *whenever* $x, y \in M$

(ii) A_1 *is compact and continuous*

(iii) A_2 *is a contraction mapping*

Then, there exists $z \in M$ *such that* $z = A_1z + A_2z$

Proof: Is given in [17].

3. Main Results

In this section we will give the main theorems both of uniqueness and existence for the problem (1.1)-(1.3) by working on the equivalent integral equation (2.6). Thus we have,

Theorem 3.1. ***[Uniqueness]*** *Let us assume that*

(H1) The function $f : [0,T] \times R^n \to R^n$ *is continuous;*

(H2) There exist three constants M, m, m_1 *such that*

$$|f(t,x) - f(t,y)| \le M|x-y|,$$
$$|J(x) - J(y)| \le m|x-y|,$$
$$|J(x)| \le m_1,$$

for each $t \in [0,T]$ *and all* $x, y \in R^n$;

(H3) There exists a constant $K \ge 0$ *such that* $|f(t,x)| \le K$ *for each* $t \in [0,T]$ *and all* $x \in R^n$ *and*

$$L = TSM + gm < 1, \tag{3.1}$$

where

$$S = \max_{[0,T]\times[0,T]} \|G(t,\tau)\|, \ g = \max_{[0,T]} \|g(t,\eta)\|.$$

Then the boundary value problem (1.1)-(1.3) *has a unique solution on* $[0,T]$.

Proof. To achieve this task, let us transform the boundary value problem (1.1)-(1.3) into a fixed point problem. Consider the operator $F : PC([0,T];R^n) \to PC([0,T];R^n)$ defined by

$$(Fx)(t) = N^{-1}\alpha + \int_0^T G(t,\tau) f(\tau, x(\tau))\, d\tau. + g(t,\eta) J(x(\eta)). \tag{3.2}$$

Evidently, the fixed points of the operator F are solutions of the boundary problem (1.1)-(1.3).

Setting $\max_{[0,T]} |f(t,0)| = M_f$ and let us select $r \ge \frac{\|N^{-1}\alpha\| + M_f TS + gm_1}{1-L}$. We show that $FB_r \subset B_r$ where

$$B_r = \{x \in PC([0,T]R^n) : \|x\| \le r\}.$$

For $x \in B_r$, using (H1) and (H2), we get

$$\|Fx(t)\| \le \|N^{-1}\alpha\| + \int_0^T |G(t,\tau)| \left(|f(\tau, x(\tau)) - f(\tau,0)| + |f(\tau,0)|\right) d\tau +$$
$$+ |g(t,\eta)| \left(|(J(x(\eta)) - J(0))| + |J(0)|\right) \le$$
$$\le \|N^{-1}d\| + S\int_0^T (M|x| + M_f)\, dt + g(m|x(\eta)| + m_1) \le$$
$$\le \|N^{-1}d\| + SMrT + M_f TS + gmr + gm_1 \le \frac{\|N^{-1}\alpha\| + M_f TS + gm_1}{1-L} \le r.$$

In order to show that the operator F is a contraction, for any $x, y \in B_r$we have

$$|Fx - Fy| \le$$
$$\le \int_0^T |G(t,\tau)\left(f(\tau, x(\tau)) - f(\tau, y(\tau))\right| d\tau + |g(t,\eta)|\, |J(x(\eta) - J(y(\eta)))| \le$$

$$\leq MS\int_0^T |x(t)-y(t)|\,dt + gm\,|x(\eta)-y(\eta)| \leq$$

$$\leq (MTS+gm)\max_{[0,T]}|x(t)-y(t)| \leq (MTS+gm)\,\|x-y\|$$

or

$$\|Fx-Fy\| \leq L\,\|x-y\|\,.$$

Thus we have that F is contraction by condition (2.6). So that, the boundary value problem (1.1)-(1.3) has a unique solution, and the proof is completed. □

Our second result is based on the Schaefer's fixed point theorem.

Theorem 3.2 (Existence)**.** *Let us assume that the conditions (H1)-(H3) hold true. Then there exists at least one solution in* $[0,T]$ *for the boundary value problem* (1.1)-(1.3).

Proof. Let F be the operator defined in (3.1). We shall use the Schaefer's fixed point theorem to prove that F has a fixed point. The proof of this theorem is based on the following four steps.

Step 1: Let us show that the operator F is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC([0,T];R^n)$. Then, for each $t\in[0,T]$

$$|(Fx)(t)-(Fx_n)(t)| =$$

$$= \left|\int_0^T G(t,\tau)\left(f(\tau,x(\tau))-f(\tau,x_n(\tau))\right)d\tau + g(t,\eta)\left(J(x(\eta))-J(x_n(\eta))\right)\right| \leq$$

$$\leq (TSM+gm)\,|x(t)-x_n(t)| \leq L\,\|x-x_n\|\,.$$

From here we get $\|(Fx)(t)-(Fx_n)(t)\| \to 0$ as $n\to\infty$, which implies that the operator F is continuous.

Step 2: Let us show that F maps bounded sets into bounded sets in $PC([0,T];R^n)$. Indeed, it is enough to show that for any $\eta>0$ there exists a positive constant ω such that for each $x\in B_\eta=\{x\in C([0,T];R^n): \|x\|\leq\eta\}$ it is $\|F(x)\|\leq\omega$. Thus we have for each $t\in[0,T]$

$$|(Fx)(t)| \leq \left\|N^{-1}\alpha\right\| + TSM + gm.$$

This implies that

$$\|F(x)\| \leq \left\|N^{-1}\alpha\right\| + TSK + gm = \omega,$$

Step 3: F maps bounded sets into equicontinuous sets of $PC([0,T];R^n)$. Let $\xi_1,\xi_2\in[0,T]$, $\xi_1<\xi_2$, and $\xi_1,\xi_2<\eta$ or $\xi_1,\xi_2>\eta$. B_r be a bounded set of $PC([0,T];R^n)$ as in Step 2, and let $x\in B_r$.

Case 1. $\xi_1,\xi_2\in[t_i,t_{i+1}]$. Then,

$$F(x(\xi_2))-F(x(\xi_1)) = \int_{t_i}^{\xi_2} N^{-1}\left(\sum_{k=0}^{i} l_i\right) f(\tau,x(\tau))\,d\tau -$$

$$-\int_{\xi_2}^{t_{i+1}} N^{-1}\left(\sum_{k=i+1}^{m} l_i\right) f(\tau,x(\tau))\,d\tau -$$

$$-\int_{t_i}^{\xi_{\backslash 1}} N^{-1}\left(\sum_{k=0}^{i} l_i\right) f(\tau,x(\tau))\,d\tau + \int_{\xi_1}^{t_{i+1}} N^{-1}\left(\sum_{k=i+1}^{m} l_i\right) f(\tau,x(\tau))\,d\tau =$$

$$= \int_{\xi_1}^{\xi_2} N^{-1}\left(\sum_{k=0}^{i} l_i\right) f\left(\tau, x\left(\tau\right)\right) d\tau + \int_{\xi_1}^{\xi_2} N^{-1}\left(\sum_{k=i+1}^{m} l_i\right) f\left(\tau, x\left(\tau\right)\right) d\tau =$$

$$= \int_{\xi_1}^{\xi_2} f\left(\tau, x\left(\tau\right)\right) d\tau.$$

Case 2. $\xi_1 \in [t_{i-1}, t_i)$, $\xi_2 \in [t_i, t_{i+1}]$. Then

$$P\left(x\left(\xi_2\right)\right) - P\left(x\left(\xi_1\right)\right) = \int_{t_{i-1}}^{t_i} N^{-1}\left(\sum_{k=0}^{i-1} l_i\right) f\left(\tau, x\left(\tau\right)\right) d\tau +$$

$$+ \int_{t_i}^{\xi_2} N^{-1}\left(\sum_{k=0}^{i} l_i\right) f\left(\tau, x\left(\tau\right)\right) d\tau - \int_{\xi_2}^{t_{i+1}} N^{-1}\left(\sum_{k=i+1}^{m} l_i\right) f\left(\tau, x\left(\tau\right)\right) d\tau -$$

$$- \int_{t_{i-1}}^{\xi_{\backslash 1}} N^{-1}\left(\sum_{k=0}^{i-1} l_i\right) f\left(\tau, x\left(\tau\right)\right) d\tau + \int_{\xi_1}^{t_i} N^{-1}\left(\sum_{k=i}^{m} l_i\right) f\left(\tau, x\left(\tau\right)\right) d\tau +$$

$$+ \int_{t_i}^{t_{i+1}} N^{-1}\left(\sum_{k=i+1}^{m} l_i\right) f\left(\tau, x\left(\tau\right)\right) d\tau =$$

$$= \int_{\xi_1}^{t_i} f\left(\tau, x\left(\tau\right)\right) d\tau + \int_{t_i}^{\xi_2} f\left(\tau, x\left(\tau\right)\right) d\tau = \int_{\xi_1}^{\xi_2} f\left(\tau, x\left(\tau\right)\right) d\tau.$$

As $t_2 \to t_1$, the right-hand side of the above equalities tends to zero. As a consequence of Steps 1 to 3 together with the Ascoli-Arzela theorem, we can conclude that $F : PC\left([0,T]; R^n\right) \to PC\left([0,T]; R^n\right)$ is completely continuous.

Step 4: Existence of a-priori bounds. Now, it remains to show that the set $\Delta = \{x \in PC\left([0,T]; R^n\right) : x = \lambda F\left(x\right) \text{ for some } 0 < \lambda < 1\}$ is bounded. Let $x \in \Delta$. Then,$x = \lambda F\left(x\right)$ for some $0 < \lambda < 1$. Thus, for each $t \in [0,T]$ we have

$$x(t) = \lambda N^{-1}\alpha + \lambda \int_0^T G(t,\tau) f(\tau, x(\tau)) d\tau + \lambda g\left(t, \eta\right) J\left(x\left(\eta\right)\right).$$

From here

$$\|x\| \le \left\|N^{-1}\alpha\right\| + SKT + gm.$$

For that reason the set Δ is bounded. The conclusion of Schaefer's fixed point theorem applies and the operator F has at least one fixed point. Thus there exists at least one solution for the problems (1.1)-(1.3) on $[0,T]$. □

Our next result is based on the Krasnoselskii's fixed point theorem and yields an important result.

Theorem 3.3. *Suppose* $\left|f\left(t,x\right)\right| \le \mu\left(t\right)$ *for* $(t,x) \in [0,T] \times R^n$, $\mu \in C\left([0,T]; R^+\right)$. *Furthermore, the conditions (H1), (H2) hold and*

$$mg < 1. \tag{3.3}$$

Then boundary value problem (1.1)-(1.3) *has at least one solution on* $[0,T]$.

Proof. Setting $\max\limits_{t\in[0,T]} |\mu(t)| = \|\mu\|$ and choosing

$$\rho \geq \|\mu\| ST + gm_1 + \left\|N^{-1}\alpha\right\|$$

and we consider $B_\rho = \{x \in PC([0,T];R^n) : \|x\| \leq \rho\}$. The operators A_1 and A_2 on B_ρ are defined as follows

$$(A_1x)(t) = \int_0^T G(t,\tau) f(\tau, x(\tau)) d\tau,$$

$$(A_2x)(t) = g(t,\eta) J(x(\eta)) + N^{-1}\alpha.$$

For any $x, y \in B_\rho$, we have

$$|(A_1x)(t) + (A_2y)(t)| \leq \left\|N^{-1}\alpha\right\| +$$

$$+ \max_{t\in[0,T]} \left\{ \int_0^T |G(t,\tau) f(\tau, x(\tau))| d\tau + |g(t,\eta) J(x(\eta))| \right\} \leq$$

$$\leq \|\mu\| ST + gm_1 + \left\|N^{-1}\alpha\right\| \leq \rho.$$

Thus $A_1x + A_2y \in B_\rho$. The condition (3.2) implies that the operator A_2 is a contraction mapping. Additionally, continuity of f implies that the operator $A_1$is continuous. Also, the operator A_1 is uniformly bounded on B_ρ where

$$\|A_1x\| \leq \|\mu\| ST \leq \rho.$$

Set $\max\limits_{[0,T]\times B_\rho} |f(t,x)| = \overline{f}$. Consequently we have (see Theorem 3.2, step 3)

$$|(A_1x)(t_2) - (A_1x)(t_1)| \leq S\bar{f} |t_2 - t_1|,$$

which tends to zero as $t_2 - t_1 \to 0$. Hence, the operator A_1 is equicontinuous. So, the operator A_1 is relatively compact on B_ρ. Then, by Arzela-Ascoli's theorem, the operator A_1 is compact on B_ρ. From here we obtain that the boundary value problem (1.1)-(1.3) has at least one solution on $[0,T]$. $\square$

4. Example

Consider the following system of differential equation

$$\begin{cases} \dot{x}_1(t) = \cos\alpha x_2(t), \ t \in [0,1], \\ \dot{x}_2(t) = \sin\beta x_2(t), \ t \in [0,1], \ t \neq 0.25 \end{cases} \qquad (A)$$

subject to

$$\begin{cases} x_1(0) + x_2(0) - x_2(0.5) = 1, \\ -x_1(0.5) + x_1(1) + x_2(1) = 0, \end{cases} \qquad (B)$$

with impulsive condition

$$\Delta x_2(0.25) = \frac{\gamma |x_1(0.25)|}{(1 + |x_1(0.25)|)}. \qquad (C)$$

Evidently,

$$l_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \ l_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \ l_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$N = l_0 + l_1 + l_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Obviously, for $t \in [0, 0.5]$ we obtain

$$G_1(t,\tau) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, & 0 \le \tau \le t, \\ \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, & t < \tau \le 0.5, \\ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, & 0.5 < \tau \le 1, \end{cases}$$

and for $t \in (0.5, 1]$

$$G_2(t,\tau) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, & 0 \le \tau \le 0.5, \\ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, & 0.5 < \tau \le t, \\ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, & t < \tau \le 1. \end{cases}$$

From here we obtain

$$T = 1,\ S \le 2,\ M = \max\{|\alpha|, |\beta|\},\ g \le 1,\ m = |\gamma|.$$

If $L = TSM + gm = 2\max\{|\alpha|, |\beta|\} + |\gamma| < 1$, then boundary value problem has unique solution on $[0, 1]$.

5. Conclusion

The method considered in this paper are general enough and can be used extensively in a wide class of problems. In this article, the existence and uniqueness of the solutions for the first-order nonlinear differential equations with multi-point and impulse conditions are established under sufficient conditions.

References

[1] V.M. Abdullayev, Numerical solution to optimal control problems with multipoint and integral conditions. *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, **44** (2018) no.2, 171–186.

[2] B. Ahmad, S. Sivasundaram, R.A. Khan, Generalized quasilinearization method for a first order differential equation with integral boundary condition. *Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal.*, **12** (2005) no.2, 289–296.

[3] K.R. Aida-zade, An approach for solving nonlinearly loaded problems for linear ordinary differential equations. *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, **44**, (2018), no.2, 338-350.

[4] A. Alsaedi, M. Alsulami, R. P. Agarwal and B. Ahmad, Some new nonlinear second-order boundary value problems on an arbitrary domain, *Adv. Differ. Equ.*, 2018, Article number: 227, 2018.

[5] A. Ashyralyev, F. Emharab, A note on the time identification nonlocal problem. *Adv. Math. Mod. Appl.,* **7** (2022), no.2, 105-120.

[6] A. Ashyralyev and Y.A. Sharifov, Existence and uniqueness of solutions for nonlinear impulsive differential equations with two-point and integral boundary conditions. *Adv. Differ. Equ.*, **2013** (2013), 1–11.

[7] A. Ashyralyev, Y.A. Sharifov, Optimal control problems for impulsive systems with integral boundary conditions, *Elec. Jour. Diff. Equ.*, **2013** (2013) no. 80, 1-11.

[8] M. Benchohra, J. Henderson, S.K. Ntouyas, Impulsive Differential Equations and Inclusions. Contemporary Mathematics and Its Application. New York, Hindawi Publishing Corporation, 2006.

[9] A.A. Boichuk, A.M. Samoilenko, *Generalized inverse operators and Fredholm boundary-value problems (2nd ed.).* Berlin, Boston, Walter de Gruyter GmbH, 2

[10] J.R. Cannon, The One-dimensional Heat Equation, *Encyclopedia of Mathematics and its Applications*, Addison-Wesley Publishing Company, advanced Book Program, Reading, MA, **23**, 1984.

[11] J.R.Cannon, S.P. Esteva, J. Van der A. Hoek, Galerkin procedure for the diffusion equation subject to the specification of mass. *SIAM J. Num. Anal.* **24**, (1987) no.3, 499–515.

[12] Y.S. Gasimov, H. Jafari, M.J. Mardanov, R.A. Sardarova, & Y.A. Sharifov, Existence and uniqueness of the solutions of the nonlinear impulse differential equations with nonlocal boundary conditions.*Quaest. Math.*, (2021), 1-14.

[13] J. R. Graef, L. Kong, Math. Solutions of second order multi-point boundary value problems. *Proc. Camb. Phil. Soc.* **145** (2008) no.2, 489-510, doi:10.1017/S0305004108001424

[14] H. Guliyev, H. Tagiev, On the determination of the coefficients of the second-order hyperbolic equation with a nonlocal condition. *Adv. Mat. Mod. & App.* **6** (2021), no.3, 218-226.

[15] A. Halanay, D. Wexler , *Teoria calitativa a sistemelor cu impulsuri.* Bucuresti, Editura Academiei Republicii Socialiste Romania, 1968.

[16] V.A. Il'in, E.I. Moiseev, Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator, *Diff. Equ.* **23** (1987) no.8, 979-987.

[17] M.A. Krasnoselskii, Two remarks on the method of successive approximations, *Usp. Math. Nauk*, **10** (1955), 123-127.

[18] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of impulsive diff. equ.* Singapore, World Scientific, 1989.

[19] H. Li and J. Zhang, Existence of Nontrivial Solutions for Some Second-Order Multipoint Boundary Value Problems. *Jour. of Fun. Spaces*, 2018, Article ID 6486135, 8 pages, 2018, https://doi.org/10.1155/2018/6486135

[20] M.J. Mardanov, Y.A. Sharifov, H.N. Aliyev, R.A. Sardarova, Existence and uniqueness of solutions for the first order non-linear differential equations with multi-point boundary conditions. *European J. Pure Appl. Math.* **13** (2020) no. 3, 414-426.

[21] M.J. Mardanov, M.J. Sharifov, K.E. Ismayilova, Existence and uniqueness of solutions for the system of integro-differential equations with three-point and nonlinear integral boundary conditions, *Bull. Karaganda univ. mathematics series* **3** (2020) no 99, 26-37.

[22] M. J. Mardanov, Y.A. Sharifov, R.A. Sardarova, H.N. Aliyev, Existence and uniqueness of solutions for nonlinear impulsive differential equations with three-point and integral boundary conditions.*Azerb. J. of Math. 10* (2020) no.1, 110-126.

[23] M. J. Mardanov, Y.A. Sharifov, F.M. Zeynalli, Existence and uniqueness of solutions for nonlinear impulsive differential equations with nonlocal boundary conditions, *Vestn. Tomsk. Gos. Univ. Mat. Mekh.* **60**, (2019), 61–72.

[24] M. J.Mardanov, Y.A. Sharifov, F. M. Zeynalli, Existence and uniqueness of the solutions to impulsive nonlinear integro-differential equations with nonlocal boundary conditions, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, **45** (2019), no. 2, 222–233.
[25] M.J. Mardanov, Y.A. Sharifov, Y.S.Gasimov, C.Cattani, Non-linear first-order differential boundary problems with multipoint and integral conditions. *Fractal and Fractional*, **5** (2021) no 1, 15.
[26] A. L. M.Martinez, E. V. Castelani, R. Hoto, Solving a second order m-point boundary value problem, *Nonlinear Studies*, **26** (2019) no. 1, 15-26.
[27] J.J. Nieto, R. Rodriguez-Lopez, Green's function for first-order multipoint boundary value problems and applications to the existence of solutions with constant sign, *J. Math. Anal. Appl.* **388**, (2012) 952-963.
[28] B. Przeradzki and R.Stanczy, Solvability of a Multi-Point Boundary Value Problem at Resonance, *J. Math. Anal. Appl.* **264** (2001), 253-261.
[29] A.R. Safari, M.F. Mekhtiyev, Y.A. Sharifov, Maximum principle in the optimal control problems for systems with integral boundary conditions and its extensions. *Abst. Appl. Anal.* VOL.2013, ID 946910.
[30] Y.A. Sharifov, Optimal control of impulsive systems with nonlocal boundary conditions, *Russian Mathematics*, **57** (2013) no.2, 65-72.
[31] A.M. Samoilenko, N.A. Perestyk, *Impulsive differential equations.* Singapore, World Sci., 1995.
[32] S. Timoshenko, *Theory of elastic stability,* McGraw-Hill, New-York, 1961.
[33] M. Urabe, An existence theorem for multi-point boundary value problems, *Funkcial. Ekvac.* **9** (1966) 43-60.
[34] Y. Zhang, F. Zhang, Multipoint boundary value problem of first order impulsive functional differential functional differential equation, *J. Appl. Math. Comput.*, **31** (2009) 267–278.

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