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# SOLVING A CLASS OF QUASILINEAR FIRST ORDER PDE

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**Abstract**. This paper focuses on a specific class of partial differential equations in three real variables, which are first order and quasilinear. Our objective is to introduce a solving method for these equations, drawing inspiration from the Lagrangian approach. We will thoroughly examine all potential cases and clarify various aspects of the method using practical exercises specifically designed for this purpose.

## 1. Introduction

The study of partial differential equations (which will be abbreviated as PDE in the following) is still an enthralling field, full of challenging equations that beg for new ideas and novel approaches. Many problems still exist, acting as continual reminders of the immense intricacy buried in these equations despite decades of research and mathematical prowess. With each unsolved equation, the need for fresh perspectives and ground-breaking concepts grows, fostering eagerness for the undiscovered truths. The field of PDEs stands as a testament to the unyielding spirit of human curiosity and the limitless potential for breakthroughs that lie just beyond the horizon.

Our comprehension of real-world phenomena and our technology today are largely based on PDEs. It is thanks to modeling through PDEs that we have been able to understand phenomena derived from other disciplines (such as biology, ecology, economics, physical sciences, astronomy, chemistry, etc.). There is no doubt that PDEs remain one of the most active areas of research due to their multiple applications in all areas of science. See Chowdhury et *al.* [2], Kruzhkov [4], Rhee et *al.* [7], Sneddon [9] and references given there.

In this paper, we restrict our work to a class of first order quasilinear PDEs with three real variables, whose general form is

$$Pp + Qq + Rr = S \tag{1.1}$$

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with

$$\begin{aligned} P &= a_1x + b_1y + c_1z + d_1u + e_1 \\ Q &= a_2x + b_2y + c_2z + d_2u + e_2 \\ R &= a_3x + b_3y + c_3z + d_3u + e_3 \\ S &= a_4x + b_4y + c_4z + d_4u + e_4 \end{aligned}$$

where $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$, $r = \frac{\partial u}{\partial z}$, and $u = u(x, y, z)$ is a smooth vector field in a domain $\Omega$ of $\mathbb{R}^3$. The functions $P$, $Q$, $R$, $S$ are linear of $(x, y, z, u)$, and $a_k$, $b_k$, $c_k$, $d_k$, $e_k$ are real numbers for all $1 \leq k \leq 4$.

In Ince [3], Kruzhkov [4] and Sneddon [9], we find a brief discussion on solving equation (1.1) in the simple case of two variables.

We recall that a PDE is said to be quasilinear, if it is linear with respect to all the highest order derivatives of the unknown function. A smooth function $u = u(x, y, z)$ is a solution of equation (1.1), if and only if $u$ is constant along the phase curves of the field $u$, i.e., it is the first integral of the associated characteristic system

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{du}{S} \tag{1.2}$$

Lagrange's method of characteristics reduces the problem of solving PDE (1.1) to the characteristic system (1.2). For further information about this method and how to apply it, see Kruzhkov [4], Rhee et *al.* [7] and Sneddon [9].

**Theorem 1.1.** *The general solution of PDE* (1.1) *is* $F(\varphi, \psi, \xi) = 0$, *where* $F$ *is an arbitrary function and* $\varphi(x, y, z) = c_1$, $\psi(x, y, z) = c_2$, $\xi(x, y, z) = c_3$ *are linearly independent first integrals of the associated characteristic system* (1.2).

In Kruzhkov [4], Mesbahi [5], Reinhard [6], Sneddon [9], we find a proof of this theorem as well as several other important theorems and properties.

First order PDEs appear frequently in stochastic process theory, as the Fokker-Planck equation, and in mathematical physics, as the Hamilton-Jacobi equation. As other examples, we mention the Hopf equation (also known as Burgers' equation without viscosity), which is used in a variety of contexts, such as the dynamics of gases without pressure and in describing the velocity field of a medium consisting of particles moving without interaction in the absence of external forces. In solid mechanics, we often find the mass conservation equation, which describes the movement of a fluid (liquid or gas) when sinks and sources are absent. We also mention the transport equation, which is the prototype of PDEs of the first order. The model is used in various sciences, such as the model for fluid infiltration through sand, where the fluid flows under gravity alone without sources or sinks, see Chechkin et *al.* [1]. The transport equation also appears in the mathematical modeling of traffic-like collective movements at different levels of biological organization. Molecular motor proteins like kinesin and dynein, which are responsible for most intracellular transport in eukayotic cells, sometimes experience traffic jams, which manifest as disease, see Chowdhury et *al.* [2] and Schadschneider [8].

Below we will present a method for solving system (1.2) and consequently equation (1.1). We will discuss all possible cases, supporting each case with an illustrative example that will explain many aspects of the method used.

## 2. Method of solving

Suppose it is possible to find constants $\lambda$, $\mu$, $\upsilon$, $\tau$ such that each ratio of system (1.2) is equal to

$$\frac{\lambda dx+\mu dy+\upsilon dz+\tau du}{\lambda P+\mu Q+\upsilon R+\tau S}$$

If $\lambda$, $\mu$, $\upsilon$ ,$\tau$ are constant multipliers, this expression will be an exact differential, if it is of the form

$$\frac{1}{\rho}\frac{\lambda dx+\mu dy+\upsilon dz+\tau du}{\lambda x+\mu y+\upsilon z+\tau u}$$

This brings us to the following system

$$\begin{aligned}&\frac{\lambda dx+\mu dy+\upsilon dz+\tau du}{(a_1\lambda+a_2\mu+a_3\upsilon+a_4\tau)x+(b_1\lambda+b_2\mu+b_3\upsilon+b_4\tau)y+(c_1\lambda+c_2\mu+c_3\upsilon+c_4\tau)z+(d_1\lambda+d_2\mu+d_3\upsilon+d_4\tau)u}\\&=\frac{1}{\rho}\frac{\lambda dx+\mu dy+\upsilon dz+\tau du}{\lambda x+\mu y+\upsilon z+\tau u}\end{aligned} \tag{2.1}$$

and this is possible only if

$$\begin{cases}\lambda\left(a_1-\rho\right)+a_2\mu+a_3\upsilon+a_4\tau=0\\ b_1\lambda+\left(b_2-\rho\right)\mu+b_3\upsilon+b_4\tau=0\\ c_1\lambda+c_2\mu+\left(c_3-\rho\right)\upsilon+c_4\tau=0\\ d_1\lambda+d_2\mu+d_3\upsilon+\left(d_4-\rho\right)\tau=0\end{cases} \tag{2.2}$$

System (2.2) can be represented in the matrix form $AX=0$, with

$$A=\begin{pmatrix}a_1-\rho & a_2 & a_3 & a_4\\ b_1 & b_2-\rho & b_3 & b_4\\ c_1 & c_2 & c_3-\rho & c_4\\ d_1 & d_2 & d_3 & d_4-\rho\end{pmatrix},\ X=\begin{pmatrix}\lambda\\ \mu\\ \upsilon\\ \tau\end{pmatrix}$$

where $\det A=0$ is required to obtain non-zero solutions for $X$ instead of just the trivial zero solution, in order to provide meaningful insight about the system's behavior. Setting $\det A=0$ allows finding a particular non-zero solution for $X$ that meets the paper's objective, which leads to

$$\Psi\left(\rho\right)=\det A=0 \tag{2.3}$$

This polynomial has four roots in $\mathbb{C}$, which we may denote by $\rho_1$, $\rho_2$, $\rho_3$, $\rho_4$. We distinguish the following possible cases:

(i) $\rho_1\neq\rho_2\neq\rho_3\neq\rho_4\in\mathbb{R}$
(ii) $\rho_1\neq\rho_2\in\mathbb{R},\ \rho_3=\overline{\rho_4}\in\mathbb{C}$
(iii) $\rho_1=\overline{\rho_2}\neq\rho_3=\overline{\rho_4}\in\mathbb{C}$
(iv) $\rho_1\neq\rho_2=\rho_3=\rho_4\in\mathbb{R}$
(v) $\rho_1=\rho_2\neq\rho_3=\rho_4\in\mathbb{R}$
(vi) $\rho_1=\rho_2\neq\rho_3\neq\rho_4\in\mathbb{R}$
(vii) $\rho_1=\rho_2\in\mathbb{R},\ \rho_3=\overline{\rho_4}\in\mathbb{C}$
(viii) $\rho_1=\overline{\rho_2}=\rho_3=\overline{\rho_4}\in\mathbb{C}$
(ix) $\rho_1=\rho_2=\rho_3=\rho_4\in\mathbb{R}$

In the following paragraphs, we will discuss all possible cases and provide an illustrative example for each. In everything that follows, we denote by $c$ or $c_j$ an arbitrary real constant.

**2.1. Case (i):** $\rho_1 \neq \rho_2 \neq \rho_3 \neq \rho_4 \in \mathbb{R}$**.** In this case, for any $\rho_j$, $1 \leq j \leq 4$, There exist real constants $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$ satisfying system (2.1), and thus we have four possible exact differentials, which gives us

$$\tfrac{\lambda_1 dx+\mu_1 dy+\upsilon_1 dz+\tau_1 du}{\rho_1(\lambda_1 x+\mu_1 y+\upsilon_1 z+\tau_1 u)} = \tfrac{\lambda_2 dx+\mu_2 dy+\upsilon_2 dz+\tau_2 du}{\rho_2(\lambda_2 x+\mu_2 y+\upsilon_2 z+\tau_2 u)} = \tfrac{\lambda_3 dx+\mu_3 dy+\upsilon_3 dz+\tau_3 du}{\rho_3(\lambda_3 x+\mu_3 y+\upsilon_3 z+\tau_4 u)}$$
$$= \tfrac{\lambda_4 x+\mu_4 dy+\upsilon_4 dz+\tau_4 du}{\rho_4(\lambda_4 x+\mu_4 y+\upsilon_4 z+\tau_4 u)}.$$

This admits as first integrals the following

$$\begin{aligned}
(\lambda_1 x+\mu_1 y+\upsilon_1 z+\tau_1 u)^{\rho_2} . (\lambda_2 x+\mu_2 y+\upsilon_2 z+\tau_2 u)^{-\rho_1} &= c_1 \\
(\lambda_2 x+\mu_2 y+\upsilon_2 z+\tau_2 u)^{\rho_3} . (\lambda_3 x+\mu_3 y+\upsilon_3 z+\tau_3 u)^{-\rho_2} &= c_2 \\
(\lambda_3 x+\mu_3 y+\upsilon_3 z+\tau_3 u)^{\rho_4} . (\lambda_4 x+\mu_4 y+\upsilon_4 z+\tau_4 u)^{-\rho_3} &= c_3
\end{aligned}$$

which are linearly independent, where $c_1$, $c_2$ and $c_3$ are arbitrary real constants. Hence the integral curves $F(c_1, c_2, c_3) = 0$, where $F$ is an arbitrary real function.

**Example 2.1.** Let the equation be given as

$$xp + (x + 2y)\, q + (3z)\, r = z + 4u$$

In this case, the polynomial $\Psi(\rho)$ of (2.3) admits four different real roots $\rho_1 = 1$, $\rho_2 = 2$, $\rho_3 = 3$, $\rho_4 = 4$. The constants $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$ associated respectively are $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 1, 1)$. The characteristic system associated with our equation becomes

$$\frac{dx}{x} = \frac{d(x+y)}{2(x+y)} = \frac{dz}{3z} = \frac{d(z+u)}{4(z+u)}$$

which gives us the following linearly independent first integrals

$$x^2(x+y)^{-1} = c_1\ ,\ (x+y)^3 z^{-2} = c_2\ ,\ z^4(z+u)^{-3} = c_3$$

where $c_1$, $c_2$ and $c_3$ are arbitrary real constants. Hence the integral curves $F(c_1, c_2, c_3) = 0$, where $F$ is an arbitrary real function.

**2.2. Case (ii):** $\rho_1 \neq \rho_2 \in \mathbb{R}$, $\rho_3 = \overline{\rho_4} \in \mathbb{C}$**.** Suppose that the roots of the polynomial $\Psi(\rho)$ are $\rho_1 \neq \rho_2 \in \mathbb{R}$ and $\rho_3 = \rho_3' + i\rho_3'' = \overline{\rho_4} \in \mathbb{C}$. In this case, we can find constants $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$, $1 \leq j \leq 4$, satisfying system (2.1), where

$$\begin{aligned}
\lambda_j, \mu_j, \upsilon_j, \tau_j &\in \mathbb{R}, \text{ for } j \in \{1, 2\} \\
(\lambda_3, \mu_3, \upsilon_3, \tau_3) &= \left(\lambda_3' + i\lambda_3'', \mu_3' + i\mu_3'', \upsilon_3' + i\upsilon_3'', \tau_3' + i\tau_3''\right) \\
(\lambda_4, \mu_4, \upsilon_4, \tau_4) &= \left(\lambda_3' - i\lambda_3'', \mu_3' - i\mu_3'', \upsilon_3' - i\upsilon_3'', \tau_3' - i\tau_3''\right)
\end{aligned}$$

with $\lambda_3', \lambda_3'', \mu_3', \mu_3'', \upsilon_3', \upsilon_3'', \tau_3', \tau_3'' \in \mathbb{R}$. The characteristic system becomes

$$\frac{df}{\rho_1 f} = \frac{dg}{\rho_2 g} = \frac{dh}{(\rho_3' + i\rho_3'')\, h} = \frac{dk}{(\rho_3' - i\rho_3'')\, k} \tag{2.4}$$

where

$$\begin{aligned}
f &= \lambda_1 x + \mu_1 y + \upsilon_1 z + \tau_1 u \\
g &= \lambda_2 x + \mu_2 y + \upsilon_2 z + \tau_2 u \\
\bar{k} &= h = \left(\lambda_3' + i\lambda_3''\right) x + \left(\mu_3' + i\mu_3''\right) y + \left(\upsilon_3' + i\upsilon_3''\right) z + \left(\tau_3' + i\tau_3''\right) u
\end{aligned}$$

From the first equality of system (2.4), we have the following first integral

$$(\lambda_1 x + \mu_1 y + \upsilon_1 z + \tau_1 u)^{\rho_2} \cdot (\lambda_2 x + \mu_2 y + \upsilon_2 z + \tau_2 u)^{-\rho_1} = c_1$$

The second equality of system (2.4) gives us

$$E = g^{\left(\rho_3' + i\rho_3''\right)} \cdot h^{-\rho_2} = c$$

and from it, we get

$$\begin{aligned} \log E &= \rho_3' \log\left(\lambda_2 x + \mu_2 y + \upsilon_2 z + \tau_2 u\right) - \rho_2 z_1 \\ &\quad + i\left(\rho_3'' \log\left(\lambda_2 x + \mu_2 y + \upsilon_2 z + \tau_2 u\right) - \rho_2 z_2\right) \end{aligned}$$

where

$$\begin{aligned} z_1 &= \frac{1}{2}\log\left(\left(\lambda_3' x + \mu_3' y + \upsilon_3' z + \tau_3' u\right)^2 + \left(\lambda_3'' x + \mu_3'' y + \upsilon_3'' z + \tau_3'' u\right)^2\right) \\ z_2 &= \arctan\left(\frac{\lambda_3'' x + \mu_3'' y + \upsilon_3'' z + \tau_3'' u}{\lambda_3' x + \mu_3' y + \upsilon_3' z + \tau_3' u}\right) \end{aligned}$$

which implies

$$E = \exp\left(\log\left(g\right)^{\rho_3'} - \rho_2 z_1\right) \cdot \exp\left[i\left(\log\left(g\right)^{\rho_3''} - \rho_2 z_2\right)\right]$$

We get the following first integral

$$E = \left(g\right)^{\rho_3'} \cdot e^{-\rho_2 z_1} \cos\left(\rho_3'' \log\left(g\right) - \rho_2 z_2\right) = c_2$$

The third equality of system (2.4) gives us

$$E' = h^{\left(\rho_3' - i\rho_3''\right)} \cdot k^{-\left(\rho_3' + i\rho_3''\right)} = c$$

which gives

$$\log E' = 2i\left(\rho_3' z_2 - \rho_3'' z_1\right)$$

and since we are looking for real solutions, we take the following first integral

$$E' = \rho_3' z_2 - \rho_3'' z_1 = c_3$$

The three first integrals obtained are linearly independent, where $c_1$, $c_2$ and $c_3$ are arbitrary real constants. Hence the integral curves $F\left(c_1, c_2, c_3\right) = 0$, where $F$ is an arbitrary real function.

**Example 2.2.** Let the equation be given as

$$\left(2x\right)p + \left(x + 2y + 2u\right)q + \left(z - y\right)r = z + u$$

The polynomial $\Psi\left(\rho\right)$ admits for roots $\rho_1 = 0$, $\rho_2 = 2$, $\rho_3 = 2 - i = \overline{\rho_4}$. The constants $\left(\lambda_j, \mu_j, \upsilon_j, \tau_j\right)$ associated respectively are

$$\left(1, -2, -4, 4\right),\ \left(1, 0, 0, 0\right),\ \left(1 + i, 1 - i, 1 + i, 2\right),\ \left(1 - i, 1 + i, 1 - i, 2\right)$$

In this way, we can obtain the following linearly independent first integrals

$$\begin{aligned} x - 2y - 4z + 4u &= c_1 \\ x^2 \exp\left(-2z_1\right) \cdot \cos\left(1\log\left(x\right) - 2z_2\right) &= c_2 \\ 2z_2 - z_1 &= c_3 \end{aligned}$$

where

$$\begin{aligned} z_1 &= \frac{1}{2}\log\left[\left(x + y + z + 2u\right)^2 + \left(x - y + z + 0u\right)^2\right] \\ z_2 &= \arctan\left(\frac{x - y + z + 0u}{x + y + z + 2u}\right) \end{aligned}$$

Hence the integral curves $F\left(c_1, c_2, c_3\right) = 0$, where $F$ is an arbitrary real function.

**2.3. Case (iii): $\rho_1 = \overline{\rho_2} \neq \rho_3 = \overline{\rho_4} \in \mathbb{C}$.** Let $\rho_1 = \rho_1' + i\rho_1'' = \overline{\rho_2}$ and $\rho_3 = \rho_3' + i\rho_3'' = \overline{\rho_4} \in \mathbb{C}$ be the roots of the polynomial $\Psi(\rho)$. In this case, we can find real constants $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$, $1 \leq j \leq 4$, satisfying system (2.1), where

$$\begin{aligned}(\lambda_j, \mu_j, \upsilon_j, \tau_j) &= \left(\lambda_j' + i\lambda_j'', \mu_j' + i\mu_j'', \upsilon_j' + i\upsilon_j'', \tau_j' + i\tau_j''\right) \ , \ j \in \{1,3\} \\ (\lambda_j, \mu_j, \upsilon_j, \tau_j) &= \left(\lambda_j' - i\lambda_j'', \mu_j' - i\mu_j'', \upsilon_j' - i\upsilon_j'', \tau_j' - i\tau_j''\right) \ , \ j \in \{2,4\}\end{aligned}$$

with $\lambda_j', \lambda_j'', \mu_j', \mu_j'', \upsilon_j', \upsilon_j'', \tau_j', \tau_j'' \in \mathbb{R}$. The characteristic system becomes

$$\frac{df}{(\rho_1' + i\rho_1'')\, f} = \frac{dg}{(\rho_1' - i\rho_1'')\, g} = \frac{dh}{(\rho_3' + i\rho_3'')\, h} = \frac{dk}{(\rho_3' - i\rho_3'')\, k} \tag{2.5}$$

where

$$\begin{aligned}\bar{g} &= f = \left(\lambda_1' + i\lambda_1''\right) x + \left(\mu_1' + i\mu_1''\right) y + \left(\upsilon_1' + i\upsilon_1''\right) z + \left(\tau_1' + i\tau_1''\right) u \\ \bar{k} &= h = \left(\lambda_3' + i\lambda_3''\right) x + \left(\mu_3' + i\mu_3''\right) y + \left(\upsilon_3' + i\upsilon_3''\right) z + \left(\tau_3' + i\tau_3''\right) u\end{aligned}$$

As in the previous case, by applying the same steps to the first and third equality of system (2.5), we obtain the following first integrals

$$\rho_1' z_2 - \rho_1'' z_1 = c_1 \quad \text{and} \quad \rho_3' z_2' - \rho_3'' z_1' = c_2$$

where

$$\begin{aligned}z_1 &= \frac{1}{2}\log\left(\left(\lambda_1' x + \mu_1' y + \upsilon_1' z + \tau_1' u\right)^2 + \left(\lambda_1'' x + \mu_1'' y + \upsilon_1'' z + \tau_1'' u\right)^2\right) \\ z_2 &= \arctan\frac{\lambda_1'' x + \mu_1'' y + \upsilon_1'' z + \tau_1'' u}{\lambda_1' x + \mu_1' y + \upsilon_1' z + \tau_1' u} \\ z_1' &= \frac{1}{2}\log\left(\left(\lambda_3' x + \mu_3' y + \upsilon_3' z + \tau_3' u\right)^2 + \left(\lambda_3'' x + \mu_3'' y + \upsilon_3'' z + \tau_3'' u\right)^2\right) \\ z_2' &= \arctan\frac{\lambda_3'' x + \mu_3'' y + \upsilon_3'' z + \tau_3'' u}{\lambda_3' x + \mu_3' y + \upsilon_3' z + \tau_3' u}\end{aligned}$$

For the second equality of system (2.5), we get

$$E = g^{\left(\rho_3' + i\rho_3''\right)} \cdot h^{-\left(\rho_1' - i\rho_1''\right)} = c$$

which gives

$$\log E = \left(\rho_3' z_1 + \rho_3'' z_2 - \rho_1' z_1' - \rho_1'' z_2'\right) + i\left(\rho_3'' z_1 - \rho_3' z_2 + \rho_1'' z_1' - \rho_1' z_2'\right)$$

then

$$E = \exp\left(\rho_3' z_1 + \rho_3'' z_2 - \rho_1' z_1' - \rho_1'' z_2'\right) . \exp\left[i\left(\rho_3'' z_1 - \rho_3' z_2 + \rho_1'' z_1' - \rho_1' z_2'\right)\right]$$

and since we are looking for real solutions, we take the following first integral

$$\exp\left(\rho_3' z_1 + \rho_3'' z_2 - \rho_1' z_1' - \rho_1'' z_2'\right) . \cos\left(\rho_3'' z_1 - \rho_3' z_2 + \rho_1'' z_1' - \rho_1' z_2'\right) = c_3$$

As a result, the integral curve $F(c_1, c_2, c_3) = 0$, where $c_1$, $c_2$, $c_3$ are arbitrary real constants and $F$ is an arbitrary real function.

**Example 2.3.** Let the equation be given as

$$(x - y + z)\, p + (x + u)\, q + (y - u)\, r = x + z$$

The polynomial $\Psi(\rho)$ admits for roots $\rho_1 = -i = \overline{\rho_2}$, $\rho_3 = \frac{1}{2} - \frac{1}{2}i\sqrt{3} = \overline{\rho_4}$. The constants $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$ associated respectively are

$$\begin{gathered}(1-2i, 2+i, 2-4i, -5), \ (1+2i, 2-i, 2+4i, -5), \\ \left(1-\sqrt{3}i, 0, 1-\sqrt{3}i, -2\right), \ \left(1+\sqrt{3}i, 0, 1+\sqrt{3}i, -2\right).\end{gathered}$$

In this way, we can obtain the following linearly independent first integrals

$$\begin{aligned} -z_1 &= c_1 \\ \exp\left(\frac{1}{2}z_1 + \frac{\sqrt{3}}{2}z_2 - 0z_1' + 1z_2'\right).\cos\left(\frac{1}{2}z_1 - \frac{\sqrt{3}}{2}z_2 - 1z_1' + 0z_2'\right) &= c_2 \\ \frac{1}{2}z_2' - \frac{\sqrt{3}}{2}z_1' &= c_3 \end{aligned}$$

where

$$\begin{aligned} z_1 &= \frac{1}{2}\log\left((x+2y+2z-5u)^2 + (2x-y+4z+0u)^2\right) \\ z_2 &= \arctan\frac{2x-y+4z+0u}{x+2y+2z-5u} \\ z_1' &= \frac{1}{2}\log\left((x+0y+z-2u)^2 + \left(\sqrt{3}x+0y+\sqrt{3}z+0u\right)^2\right) \\ z_2' &= \arctan\frac{\sqrt{3}x+0y+\sqrt{3}z+0u}{x+0y+z-2u} \end{aligned}$$

Hence the integral curves $F(c_1, c_2, c_3) = 0$, where $F$ is an arbitrary real function.

The remaining cases will only be treated through examples. This approach aims to streamline concepts and circumvent specific computational challenges arising from the nature of the equation to be resolved. By employing the same methodology as in prior cases, we arrive each time at an insufficient number of first integrals. This is the basic difficulty in this work, and to overcome it, the suitable approach has been identified, which we will explain through the examples that we will present later in this paper. First, we proceed as in the previous examples, we will find at most two first integrals, then we work to complete them to obtain three linearly independent first integrals, which will make it possible to find the general solution of the proposed equation. We will treat each case separately with an illustrative example. It is worth noting that in each of the subsequent cases, obtaining three first integrals that are linearly independent is sufficient, and our objective is not to discover all potential first integrals. It is important to mention that all the equations we will examine below have been meticulously formulated to be compatible with all conceivable cases. We will develop the method used previously in the first three cases to obtain additional linearly independent first integrals. For this, suppose that it is possible to find constants $(\alpha_j, \beta_j, \gamma_j, \delta_j)$, $1 \leq j \leq 4$, and $\rho$ such that each ratio of (1.2) is equal to the following exact differential form

$$\frac{\phi_1}{\phi_2} = \frac{dD}{\rho D} \tag{2.6}$$

with

$$\begin{aligned}\phi_1 &= (\alpha_1 x + \beta_1 y + \gamma_1 z + \tau_1 u)\,dx + (\alpha_2 x + \beta_2 y + \gamma_2 z + \tau_2 u)\,dy \\ &\quad + (\alpha_3 x + \beta_3 y + \gamma_3 z + \tau_3 u)\,dz + (\alpha_4 x + \beta_4 y + \gamma_4 z + \tau_4 u)\,du \\ \phi_2 &= (\alpha_1 x + \beta_1 y + \gamma_1 z + \tau_1 u)\,x + (\alpha_2 x + \beta_2 y + \gamma_2 z + \tau_2 u)\,(x+2y) \\ &\quad + (\alpha_3 x + \beta_3 y + \gamma_3 z + \tau_3 u)\,z + (\alpha_4 x + \beta_4 y + \gamma_4 z + \tau_4 u)\,(z+u)\end{aligned}$$

where $dD$ denotes the total derivative of $D$.

We will work to clarify all possible cases through the examples that we will address below.

### 2.4. Case (iv): $\rho_1 \neq \rho_2 = \rho_3 = \rho_4 \in \mathbb{R}$.

**Example 2.4.** Let the equation be given as

$$xp + (x+2y)\,q + zr = z + u \tag{2.7}$$

The polynomial $\Psi(\rho)$ admits a triple root $\rho_1 = 1$ and a simple root $\rho_2 = 2$. The constants $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$ associated respectively are $(1, 0, -1, 0)$ and $(1, 1, 0, 0)$. The characteristic system corresponding to equation (2.7) can therefore be written in the form

$$\frac{dx}{x} = \frac{dy}{x+2y} = \frac{dz}{z} = \frac{du}{z+u} = \frac{d\,(x-z)}{x-z} = \frac{d\,(x+y)}{2\,(x+y)} \tag{2.8}$$

This leads to a single first integral

$$(x+y)^1\,(x-z)^{-2} = c_1$$

We only need two more first integrals. For that, suppose it is possible to find constants $(\alpha_j, \beta_j, \gamma_j, \delta_j)$, $1 \leq j \leq 4$, and $\rho$ such that each ratio of (2.8) is of the form (2.6), with

$$\begin{aligned}D &= (\alpha_1 + \alpha_2)\,x^2 + 2\beta_2 y^2 + (\gamma_3 + \gamma_4)\,z^2 + \gamma_4 u^2 + (\beta_1 + \beta_2 + 2\alpha_2)\,xy \\ &\quad + (\gamma_1 + \gamma_2 + \alpha_3 + \alpha_4)\,xz + (\delta_1 + \delta_2 + \alpha_4)\,xu + (2\gamma_2 + \beta_3 + \beta_4)\,yz \\ &\quad + (2\delta_2 + \beta_4)\,yu + (\delta_3 + \delta_4 + \gamma_4)\,zu\end{aligned}$$

That's right, if

$$\left\{\begin{array}{rr} (2-\rho)\,\alpha_1 + 2\alpha_2 = 0, & \gamma_1 + \gamma_2 + (1-\rho)\,\alpha_3 + \alpha_4 = 0 \\ (1-\rho)\,\beta_1 + \beta_2 + 2\alpha_2 = 0, & 2\gamma_2 + (1-\rho)\,\beta_3 + \beta_4 = 0 \\ (1-\rho)\,\gamma_1 + \gamma_2 + \alpha_3 + \alpha_4 = 0, & (2-\rho)\,\gamma_3 + 2\gamma_4 = 0 \\ (1-\rho)\,\delta_1 + \delta_2 + \alpha_4 = 0, & (1-\rho)\,\delta_3 + \delta_4 + \gamma_4 = 0 \\ \beta_1 + \beta_2 + (2-\rho)\,\alpha_2 = 0, & \delta_1 + \delta_2 + (1-\rho)\,\alpha_4 = 0 \\ (4-\rho)\,\beta_2 = 0, & 2\delta_2 + (1-\rho)\,\beta_4 = 0 \\ (2-\rho)\,\gamma_2 + \beta_3 + \beta_4 = 0, & \delta_3 + \delta_4 + (1-\rho)\,\gamma_4 = 0 \\ (2-\rho)\,\delta_2 + \beta_4 = 0, & 2\gamma_4 - \rho\delta_4 = 0 \end{array}\right.$$

This linear homogeneous system will have a non-trivial solution, if the determinant of its coefficient matrix is zero, i.e.,

$$\begin{aligned}&\rho^{16} - 23\rho^{15} + 231\rho^{14} - 1323\rho^{13} + 4716\rho^{12} - 10\,626\rho^{11} \\ &+14\,312\rho^{10} - 8800\rho^9 - 2880\rho^8 + 7776\rho^7 - 3456\rho^6 = 0\end{aligned}$$

The roots of this polynomial are : 2 (quadruple), 4, $1+\sqrt{3}$, $1-\sqrt{3}$, 3 (triple), 0 (of order 6). We can find several first integrals, but we only need two. For this, we will choose enough values of $\rho$. We can take two different values, and we can be satisfied with a root which is at least double, then we can take $\rho \in \{2,3\}$. If we substitute the values of $\rho$ in (2.6) and solve the resulting system, we find $(\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \alpha_3, \beta_3, \gamma_3, \delta_3, \alpha_4, \beta_4, \gamma_4, \delta_4)$ as following

$$\begin{aligned}(-1,0,1,0,0,0,0,0,1,0,2,0,0,0,0,0) \text{ for } \rho &= 2\\ (2,1,-1,0,1,0,-1,0,-1,-1,0,0,0,0,0,0) \text{ for } \rho &= 3\end{aligned}$$

These values would transform system (2.8) into the following exact differential form

$$\frac{d(x-z)}{x-z} = \frac{d(x+y)}{2(x+y)} = \frac{d\left(-x^2+2z^2+2xz\right)}{2\left(-x^2+2z^2+2xz\right)} = \frac{d\left(3x^2+3xy-3xz-3yz\right)}{3\left(3x^2+3xy-3xz-3yz\right)}$$

which gives us these two new first integrals

$$\begin{aligned}\left(-x^2+2z^2+2xz\right)^3 . \left(3x^2+3xy-3xz-3yz\right)^{-2} &= c_2\\ \left(3x^2+3xy-3xz-3yz\right)^4 . \left(2x^2+4y^2+4xy\right)^{-3} &= c_3\end{aligned}$$

Hence the integral curves $F(c_1, c_2, c_3) = 0$, where $F$ is an arbitrary real function.

### 2.5. Case (v): $\rho_1 = \rho_2 \neq \rho_3 = \rho_4 \in \mathbb{R}$.

**Example 2.5.** Let the equation be given as

$$(x+y-z)\,p + (2y)\,q + (2z+u)\,r = u \tag{2.9}$$

The polynomial $\Psi(\rho)$ admits two double roots $\rho_1 = 1$ and $\rho_2 = 2$. The constants $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$ associated respectively are $(0,0,0,1)$ and $(0,-1,1,1)$. Therefore, the characteristic system that corresponds to equation (2.9) can be expressed as follows

$$\frac{dx}{x+y-z} = \frac{dy}{2y} = \frac{dz}{2z+u} = \frac{du}{u} = \frac{d(-y+z+u)}{2(-y+z+u)} \tag{2.10}$$

This leads to a single first integral

$$\frac{u^2}{-y+z+u} = c_1$$

From system (2.10), other first integrals can be extracted, but we will find this using our method for the sake of better understanding. Suppose it is possible to find constants $(\alpha_j, \beta_j, \gamma_j, \delta_j)$, $1 \leq j \leq 4$, and $\rho$ such that each ratio of (2.10) is of the form (2.6), with

$$\begin{aligned}D \;=\;& (\alpha_1)\,x^2 + (\beta_1 + 2\beta_2)\,y^2 + (-\gamma_1 + 2\gamma_3)\,z^2 + (\beta_1 + \alpha_1 + 2\alpha_2)\,xy\\ &+ (\gamma_1 - \alpha_1 + 2\alpha_3)\,xz + (\delta_1 + \alpha_3 + \alpha_4)\,xu + (\gamma_1 - \beta_1 + 2\beta_3 + 2\gamma_2)\,yz\\ &+ (\delta_1 + 2\delta_2 + \beta_3 + \beta_4)\,yu + (-\delta_1 + 2\delta_3 + \gamma_3 + \gamma_4)\,zu + (\delta_3 + \delta_4)\,u^2\end{aligned}$$

That's right, if

$$\begin{cases} (2-\rho)\,\alpha_1 = 0, & \gamma_1 - \alpha_1 + (2-\rho)\,\alpha_3 = 0 \\ (1-\rho)\,\beta_1 + \alpha_1 + 2\alpha_2 = 0, & \gamma_1 - \beta_1 + (2-\rho)\,\beta_3 + 2\gamma_2 = 0 \\ (1-\rho)\,\gamma_1 - \alpha_1 + 2\alpha_3 = 0, & -2\gamma_1 + (4-\rho)\,\gamma_3 = 0 \\ (1-\rho)\,\delta_1 + \alpha_3 + \alpha_4 = 0, & -\delta_1 + (2-\rho)\,\delta_3 + \gamma_3 + \gamma_4 = 0 \\ \beta_1 + \alpha_1 + (2-\rho)\,\alpha_2 = 0, & \delta_1 + \alpha_3 + (1-\rho)\,\alpha_4 = 0 \\ 2\beta_1 + (4-\rho)\,\beta_2 = 0, & \delta_1 + 2\delta_2 + \beta_3 + (1-\rho)\,\beta_4 = 0 \\ \gamma_1 - \beta_1 + 2\beta_3 + (2-\rho)\,\gamma_2 = 0, & -\delta_1 + 2\delta_3 + \gamma_3 + (1-\rho)\,\gamma_4 = 0 \\ \delta_1 + (2-\rho)\,\delta_2 + \beta_3 + \beta_4 = 0, & 2\delta_3 + (2-\rho)\,\delta_4 = 0 \end{cases}$$

If the determinant of the coefficient matrix of this linear homogeneous system is zero, it indicates that the system will possess a non-trivial solution, i.e.,

$$\begin{gathered} \rho^{16} - 30\rho^{15} + 406\rho^{14} - 3272\rho^{13} + 17\,449\rho^{12} - 64\,658\rho^{11} + 169\,896\rho^{10} \\ -316\,592\rho^{9} + 410\,064\rho^{8} - 351\,648\rho^{7} + 179\,712\rho^{6} - 41\,472\rho^{5} = 0 \end{gathered}$$

This polynomial has as roots : 2 (quintuple), 3 (quadruple), 4 (double), 0 (quintuple). We only need two additional first integrals, for this we will choose enough values of $\rho$. As in the previous case, we avoid $\rho = 0$, then we can take $\rho \in \{2,3\}$. If we substitute the values of $\rho$ in (2.6) and solve the resulting system, we find $(\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \alpha_3, \beta_3, \gamma_3, \delta_3, \alpha_4, \beta_4, \gamma_4, \delta_4)$ as following

$$\begin{aligned} (0,0,0,0,0,0,0,1,0,-1,0,0,0,1,0,2) \text{ for } \rho &= 2 \\ (0,0,0,0,0,0,0,0,0,0,4,1,0,0,2,1) \text{ for } \rho &= 3 \end{aligned}$$

These values would transform system (2.10) into the following exact differential form

$$\frac{du}{u} = \frac{d\,(-y+z+u)}{2\,(-y+z+u)} = \frac{d\left(2u^2\right)}{2\left(2u^2\right)} = \frac{d\left(8z^2+2u^2+8zu\right)}{3\left(8z^2+2u^2+8zu\right)}$$

which gives us these two first integrals

$$\begin{aligned} \left(2u^2\right)^3 \cdot \left(8z^2+2u^2+8zu\right)^{-2} &= c_2 \\ \left(8z^2+2u^2+8zu\right)^4 \cdot \left(3u^2+3yu+3zu\right)^{-3} &= c_3 \end{aligned}$$

Hence the integral curves $F\,(c_1, c_2, c_3) = 0$, where $F$ is an arbitrary real function.

### 2.6. Case (vi): $\rho_1 = \rho_2 \neq \rho_3 \neq \rho_4 \in \mathbb{R}$.

**Example 2.6.** Let the equation be given as

$$xp + (x+2y)\,q + (3z)\,r = z + u \tag{2.11}$$

The characteristic system corresponding to equation (2.11) is

$$\frac{dx}{x} = \frac{dy}{x+2y} = \frac{dz}{3z} = \frac{du}{z+u} \tag{2.12}$$

The polynomial $\Psi\,(\rho)$ admits one double root $\rho_1 = \rho_2 = 1$ and two simple roots $\rho_3 = 2$, $\rho_4 = 3$. The constants $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$ associated respectively are $(1,0,-1,2)$, $(1,1,0,0)$ and $(0,0,1,0)$. Then, system (2.12) gives us

$$\frac{d\,(x-z+2u)}{(x-z+2u)} = \frac{d\,(x+y)}{2\,(x+y)} = \frac{dz}{3z}$$

This leads to these two first integrals

$$(x-z+2u)^2 \cdot (x+y)^{-1} = c_1 \quad \text{and} \quad (x+y)^3 \cdot (z)^{-2} = c_2$$

So, we need another first integral. For that, suppose it is possible to find constants $(\alpha_j, \beta_j, \gamma_j, \delta_j)$, $1 \le j \le 4$, and $\rho$ such that each ratio of (2.12) is of the form (2.6), with

$$\begin{aligned} D \ = \ & (\alpha_1+\alpha_2)\, x^2 + (\beta_1+2\beta_2)\, y^2 + (3\gamma_3+\gamma_4)\, z^2 + \delta_4 u^2 + (3\delta_3+\gamma_4)\, zu \\ & + (\gamma_1+\gamma_2+3\alpha_3+\alpha_4)\, xz + (\gamma_1+2\gamma_2+3\beta_3+\beta_4)\, yz \\ & + (\beta_1+\beta_2+2\alpha_2+\alpha_1)\, xy + (\delta_1+2\delta_2+\beta_4)\, yu + (\delta_1+\delta_2+\alpha_4)\, xu \end{aligned}$$

That's right, if

$$\begin{cases} 2\alpha_1+2\alpha_2=0, & \gamma_1+\gamma_2+3\alpha_3+\alpha_4=0 \\ \beta_1+\beta_2+2\alpha_2+\alpha_1=0, & \gamma_1+2\gamma_2+3\beta_3+\beta_4=0 \\ \gamma_1+\gamma_2+3\alpha_3+\alpha_4=0, & 6\gamma_3+2\gamma_4=0 \\ \delta_1+\delta_2+\alpha_4=0, & 3\delta_3+\gamma_4=0 \\ \beta_1+\beta_2+2\alpha_2+\alpha_1=0, & \delta_1+\delta_2+\alpha_4=0 \\ 2\beta_1+4\beta_2=0, & \delta_1+2\delta_2+\beta_4=0 \\ \gamma_1+2\gamma_2+3\beta_3+\beta_4=0, & 3\delta_3+\gamma_4=0 \\ \delta_1+2\delta_2+\beta_4=0, & 2\delta_4=0 \end{cases}$$

This system will have a non-trivial solution, if the determinant of its coefficient matrix is zero, i.e.,

$$\begin{gathered} \rho^{16} - 35\rho^{15} + 541\rho^{14} - 4866\rho^{13} + 28\,240\rho^{12} - 110\,780\rho^{11} + 298\,955\rho^{10} \\ -553\,425\rho^9 + 686\,695\rho^8 - 542\,426\rho^7 + 244\,980\rho^6 - 47\,880\rho^5 = 0 \end{gathered}$$

Among the roots of this polynomial are the following : 1, 2, 3, 6, 0, $\frac{5}{2} \pm \frac{1}{2}\sqrt{5}$, $\frac{9}{2} \pm \frac{1}{2}\sqrt{5}$. Just choose a single value for $\rho$, then we can take $\rho = 2$. If we substitute this value of $\rho$ in (2.6) and solve the resulting system, we find

$$(\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \alpha_3, \beta_3, \gamma_3, \delta_3, \alpha_4, \beta_4, \gamma_4, \delta_4)$$

as following

$$(0,0,0,0,0,0,0,0,0,0,0,1,0,0,2,-1) \ \text{ for } \rho = 2$$

Then system (2.12) becomes

$$\frac{d\,(x-z+2u)}{(x-z+2u)} = \frac{d\,(x+y)}{2\,(x+y)} = \frac{dz}{3z} = \frac{d\left(u^2+zu\right)}{2\left(u^2+zu\right)}$$

which gives us another first integral

$$(x-z+2u)^2 \cdot \left(u^2+zu\right)^{-1} = c_3$$

Hence the integral curves $F(c_1, c_2, c_3) = 0$, where $F$ is an arbitrary real function.

### 2.7. Case (vii): $\rho_1 = \rho_2 \in \mathbb{R}$, $\rho_3 = \overline{\rho_4} \in \mathbb{C}$.

**Example 2.7.** Let the equation be given as

$$(2x-y)\, p + (x+2y+2u)\, q + zr = z+u \tag{2.13}$$

The characteristic system corresponding to equation (2.13) is

$$\frac{dx}{2x-y} = \frac{dy}{x+2y+2u} = \frac{dz}{z} = \frac{du}{z+u} \tag{2.14}$$

The polynomial $\Psi(\rho)$ admits one double root $\rho_1 = \rho_2 = 1$ and two complex roots conjugate $\rho_3 = 2 + i = \overline{\rho_4}$. The constants $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$ associated respectively are $(0, 0, 1, 0)$, $(1 + i, 1 - i, 1 + i, 2)$ and $(1 - i, 1 + i, 1 - i, 2)$. Then, each ratio of (2.14) is equal to

$$\frac{dz}{z} = \frac{d((1-i)x+(1+i)+y(1-i)z+2u)}{(2+i)((1-i)x+(1+i)+y(1-i)z+2u)} = \frac{d((1+i)x+(1-i)+y(1+i)z+2u)}{(2-i)((1+i)x+(1-i)+y(1+i)z+2u)} \tag{2.15}$$

This leads to two first integrals, we can take the following

$$z^2 \exp(-z_1) \cos(1 \log(g) - z_2) = c_1 \quad \text{and} \quad 2z_2 - z_1 = c_2$$

where

$$\begin{aligned} z_1 &= \frac{1}{2} \log\left((x + y + z + 2u)^2 + (-x + y - z)^2\right) \\ z_2 &= \arctan \frac{-x + y - z}{x + y + z + 2u} \end{aligned}$$

In order to obtain an additional first integral, we use the same procedure as in the previous examples. This approach also enables us to obtain other linearly independent first integrals, but we are limited to finding just one. For that, suppose it is possible to find constants $(\alpha_j, \beta_j, \gamma_j, \delta_j)$, $1 \le j \le 4$, and $\rho$ such that each ratio of (2.14) is of the form (2.6), with

$$\begin{aligned} D = {} & (2\alpha_1 + \alpha_2) x^2 + (-\beta_1 + 2\beta_2) y^2 + (2\gamma_2 + \delta_3 + \delta_4 + \gamma_4) zu \\ & + (\gamma_3 + \gamma_4) z^2 + (2\delta_2 + \delta_4) u^2 + (2\beta_1 + \beta_2 + 2\alpha_2 - \alpha_1) xy \\ & + (-\delta_1 + 2\delta_2 + 2\beta_2 + \beta_4) yu + (2\delta_1 + \delta_2 + 2\alpha_2 + \alpha_4) xu \\ & + (-\gamma_1 + 2\gamma_2 + \beta_3 + \beta_4) yz + (2\gamma_1 + \gamma_2 + \alpha_3 + \alpha_4) xz \end{aligned}$$

That's right, if

$$\left\{ \begin{array}{rr} (4-\rho)\alpha_1 + 2\alpha_2 = 0 \ , & 2\gamma_1 + \gamma_2 + (1-\rho)\alpha_3 + \alpha_4 = 0 \\ (2-\rho)\beta_1 + \beta_2 + 2\alpha_2 - \alpha_1 = 0 \ , & -\gamma_1 + 2\gamma_2 + (1-\rho)\beta_3 + \beta_4 = 0 \\ (2-\rho)\gamma_1 + \gamma_2 + \alpha_3 + \alpha_4 = 0 \ , & (2-\rho)\gamma_3 + 2\gamma_4 = 0 \\ (2-\rho)\delta_1 + \delta_2 + 2\alpha_2 + \alpha_4 = 0 \ , & 2\gamma_2 + (1-\rho)\delta_3 + \delta_4 + \gamma_4 = 0 \\ 2\beta_1 + \beta_2 + (2-\rho)\alpha_2 - \alpha_1 = 0 \ , & 2\delta_1 + \delta_2 + 2\alpha_2 + (1-\rho)\alpha_4 = 0 \\ -2\beta_1 + (4-\rho)\beta_2 = 0 \ , & -\delta_1 + 2\delta_2 + 2\beta_2 + (1-\rho)\beta_4 = 0 \\ -\gamma_1 + (2-\rho)\gamma_2 + \beta_3 + \beta_4 = 0 \ , & 2\gamma_2 + \delta_3 + \delta_4 + (1-\rho)\gamma_4 = 0 \\ -\delta_1 + (2-\rho)\delta_2 + 2\beta_2 + \beta_4 = 0 \ , & 4\delta_2 + (2-\rho)\delta_4 = 0 \end{array} \right.$$

This system will have a non-trivial solution, if the determinant of its coefficient matrix is zero, i.e.,

$$\begin{gathered} \rho^{16} - 31\rho^{15} + 438\rho^{14} - 3712\rho^{13} + 20\,908\rho^{12} - 81\,964\rho^{11} + 227\,592\rho^{10} \\ -446\,368\rho^9 + 604\,096\rho^8 - 535\,360\rho^7 + 278\,400\rho^6 - 64\,000\rho^5 = 0 \end{gathered}$$

This polynomial has the following roots : 0, 1, 2, 4, $4 \pm 2i$, $3 \pm i$. Just choose a single value for $\rho$, then we can take $\rho = 1$. substitute it into (2.6) and solve the resulting system, we find

$$(\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \alpha_3, \beta_3, \gamma_3, \delta_3, \alpha_4, \beta_4, \gamma_4, \delta_4)$$

as following

$$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2, 0, 0, 0, 1, 0) \ \text{for} \ \rho = 1$$

Then system (2.15) becomes

$$\frac{d\left(-z^2+zu\right)}{-z^2+zu}=\frac{dz}{z}=\frac{d((1-i)x+(1+i)+y(1-i)z+2u)}{(2+i)((1-i)x+(1+i)+y(1-i)z+2u)}=\frac{d((1+i)x+(1-i)+y(1+i)z+2u)}{(2-i)((1+i)x+(1-i)+y(1+i)z+2u)}$$

which gives us another first integral

$$\left(-z^2+zu\right).z^{-1}=c_3$$

Hence the integral curves $F\left(c_1,c_2,c_3\right)=0$, where $F$ is an arbitrary real function.

**2.8. Case (viii):** $\rho_1=\overline{\rho_2}=\rho_3=\overline{\rho_4}\in\mathbb{C}$**.**

**Example 2.8.** Let the equation be given as

$$(x+y)\,p+(-x+y)\,q+(z+u)\,r=-z+u \tag{2.16}$$

The characteristic system corresponding to equation (2.16) is

$$\frac{dx}{x+y}=\frac{dy}{-x+y}=\frac{dz}{z+u}=\frac{du}{-z+u} \tag{2.17}$$

The polynomial $\Psi\left(\rho\right)$ admits two double conjugate complex roots $\rho_1=1-i=\overline{\rho_2}=\rho_3=\overline{\rho_4}$. The constants $\left(\lambda_j,\mu_j,\upsilon_j,\tau_j\right)$ associated respectively are $(i,1,0,0)$ and $(-i,1,0,0)$. Each ratio of (2.17) is equal to

$$\frac{d\left(ix+y\right)}{\left(1-i\right)\left(ix+y\right)}=\frac{d\left(-ix+y\right)}{\left(1+i\right)\left(-ix+y\right)}$$

This leads to the following first integral

$$z_2-z_1=c_1$$

where

$$z_1=\frac{1}{2}\log\left(y^2+x^2\right)\quad\text{and}\quad z_2=\arctan\frac{x}{y}$$

We use the same procedure as before to obtain two additional first integrals. This approach also enables us to obtain other first integrals, but we are limited to finding only two. For that, suppose it is possible to find constants $\left(\alpha_j,\beta_j,\gamma_j,\delta_j\right)$, $1\leq j\leq 4$, and $\rho$ such that each ratio of (2.17) is of the form (2.6), with

$$\begin{aligned} D \;=\; & \left(\alpha_1-\alpha_2\right)x^2+\left(\beta_1+\beta_2\right)y^2+\left(\delta_3+\gamma_3+\gamma_4-\delta_4\right)zu \\ & +\left(\gamma_3-\gamma_4\right)z^2+\left(\delta_1-\delta_2+\alpha_3+\alpha_4\right)xu+\left(\gamma_1-\gamma_2+\alpha_3-\alpha_4\right)xz \\ & +\left(\delta_1+\delta_2+\beta_3+\beta_4\right)yu+\left(\delta_3+\delta_4\right)u^2+\left(\gamma_1+\gamma_2+\beta_3-\beta_4\right)yz \\ & +\left(\beta_1+\alpha_1-\beta_2+\alpha_2\right)xy \end{aligned}$$

That's right, if

$$\left\{\begin{array}{rr} \left(2-\rho\right)\alpha_1-2\alpha_2=0, & \gamma_1-\gamma_2+\left(1-\rho\right)\alpha_3-\alpha_4=0 \\ \left(1-\rho\right)\beta_1+\alpha_1-\beta_2+\alpha_2=0, & \gamma_1+\gamma_2+\left(1-\rho\right)\beta_3-\beta_4=0 \\ \left(1-\rho\right)\gamma_1-\gamma_2+\alpha_3-\alpha_4=0, & \left(2-\rho\right)\gamma_3-2\gamma_4=0 \\ \left(1-\rho\right)\delta_1-\delta_2+\alpha_3+\alpha_4=0, & \left(1-\rho\right)\delta_3+\gamma_3+\gamma_4-\delta_4=0 \\ \beta_1+\alpha_1-\beta_2+\left(1-\rho\right)\alpha_2=0, & \delta_1-\delta_2+\alpha_3+\left(1-\rho\right)\alpha_4=0 \\ 2\beta_1+\left(2-\rho\right)\beta_2=0, & \delta_1+\delta_2+\beta_3+\left(1-\rho\right)\beta_4=0 \\ \gamma_1+\left(1-\rho\right)\gamma_2+\beta_2-\beta_4=0, & \delta_3+\gamma_3+\left(1-\rho\right)\gamma_4-\delta_4=0 \\ \delta_1+\left(1-\rho\right)\delta_2+\beta_3+\beta_4=0, & 2\delta_3+\left(2-\rho\right)\delta_4=0 \end{array}\right.$$

This system will have a non-trivial solution, if the determinant of its coefficient matrix is zero, i.e.,

$$\begin{gathered}\rho^{16}-20\rho^{15}+193\rho^{14}-1173\rho^{13}+4957\rho^{12}-15\,232\rho^{11}+34\,650\rho^{10}\\ -58\,344\rho^{9}+71\,512\rho^{8}-61\,504\rho^{7}+34\,688\rho^{6}-11\,264\rho^{5}+1536\rho^{4}=0\end{gathered}$$

Among the roots of this polynomial, we take 1 and 2, and we substitute them in (2.6). This allows us to find the constants

$$(\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha_2,\beta_2,\gamma_2,\delta_2,\alpha_3,\beta_3,\gamma_3,\delta_3,\alpha_4,\beta_4,\gamma_4,\delta_4)$$

as following

$$\begin{aligned}(0,0,1,1,0,0,-1,1,0,-2,0,0,1,1,0,0)\ \text{for}\ \rho &= 1\\ (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1)\ \text{for}\ \rho &= 2\end{aligned}$$

Each ratio of (2.17) is equal to

$$\frac{d\,(xz+xu+-3yz+yu)}{1\,(xz+xu+-3yz+yu)}=\frac{d\left(u^2-zu\right)}{2\,(u^2-zu)}=\frac{d\,(-ix+y)}{(1+i)\,(-ix+y)}=\frac{d\,(ix+y)}{(1-i)\,(ix+y)}$$

which gives us the following first integrals

$$\begin{aligned}(xz+xu+-3yz+yu)^2\cdot\left(u^2-zu\right)^{-1} &= c_2\\ \left(u^2-zu\right)\cdot e^{-2z_1}\cos\left(\log\left(u^2-zu\right)-2z_2\right) &= c_3\end{aligned}$$

Hence the integral curves $F\,(c_1,c_2,c_3)=0$, where $F$ is an arbitrary real function.

**2.9. Case (ix):** $\rho_1=\rho_2=\rho_3=\rho_4\in\mathbb{R}$.

**Example 2.9.** Let the equation be given as

$$(2x)\,p+(2y)\,q+(2y+2z)\,r=-x+z+2u \tag{2.18}$$

The characteristic system corresponding to equation (2.18) is

$$\frac{dx}{2x}=\frac{dy}{2y}=\frac{dz}{2y+2z}=\frac{du}{-x+z+2u} \tag{2.19}$$

The polynomial $\Psi\,(\rho)$ admits one quadruple root $\rho=2$. The constants $(\lambda,\mu,\upsilon,\tau)$ associated respectively are $(1,-1,0,0)$. Each ratio of (2.19) is equal to

$$\frac{d\,(x-y)}{2\,(x-y)}$$

We use the same procedure as in the previous example to obtain other first integrals. For this, suppose that it is possible to find constants $(\alpha_j,\beta_j,\gamma_j,\delta_j)$, $1\le j\le 4$, and $\rho$ such that each ratio of (2.19) is of the form (2.6), with

$$\begin{aligned}D \;=\;& (2\alpha_1-\alpha_4)\,x^2+(2\gamma_3+\gamma_4)\,z^2+(\delta_4+2\delta_3+2\gamma_4)\,zu+(2\delta_1-\delta_4+2\alpha_4)\,xu\\ &+(2\beta_1+2\alpha_2+2\alpha_3-\beta_4)\,xy+(2\gamma_1+2\alpha_3-\gamma_4+\alpha_4)\,xz+(2\delta_4)\,u^2\\ &+(2\gamma_3+\beta_4+2\beta_3+2\gamma_2)\,yz+(2\delta_3+2\delta_2+2\beta_4)\,yu+(2\beta_2+2\beta_3)\,y^2\end{aligned}$$

That's right, if

$$\begin{cases} (4-\rho)\,\alpha_1 - 2\alpha_4 = 0, & 2\gamma_1 + (2-\rho)\,\alpha_3 - \gamma_4 + \alpha_4 = 0 \\ (2-\rho)\,\beta_1 + 2\alpha_2 + 2\alpha_3 - \beta_4 = 0, & 2\gamma_3 + \beta_4 + (2-\rho)\,\beta_3 + 2\gamma_2 = 0 \\ (2-\rho)\,\gamma_1 + 2\alpha_3 - \gamma_4 + \alpha_4 = 0, & (4-\rho)\,\gamma_3 + 2\gamma_4 = 0 \\ (2-\rho)\,\delta_1 - \delta_4 + 2\alpha_4 = 0, & \delta_4 + (2-\rho)\,\delta_3 + 2\gamma_4 = 0 \\ 2\beta_1 + (2-\rho)\,\alpha_2 + 2\alpha_3 - \beta_4 = 0, & 2\delta_1 - \delta_4 + (2-\rho)\,\alpha_4 = 0 \\ (4-\rho)\,\beta_2 + 4\beta_3 = 0, & 2\delta_3 + 2\delta_2 + (2-\rho)\,\beta_4 = 0 \\ 2\gamma_3 + \beta_4 + 2\beta_3 + (2-\rho)\,\gamma_2 = 0, & \delta_4 + 2\delta_3 + (2-\rho)\,\gamma_4 = 0 \\ 2\delta_3 + (2-\rho)\,\delta_2 + 2\beta_4 = 0, & (4-\rho)\,\delta_4 = 0 \end{cases}$$

This system will have a non-trivial solution, if the determinant of its coefficient matrix is zero, i.e.,

$$\begin{gathered}\rho^{16} - 40\rho^{15} + 720\rho^{14} - 7676\rho^{13} + 53\,632\rho^{12} - 256\,256\rho^{11} + 845\,824\rho^{10} \\ -1894\,400\rho^{9} + 2719\,744\rho^{8} - 2162\,688\rho^{7} + 524\,288\rho^{6} + 262\,144\rho^{5} = 0\end{gathered}$$

Among the roots of this polynomial, we take $\rho = 4$, and we substitute it in (2.6), This allows us to find the constants

$$(\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \alpha_3, \beta_3, \gamma_3, \delta_3, \alpha_4, \beta_4, \gamma_4, \delta_4)$$

as following

$$\begin{gathered}(1,1,0,0,1,1,0,0,0,0,0,0,0,0,0,0) \\ (1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\ (0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0)\end{gathered}$$

These values would transform system (2.19) into the following exact differential form

$$\frac{d\left(2x^2+2y^2+4xy\right)}{4\left(2x^2+2y^2+4xy\right)} = \frac{d\left(2x^2\right)}{4\left(2x^2\right)} = \frac{d\left(4xy\right)}{4\left(4xy\right)} = \frac{d\left(x-y\right)}{2\left(x-y\right)}$$

which gives us these linearly independent first integrals

$$\begin{aligned} \left(2x^2+2y^2+4xy\right)\cdot\left(2x^2\right)^{-1} &= c_1 \\ \left(2x^2\right)\cdot\left(4xy\right)^{-1} &= c_2 \\ \left(4xy\right)\cdot\left(x-y\right)^{-2} &= c_3 \end{aligned}$$

Hence the integral curves $F(c_1, c_2, c_3) = 0$, where $F$ is an arbitrary real function.

## 3. Conclusion

Our paper introduces a versatile approach that can effectively solve a wide range of first order quasilinear equations, as described within the paper. The method's relevance transcends practical applications, encompassing various mathematical equations exhibiting quasilinearity. Through providing a comprehensive framework, our research establishes a solid groundwork for addressing such equations and ensuring their successful resolution, irrespective of their immediate real-world significance. The broad scope and applicability of our method highlight its importance and potential in advancing the field of mathematical analysis. As a practical application, we can use it to determine the surfaces orthogonal to a given system of surfaces and to solve the Hamilton-Jacobi equation, which is of great importance as a first order partial differential equation in mathematical physics. Additionally, it is worth noting that our methodology can be extended to tackle equations with two variables of the same type. This extension finds relevance in several renowned examples, including the transport equation, Maxwell's equation, and others.

In conclusion, we emphasize the challenging nature of unsolved PDEs and the need for fresh insights and innovative concepts to address them. We assert that the field of PDEs is rich with unresolved equations awaiting breakthroughs. In the near future, we will endeavor to study some classes of first order nonlinear PDEs.

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