Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 50, Number 1, 2024, Pages 78–95
https://doi.org/10.30546/2409-4994.2024.50.1.78

THE CAUCHY PROBLEM FOR THE MODIFIED KORTEWEG-DE VRIES-LIOUVILLE (MKDV-L) EQUATION WITH AN ADDITIONAL TERM IN THE CLASS OF PERIODIC INFINITE-GAP FUNCTIONS

AKNAZAR KHASANOV, ULUGHBEK KHUDAYOROV, AND TEMUR KHASANOV

Abstract. In this paper, the inverse spectral problem method is used to integrate a modified Korteweg-de Vries-Liouville (mKdV-L) equation with an additional term in the class of periodic infinite-gap functions. The evolution of the spectral data of the periodic Dirac operator is introduced, and the coefficient of the Dirac operator is a solution for a modified Korteweg-de Vries-Liouville equation with an additional term. A simple algorithm for deriving the Dubrovin system of differential equations is proposed. The solvability of the Cauchy problem for a Dubrovin infinite system of differential equations in the class of six times continuously differentiable periodic infinite-gap functions is proven. It is proven that there is a global solution of the Cauchy problem for a modified Korteweg-de Vries-Liouville equation with an additional term for sufficiently smooth initial conditions.

1. Introduction

In this paper, we consider the Cauchy problem for a combination of the modified Korteweg-de Vries and Liouville equation (mKdV-L), with an additional term of the form

$$q_{xt} = a(t)\left\{q_{xxxx} - \frac{3}{2}q_x^2 q_{xx}\right\} + b(t)e^q - c(t)q_{xx}, \quad q = q(x,t),\ x \in \mathbb{R},\ t > 0 \quad (1.1)$$

with initial condition

$$q(x,t)|_{t=0} = q_0(x), \quad q_0(x+\pi) = q_0(x) \in C^6(\mathbb{R}) \quad (1.2)$$

in the class of real infinite-gap π-periodic functions with respect to x:

$$q(x+\pi,t) = q(x,t), \quad q(x,t) \in C^{4,1}_{x,t}(t>0) \cap C(t \geq 0). \quad (1.3)$$

Here $a(t)$, $b(t)$, $c(t) \in C\left([0,\infty)\right)$ are given continuous bounded functions.

2010 *Mathematics Subject Classification.* 35J10, 34A55.
Key words and phrases. modified Korteweg-de Vries-Liouville (mKdV-L) equation with additional terms, Dirac operator, spectral data, Dubrovin system of equations, trace formulas.

It is easy to verify that the compatibility conditions for the following linear equations

$$y_x = \begin{pmatrix} \frac{1}{2}q_x & -\lambda \\ \lambda & -\frac{1}{2}q_x \end{pmatrix} y, \ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

$$y_t = \left\{ a(t) \begin{pmatrix} A & B \\ C & -A \end{pmatrix} + \frac{b(t)}{2\lambda} \begin{pmatrix} 0 & e^q \\ 0 & 0 \end{pmatrix} + c(t) \begin{pmatrix} -\frac{1}{2}q_x & \lambda \\ -\lambda & \frac{1}{2}q_x \end{pmatrix} \right\} y,$$

are equivalent to the equation (1.1) for the function $q = q(x,t), x \in \mathbb{R}, t > 0$. Here

$$A = -2\lambda^2 q_x - \frac{1}{4}q_x^3 + \frac{1}{2}q_{xxx}, B = \lambda q_{xx} + 4\lambda^3 + \frac{\lambda}{2}q_x^2, C = \lambda q_{xx} - 4\lambda^3 - \frac{\lambda}{2}q_x^2.$$

Note that from the equation (1.1), for the case $a(t) = 1$, $b(t) = 0$, $c(t) = 0$ we obtain the modified Korteweg-de Vries equation (mKdV) [17], [45] and for $a(t) = 0$, $b(t) = 1$, $c(t) = 0$ we have the Liouville equations (see [15], p. 14 and [16]). In this paper, we propose an algorithm for constructing exact solutions $q(x,t)$, $x \in \mathbb{R}$, $t > 0$, of the (1.1)-(1.3) by reducing to the inverse spectral problem for the following Dirac operator:

$$\mathfrak{L}(\tau,t)y \equiv B\frac{dy}{dx} + \Omega(x+\tau,t)y = \lambda y, \ x \in \mathbb{R}, \ t > 0, \ \tau \in \mathbb{R} \tag{1.4}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \Omega(x,t) = \begin{pmatrix} P(x,t) & Q(x,t) \\ Q(x,t) & -P(x,t) \end{pmatrix},$$

$$P(x,t) = 0, Q(x,t) = \frac{1}{2}q_x'(x,t).$$

It is well known that finding an explicit formula for solving the nonlinear evolutionary Korteweg-de Vries equation (KdV), the modified Korteweg-de Vries equation (mKdV), the nonlinear Schrodinger equation (NSE), the sine-Gardon equation(s-G), the Hirota equation and etc. in the class of periodic functions essentially depends on the number of nontrivial gaps in the spectrum of the periodic Sturm-Liouville and Dirac operator. In this regard, the class of periodic functions is conveniently divided into two sets:

1. The class of periodic finite-gap functions;
2. The class of periodic infinite-gap functions.

The complete integrability of nonlinear evolution equations (KdV, NSE, mKdV, s-G, Hirota) in the class of finite-gap periodic and quasi-periodic functions was studied using the method of the inverse spectral problem for the Sturm-Liouville and Dirac operator with a periodic potential, when the spectrum has only a finite number of non-trivial gaps, in the works of Its-Matveev [12], Dubrovin-Novikov [8], Its-Kotlyarov[13], Its [14], Smirnov [42], Matveev-Smirnov [33], [34]. In addition, an explicit formula was derived in terms of the Riemann theta functions for finite-gap solutions of nonlinear evolution equations (KdV, NSE, mKdV, s-G, Hirota, etc.).

Thus, in these papers (see [8],[12], [13], [14], [33], [34] and [42], the solvability of the Cauchy problem for nonlinear evolution equations (KdV, NSE, mKdV, s-G, etc.) was proved for any finite-gap periodic and quasi-periodic initial data. This theory is described in more detail in the monographs [38] and [46], as well as in the works [24], [28], [32].

It is known from [11] that if $q(x) = 2a\cos 2x,\ a \neq 0$, then all gaps are open in the spectrum of the Sturm-Liouville operator $\mathfrak{L}y \equiv -y'' + q(x)y,\ x \in \mathbb{R}$, in other words, $q(x)$ is a periodic infinite-gap potential. There are similar examples for the periodic Dirac operator in [6].

A method for constructing exact solutions of the Cauchy problem for combining the defocusing nonlinear Schrodinger equation and the complex modified Korteweg-de Vries equation (DNSE-cmKdV),

$$iq_t + b(t)(q_{xx} - 2|q|^2 q) - ia(t)(q_{xxx} - 6|q|^2 q_x) = 0,\ a(t), b(t) \in C[0,\infty), x \in \mathbb{R},\ t > 0.$$

$$q(x,t)|_{t=0} = q_0(x),\ q_0(x+\pi) = q_0(x) \in C^6(\mathbb{R})$$

in the class of π-periodic infinite-gap functions was proposed in [31]. Based on the ideas of [31], the solvability of the Cauchy problem for the mKdV-sG equation of the following form

$$q_{xt} = a(t)\left\{q_{xxxx} - \frac{3}{2}q_x^2 q_{xx}\right\} + b(t)chq,\ \ q = q(x,t),\ x \in \mathbb{R},\ t > 0,$$

$$q(x,t)|_{t=0} = q_0(x),\ \ q_0(x+\pi) = q_0(x) \in C^6(\mathbb{R})$$

in the class of π-periodic infinite-gap functions was studied in [21]. Here $a(t), b(t) \in C([0,\infty))$ are given continuous bounded functions.

This statement of the problem has not been studied before for equation (1.1).

We note that the Cauchy problem for nonlinear evolution equations without a source and with a source, as well as with an additional term in various formulations in the class of periodic and almost periodic infinite-gap functions was studied in [2], [7], [19], [20], [22] and [27], [29], [35], [36], [39].

2. Evolution of spectral data

Denote by $c(x,\lambda,\tau,t) = (c_1(x,\lambda,\tau,t), c_2(x,\lambda,\tau,t))^T$ and $s(x,\lambda,\tau,t) = (s_1(x,\lambda,\tau,t), s_2(x,\lambda,\tau,t))^T$ solutions of the equation (1.4) with initial conditions $c(0,\lambda,\tau,t) = (1,0)^T$ and $s(0,\lambda,\tau,t) = (0,1)^T$. Function

$$\Delta(\lambda,\tau,t) = c_1(\pi,\lambda,\tau,t) + s_2(\pi,\lambda,\tau,t)$$

is called the Lyapunov function for the equation (1.4).

Moreover, for solutions $c(x,\lambda,\tau,t)$ and $s(x,\lambda,\tau,t)$ for large $|\lambda|$ the following asymptotics hold:

$$c(x,\lambda,\tau,t) = \begin{pmatrix} \cos\lambda x \\ \sin\lambda x \end{pmatrix} +$$

$$+\frac{1}{2\lambda}\begin{pmatrix} \frac{1}{2}[q'_x(x+\tau,t) + q'_x(\tau,t)]\sin\lambda x + a(x,\tau,t)\sin\lambda x \\ -\frac{1}{2}[q'_x(x+\tau,t) - q'_x(\tau,t)]\cos\lambda x - a(x,\tau,t)\cos\lambda x \end{pmatrix} + O\left(\frac{1}{\lambda^2}\right),\ |\lambda| \to \infty,$$

$$s(x,\lambda,\tau,t) = \begin{pmatrix} -\sin\lambda x \\ \cos\lambda x \end{pmatrix} +$$

$$+\frac{1}{2\lambda}\begin{pmatrix} \frac{1}{2}[q'_x(x+\tau,t) - q'_x(\tau,t)]\cos\lambda x + a(x,\tau,t)\cos\lambda x \\ -\frac{1}{2}[q'_x(x+\tau,t) + q'_x(\tau,t)]\sin\lambda x + a(x,\tau,t)\sin\lambda x \end{pmatrix} + O\left(\frac{1}{\lambda^2}\right),\ |\lambda| \to \infty,$$

where

$$a(x,\tau,t) = \frac{1}{4}\int_0^x \left[q'_s(s+\tau,t)\right]^2 ds.$$

From these asymptotics for real λ, we have

$$\Delta(\lambda,\tau,t)=2\cos\lambda\pi+\frac{1}{\lambda}a(\pi,\tau,t)\sin\lambda\pi+O\left(\frac{1}{\lambda^2}\right),\ |\lambda|\to\infty,$$

$$\Delta^2(\lambda,\tau,t)-4=-4\sin^2\lambda\pi+\frac{4a(\pi,\tau,t)}{\lambda}\cos\lambda\pi\sin\lambda\pi+O\left(\frac{1}{\lambda^2}\right),\ |\lambda|\to\infty.$$

Vector functions

$$\psi^{\pm}(x,\lambda,\tau,t)=(\psi_1^{\pm}(x,\lambda,\tau,t),\psi_2^{\pm}(x,\lambda,\tau,t))^T=c(x,\lambda,\tau,t)+m^{\pm}(\lambda,\tau,t)s(x,\lambda,\tau,t)$$

are called Floquet solutions of the equation (1.4). The Weyl-Titchmarsh functions are defined by the following formulas [5]

$$m^{\pm}(\lambda,\tau,t)=\frac{s_2(\pi,\lambda,\tau,t)-c_1(\pi,\lambda,\tau,t)\mp\sqrt{\Delta^2(\lambda,\tau,t)-4}}{2s_1(\pi,\lambda,\tau,t)}.$$

The spectrum of the Dirac operator $\mathfrak{L}(\tau,t)$ is purely continuous and consists of the following set

$$\sigma\left(\mathfrak{L}\right)=\{\lambda\in\mathbb{R}:\ |\Delta(\lambda)|\le 2\}=\mathbb{R}\backslash\left(\bigcup_{n=-\infty}^{\infty}(\lambda_{2n-1},\,\lambda_{2n})\right).$$

The intervals $(\lambda_{2n-1},\lambda_{2n}),\ n\in\mathbb{Z}\backslash\{0\}$, are called gaps, where λ_n are the roots of the equation $\Delta(\lambda)\mp 2=0$. They coincide with the eigenvalues of the periodic or antiperiodic $(y(0)=\pm y(\pi))$ problem for the equation (1.4). It is easy to prove that $\lambda_{-1}=\lambda_0=0$, i.e. $\lambda=0$ is the double eigenvalue of the periodic problem for the equation (1.4).

The roots of the equation $s_1(\pi,\lambda,\tau,t)=0$ will be denoted by $\xi_n(\tau,t),n\in\mathbb{Z}\backslash\{0\}$, while $\xi_n(\tau,t)\in[\lambda_{2n-1},\lambda_{2n}],\ n\in\mathbb{Z}\backslash\{0\}$. Since the coefficient in the equation (1.4) has the form $P(x,t)\equiv 0,\ Q(x,t)=\frac{1}{2}q'_x(x,t)$, then $\lambda_{-1}=\lambda_0=\xi_0=0$, i.e. $\xi=0$ is an eigenvalue of the Dirichlet problem.

Numbers $\xi_n(\tau,t),n\in\mathbb{Z}\backslash\{0\}$ and $\sigma_n(\tau,t)=sgn\{s_2(\pi,\xi_n,\tau,t)-c_1(\pi,\xi_n,\tau,t)\}$, $n\in\mathbb{Z}\backslash\{0\}$, are called the spectral parameters of the operator $\mathfrak{L}(\tau,t)$. Spectral parameters $\xi_n(\tau,t),\sigma_n(\tau,t)=\pm1,\ n\in\mathbb{Z}\backslash\{0\}$, and the spectrum boundaries $\lambda_n(\tau,t),n\in\mathbb{Z}\backslash\{0\}$, are called the spectral data of the Dirac operator $\mathfrak{L}(\tau,t)$.

The problem of recovering the coefficient $\Omega(x,t)$ of the operator $\mathfrak{L}(\tau,t)$ from spectral data is called the *inverse problem.*

Using the initial function $q_0(x+\tau),\ \tau\in\mathbb{R}$, we construct the Dirac operator $\mathfrak{L}(\tau,0)$ of the following form

$$\mathfrak{L}(\tau,0)y\equiv B\frac{dy}{dx}+\Omega_0(x+\tau)y=\lambda y,\ x\in\mathbb{R},\ \tau\in\mathbb{R}$$

$$\Omega_0(x+\tau)=\begin{pmatrix}0 & \frac{1}{2}q'_0(x+\tau)\\ \frac{1}{2}q'_0(x+\tau) & 0\end{pmatrix},\ y=\begin{pmatrix}y_1\\ y_2\end{pmatrix},$$

then we will see that the boundaries of the spectrum $\lambda_n(\tau),n\in\mathbb{Z}$, of the obtained problem do not depend on the parameter $\tau\in\mathbb{R}$, i.e. $\lambda_n(\tau)=\lambda_n,n\in\mathbb{Z}$, and the spectral parameters depend on the parameter τ: $\xi_n^0=\xi_n^0(\tau),\sigma_n^0=\sigma_n^0(\tau)=\pm1,n\in\mathbb{Z}$, and they are periodic functions: $\xi_n^0(\tau+\pi)=\xi_n^0(\tau),\sigma_n^0(\tau+\pi)=\sigma_n^0(\tau),\tau\in\mathbb{R},n\in\mathbb{Z}$.

Solving the direct problem, find the spectral data $\left\{\lambda_n, \xi_n^0(\tau), \sigma_n^0(\tau), n \in \mathbb{Z}\setminus\{0\}\right\}$ of the operator $\mathfrak{L}(\tau,0)$.

The inverse problem for the Dirac operator of the following form

$$Ly \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda y, x \in \mathbb{R}$$

with periodic coefficients $p(x) = p(x+\pi), q(x) = q(x+\pi)$, were studied in various formulations in [3], [4], [10], [18], [23], [25], [26], [30], [37], [40], [41]. It should be noted that the inverse problem in terms of the spectral data $\{\lambda_n,\ \xi_n,\ \sigma_n,\ n \geq 1\}$ for the Hill operator was studied in [1], [9] and [43], [44].

3. Main results

The main result of this paper is contained in the following theorem.

Theorem 3.1. *Let* $q(x,t),\ x \in \mathbb{R},\ t > 0$, *function be a solution of the Cauchy problem* (1.1)-(1.3). *Then the boundaries of the spectrum* $\lambda_n(\tau,t), n \in \mathbb{Z}\setminus\{0\}, \tau \in \mathbb{R}$, *of the operator* $\mathfrak{L}(\tau,t)$ *do not depend on the parameters* τ *and* t, *i.e.* $\lambda_n(\tau,t) = \lambda_n, n \in \mathbb{Z}\setminus\{0\}$, *and the spectral parameters* $\xi_n(\tau,t),\ \sigma_n(\tau,t) = \pm 1, n \in \mathbb{Z}\setminus\{0\}$, *satisfy the first and second systems of Dubrovin differential equations, respectively:*

$$1. \begin{cases} \frac{\partial \lambda_n(\tau,t)}{\partial \tau} = 0, \quad n \in \mathbb{Z}\setminus\{0\}, \\ \frac{\partial \xi_n(\tau,t)}{\partial \tau} = 2(-1)^{n-1}\sigma_n(\tau,t)h_n\left(\xi(\tau,t)\right)\xi_n(\tau,t), \quad n \in \mathbb{Z}\setminus\{0\}; \end{cases} \tag{3.1}$$

$$2. \begin{cases} \frac{\partial \lambda_n(\tau,t)}{\partial t} = 0, \quad n \in \mathbb{Z}\setminus\{0\}, \\ \frac{\partial \xi_n(\tau,t)}{\partial t} = 2(-1)^{n}\sigma_n(\tau,t)h_n(\xi(\tau,t))g_n(\xi(\tau,t)), \ n \in \mathbb{Z}\setminus\{0\}. \end{cases} \tag{3.2}$$

Here the signs $\sigma_n(\tau,t) = \pm 1,\ n \in \mathbb{Z}\setminus\{0\}$, are reversed for each point collision $\xi_n(\tau,t), n \in \mathbb{Z}\setminus\{0\}$, with the boundaries of its gap $[\lambda_{2n-1}, \lambda_{2n}]$. In addition, the following initial conditions are satisfied

$$\xi_n(\tau,t)|_{t=0} = \xi_n^0(\tau), \quad \sigma_n(\tau,t)|_{t=0} = \sigma_n^0(\tau),\ n \in \mathbb{Z}\setminus\{0\} \tag{3.3}$$

where $\xi_n^0(\tau),\ \sigma_n^0(\tau) = \pm 1,\ n \in \mathbb{Z}\setminus\{0\}$ are spectral parameters of the Dirac operator $\mathfrak{L}(\tau,0)$. The sequences $h_n(\xi)$ and $g_n(\xi),\ n \in \mathbb{Z}\setminus\{0\}$, involved in the equation (3.2) are determined by the following formulas:

$$\begin{gathered} h_n(\xi) = \sqrt{(\xi_n(\tau,t) - \lambda_{2n-1})(\lambda_{2n} - \xi_n(t,\tau))} \times f_n(\xi), \\ f_n(\xi) = \sqrt{\prod_{k=-\infty k\neq n}^{\infty} \frac{(\lambda_{2k-1} - \xi_n(\tau,t))(\lambda_{2k} - \xi_n(\tau,t))}{(\xi_k(\tau,t) - \xi_n(\tau,t))^2}}, \\ g_n(\xi) = a(t)\left\{4\xi_n^3(\tau,t) + \xi_n(\tau,t)\left[\frac{1}{2}q_\tau^2 + q_{\tau\tau}\right]\right\} + \\ + \frac{b(t)}{2\xi_n(\tau,t)}\exp\left\{q(\tau,t)\right\} + c(t)\xi_n(\tau,t). \end{gathered} \tag{3.4}$$

Proof. Let π periodic with respect to x function $q(x,t),\ \ x\in\mathbb{R},\ t>0$ satisfy the equation (1.1). Denote by $y_n=(y_{n,1}(x,\tau,t),y_{n,2}(x,\tau,t))^T,\ n\in\mathbb{Z}\backslash\{0\}$ orthonormal eigenfunctions of the operator $\mathfrak{L}(\tau,t)$ considered on the segment $[0,\pi]$, with Dirichlet boundary conditions

$$y_1(0,\tau,t)=0,y_1(\pi,\tau,t)=0, \tag{3.5}$$

corresponding to eigenvalues $\xi_n=\xi_n(\tau,t),\ n\in\mathbb{Z}\backslash\{0\}$. Differentiating with respect to t the following equality

$$\xi_n(\tau,t)=(\mathfrak{L}(\tau,t)y_n,y_n),\ n\in\mathbb{Z}\backslash\{0\},$$

and using the symmetry operator $\mathfrak{L}(\tau,t)$, we have

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=(\dot{\Omega}(x+\tau,t)y_n,y_n),\ n\in\mathbb{Z}\backslash\{0\}. \tag{3.6}$$

Using the following explicit form of the dot product

$$(y,z)=\int_0^\pi[y_1(x)\overline{z_1(x)}+y_2(x)\overline{z_2(x)}]dx,\ y=\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix},\ z=\begin{pmatrix} z_1(x)\\ z_2(x)\end{pmatrix},$$

we rewrite the equality (3.6) in the following form

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=\int_0^\pi y_{n,1}y_{n,2}q_{xt}dx. \tag{3.7}$$

Substituting the expressions (1.1) into (3.7), we obtain the following equality

$$\frac{\partial\xi_n\left(\tau,t\right)}{\partial t}=a(t)I_1(\tau,t)+b(t)I_2(\tau,t)+c(t)I_3(\tau,t) \tag{3.8}$$

where

$$I_1(\tau,t)=\int_0^\pi\left[y_{n,1}y_{n,2}\left\{q_{xxxx}(x+\tau,t)-\frac{3}{2}q_x^2(x+\tau,t)q_{xx}(x+\tau,t)\right\}\right]dx, \tag{3.9}$$

$$I_2(\tau,t)=\int_0^\pi\left[y_{n,1}y_{n,2}e^{q(x+\tau,t)}\right]dx, \tag{3.10}$$

$$I_3(\tau,t)=-\int_0^\pi\left[y_{n,1}y_{n,2}q_{xx}\right]dx. \tag{3.11}$$

Using the following equalities

$$\begin{cases} y_{n,1}(x,\tau,t)=\frac{1}{\xi_n(\tau,t)}\left(y'_{n,2}(x,\tau,t)+\frac{1}{2}q'_x(x+\tau,t)y_{n,2}(x,\tau,t)\right), \\ y_{n,2}(x,\tau,t)=\frac{1}{\xi_n(\tau,t)}\left(-y'_{n,1}(x,\tau,t)+\frac{1}{2}q'_x(x+\tau,t)y_{n,1}(x,\tau,t)\right), \end{cases}$$

it is easy to derive the following equality

$$\begin{aligned} &y_{n,1}y_{n,2}\left\{q_{xxxx}(x+\tau,t)-\frac{3}{2}q_x^2(x+\tau,t)q_{xx}(x+\tau,t)\right\}= \\ &=\left\{\left(\frac{1}{2}\xi_nq_x^2+4\xi_n^3-\xi_nq_{xx}\right)y_{n,1}^2+\right. \\ &\left.+\left(q_{xxx}-\frac{1}{2}q_x^3-4\xi_n^2q_x\right)y_{n,1}y_{n,2}+\left(\frac{1}{2}\xi_nq_x^2+4\xi_n^3+\xi_nq_{xx}\right)y_{n,2}^2\right\}'. \end{aligned}$$

Using this equality in the above, it is easy to calculate $I_1(\tau,t)$:

$$I_1(\tau,t)=\left(\frac{1}{2}\xi_n q_\tau^2(\tau,t)+4\xi_n^3(\tau,t)+\xi_n q_{\tau\tau}(\tau,t)\right)\left[y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)\right]. \tag{3.12}$$

Then calculate $I_2(\tau,t)$ and $I_3(\tau,t)$:

$$I_2(\tau,t)=\int_0^\pi\left[y_{n,1}y_{n,2}e^{q(x+\tau,t)}\right]dx=$$
$$=\frac{1}{\xi_n(\tau,t)}\int_0^\pi\left[y_{n,2}e^{q(x+\tau,t)}\left(y_{n,2}'(x,\tau,t)+\frac{1}{2}q_x'(x+\tau,t)y_{n,2}(x,\tau,t)\right)\right]dx=$$
$$=\frac{1}{2\xi_n(\tau,t)}\left(\int_0^\pi\left[y_{n,2}^2(x,\tau,t)e^{q(x+\tau,t)}\right]'dx\right)=$$
$$=\frac{1}{2\xi_n(\tau,t)}e^{q(\tau,t)}\left[y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)\right],$$

$$I_3(\tau,t)=-\int_0^\pi\left[y_{n,1}y_{n,2}q_{xx}\right]dx=\int_0^\pi\left[\{y_{n,1}'y_{n,2}+y_{n,1}y_{n,2}'\}q_x\right]dx=$$
$$=\xi_n\int_0^\pi\left[\{y_{n,1}^2-y_{n,2}^2\}q_x\right]dx=\xi_n\int_0^\pi\left[y_{n,1}^2+y_{n,2}^2\right]'dx=$$
$$=\xi_n\left[y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)\right],$$

i.e.

$$I_2(\tau,t)=\frac{1}{2\xi_n(\tau,t)}e^{q(\tau,t)}\left[y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)\right], \tag{3.13}$$

$$I_3(\tau,t)=\xi_n\left[y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)\right]. \tag{3.14}$$

Substituting (3.12), (3.13) and (3.14) into (3.8), we have

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=\left[a(t)\left\{4\xi_n^3(\tau,t)+\xi_n(\tau,t)\left[\frac{1}{2}q_\tau^2+q_{\tau\tau}\right]\right\}+\right.$$
$$\left.+\frac{b(t)}{2\xi_n(\tau,t)}\exp\{q(\tau,t)\}+c(t)\xi_n\right]\times\left[y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)\right]. \tag{3.15}$$

Since the eigenvalues $\xi_n=\xi_n(\tau,t)$ of the Dirichlet problem for the equation (1.4) are simple, the following equality holds:

$$y_n(x,\tau,t)=\frac{1}{c_n(\tau,t)}s(x,\xi_n,\tau,t),$$

where

$$c_n^2(\tau,t)=\int_0^\pi[s_1^2(x,\xi_n,\tau,t)+s_2^2(x,\xi_n,\tau,t)]dx=$$
$$=-\frac{\partial s_1(\pi,\xi_n,\tau,t)}{\partial\lambda}\cdot s_2(\pi,\xi_n,\tau,t).$$

Using these equalities, we have

$$y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)=-\frac{s_2(\pi,\xi_n,\tau,t)-\frac{1}{s_2(\pi,\xi_n,\tau,t)}}{\frac{\partial s_1(\pi,\xi_n,\tau,t)}{\partial\lambda}}.$$

Substituting into this relation the following equality

$$s_2(\pi,\xi_n,\tau,t)-\frac{1}{s_2(\pi,\xi_n,\tau,t)}=\sigma_n(\tau,t)\sqrt{\Delta^2(\xi_n)-4},$$

we have

$$y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)=-\frac{\sigma_n(\tau,t)\sqrt{\Delta^2(\xi_n)-4}}{\frac{\partial s_1(\pi,\xi_n,\tau,t)}{\partial\lambda}}. \tag{3.16}$$

Using the known expansions

$$\Delta^2(\lambda)-4=-4\pi^2\prod_{k=-\infty}^{\infty}\frac{(\lambda-\lambda_{2k-1})(\lambda-\lambda_{2k})}{a_k^2},$$

$$s_1(\pi,\lambda,t)=\pi\prod_{k=-\infty}^{\infty}\frac{\xi_k-\lambda}{a_k},$$

where $a_0=1$ and $a_k=k$ at $k\neq 0$, rewrite the equality (3.16) as the following form:

$$y_{n,2}^2(\pi,t)-y_{n,2}^2(0,t)=2(-1)^n\sigma_n(\tau,t)h_n(\xi).$$

Substituting this expression into the equality (3.15), we get (3.2). Similarly, the equality (3.1) can be proved.

If we replace the Dirichlet boundary conditions with periodic ($y(0,t)=y(\pi,t)$) or antiperiodic ($y(0,t)=-y(\pi,t)$) boundary conditions, then instead of equation (3.15), we have

$$\frac{\partial\lambda_n(\tau,t)}{\partial t}=0\,,\text{ i.e. }\lambda_n(\tau,t)=\lambda_n(\tau,0),\ \ n\in\mathbb{Z}\setminus\{0\}.$$

Now put $t=0$, in the equation $\mathfrak{L}(\tau,t)\nu_n=\lambda_n(\tau,t)\nu_n,\ \ n\in\mathbb{Z}\setminus\{0\}$. Since the eigenvalues $\lambda_n(\tau)=\lambda_n(\tau,0),\ \ n\in\mathbb{Z}\setminus\{0\}$, periodic or antiperiodic problem for the equation $\mathfrak{L}(\tau,0)\nu_n=\lambda_n(\tau)\nu_n,\ \ n\in\mathbb{Z}\setminus\{0\}$ do not depend on the parameter $\tau\in\mathbb{R}$, we have $\lambda_n(\tau,t)=\lambda_n(\tau)=\lambda_n,\ \ n\in\mathbb{Z}\setminus\{0\}$.

Theorem 3.1 is proved. ■

Then, taking into account the following trace formulas

$$q_\tau'(\tau,t)=2\sum_{k=-\infty,k\neq 0}^{\infty}(-1)^{k-1}\sigma_k(\tau,t)\,h_k(\xi(\tau,t)), \tag{3.17}$$

$$q(\tau,t)=C(t)+2\int_0^\tau\left(\sum_{k=-\infty,\,k\neq 0}^{\infty}(-1)^{k-1}\sigma_k(s,t)\,h_k(\xi(s,t))\right)ds, \tag{3.18}$$

$$\left(\frac{1}{2}q_\tau(\tau,t)\right)^2+\frac{1}{2}q_{\tau\tau}(\tau,t)=\sum_{k=-\infty,k\neq 0}^{\infty}\left(\frac{\lambda_{2k-1}^2+\lambda_{2k}^2}{2}-\xi_k^2(\tau,t)\right), \tag{3.19}$$

where $C(t)$ is some bounded continuous function, the (3.2) system can be rewritten in closed form:

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=2(-1)^n\sigma_n(\tau,t)\sqrt{(\xi_n(\tau,t)-\lambda_{2n-1})(\lambda_{2n}-\xi_n(t,\tau))}f_n(\xi)g_n(\xi), \tag{3.20}$$

where

$$g_n(\xi) = a(t)\left[4\xi_n^3(\tau,t) + 2\xi_n(\tau,t)\sum_{k=-\infty,k\neq 0}^{\infty}\left(\frac{\lambda_{2k-1}^2+\lambda_{2k}^2}{2} - \xi_k^2(\tau,t)\right)\right] +$$
$$+\frac{b(t)}{2\xi_n(\tau,t)}\exp\left\{C(t) + 2\int_0^\tau\left(\sum_{k=-\infty,\,k\neq 0}^{\infty}(-1)^{k-1}\sigma_k(s,t)\,h_k(\xi(s,t))\right)ds+\right.$$
$$\left. + c(t)\xi_n(\tau,t)\right\}. \tag{3.21}$$

Applying the Mittag-Leffler theorem, we have

$$\frac{s_2(\pi,\lambda,\tau,t) - c_1(\pi,\lambda,\tau,t)}{s_1(\pi,\lambda,\tau,t)} =$$
$$= \frac{2}{\lambda}\sum_{k=-\infty,k\neq 0}^{\infty}(-1)^{k-1}\sigma_k(\tau,t)\,h_k(\xi(\tau,t)) + O\left(\frac{1}{\lambda^2}\right),\ |\lambda|\to\infty.$$

On the other hand, using the asymptotics for the solutions $c(x,\lambda,\tau,t)$ and $s(x,\lambda,\tau,t)$ for large $|\lambda|$, we have

$$\frac{s_2(\pi,\lambda,\tau,t) - c_1(\pi,\lambda,\tau,t)}{s_1(\pi,\lambda,\tau,t)} = \frac{1}{\lambda}q'_\tau(\tau,t) + O\left(\frac{1}{\lambda^2}\right),\ |\lambda|\to\infty.$$

Comparing these asymptotics, we obtain the trace formula (3.17).

If $y(x,\tau,t) = (y_1(x,\tau,t), y_2(x,\tau,t))^T$ is a solution of a periodic or antiperiodic problem for the equation (1.4) corresponding to the spectral parameter $\lambda \neq 0$, then $y_1(x,\tau,t)$ is a solution to the following boundary value problems

$$-y_1'' + q_1(x+\tau,t)y_1 = \lambda^2 y_1, y_1(0,\tau,t) = \pm y_1(\pi,\tau,t)$$

where

$$q_1(x,t) = \frac{1}{4}q_x^2(x,t) + \frac{1}{2}q_{xx}(x,t).$$

Since, the function $q_1(x+\tau,t)$ satisfies the following equality

$$q(\tau,t) = \lambda_0^2 + \sum_{k=1}^{\infty}(\lambda_{2k-1}^2 + \lambda_{2k}^2 - 2\xi_k^2(\tau,t)).$$

From this follows the formula (3.19).

As a result of the change of variables

$$\xi_n(\tau,t) = \lambda_{2n-1} + (\lambda_{2n} - \lambda_{2n-1})\sin^2 x_n(\tau,t),\ n\in\mathbb{Z}\setminus\{0\} \tag{3.22}$$

Dubrovin system of differential equations (3.20) and the initial conditions (3.3) can be rewritten as a single equation in the Banach space $\mathbb{K}$:

$$\frac{dx(\tau,t)}{dt} = H(x(\tau,t)),\quad x(\tau,t)|_{t=0} = x^0(\tau)\in\mathbb{K}, \tag{3.23}$$

where

$$\mathbb{K} = \{x(\tau,t) = (..., x_{-1}(\tau,t), x_1(\tau,t), ...) :$$
$$\left.\|x(\tau,t)\| = \sum_{n=-\infty, n\neq 0}^{\infty} (1+|n|)\left(\lambda_{2n}-\lambda_{2n-1}\right)|x_n(\tau,t)| < \infty\right\},$$
$$H(x) = (..., H_{-1}(x),\ H_1(x), ...),$$
$$H_n(x) = (-1)^n \sigma_n^0(\tau)\cdot g_n(...,\ \lambda_1 + (\lambda_2-\lambda_1)\sin^2 x_1(\tau,t),\ ...)\times$$
$$\times f_n(...,\ \lambda_1 + (\lambda_2-\lambda_1)\sin^2 x_1(\tau,t),\ , ...) = (-1)^n\sigma_n^0(\tau) g_n\left(x(\tau,t)\right) f_n\left(x(\tau,t)\right).$$

It is known that if $q_0(x+\pi) = q_0(x) \in C^6(\mathbb{R})$ then $(q_0(x))' \in C^5(\mathbb{R})$. Therefore, for the gap length of the operator $\mathfrak{L}(\tau,0)$, we have the following equality (see [37], p. 98):

$$\gamma_k \equiv \lambda_{2k} - \lambda_{2k-1} = \frac{\left|q_{2k}^5\right|}{2^4|k|^5} + \frac{\delta_k}{|k|^6},$$

where

$$\lambda_{2k}, \lambda_{2k-1} = k + \sum_{j=1}^{6} c_j k^{-j} \pm 2^{-5}|k|^{-5}\left|q_{2k}^5\right| + |k|^{-6}\varepsilon_k^{\pm}, \tag{3.24}$$

$$\sum_{k=-\infty}^{\infty}\left|q_{2k}^5\right|^2 < \infty, \quad \sum_{k=-\infty}^{\infty}(\varepsilon_k^{\pm})^2 < \infty, \quad \delta_k = \varepsilon_k^+ - \varepsilon_k^-.$$

Hence, taking into account $\xi_n(\tau,t) \in [\lambda_{2n-1}, \lambda_{2n}]$, we have

$$\inf_{k\neq n}|\xi_n(\tau,t) - \xi_k(\tau,t)| \geq a > 0.$$

Now, using this inequality and (3.24), we estimate the following functions:

$$|f_n(x(\tau,t))|,\ \left|\tfrac{\partial f_n(x(\tau,t))}{\partial x_m}\right|,\ |g_n(x(\tau,t))|,\ \left|\tfrac{\partial g_n(x(\tau,t))}{\partial x_m}\right|.$$

Lemma 3.1. *The following estimates are valid:*

$$1. C_1 \leq |f_n(x)| \leq C_2, \left|\frac{\partial f_n(x)}{\partial x_m}\right| \leq C_3\gamma_m, \tag{3.25}$$

$$2. |g_n(x)| \leq C_4|n|^3, \left|\frac{\partial g_n(x)}{\partial x_m}\right| \leq C_5\gamma_m|m||n|, m, n \in \mathbb{Z}\backslash\{0\} \tag{3.26}$$

where $C_j > 0, j = 1,2,3,4,5,$ *do not depend on the parameters* m *and* n.

Proof. The inequality (3.25) was proved in [39]. Therefore, we prove the inequalities (3.26). Since the functions $a(t), b(t)$ and $c(t)$ are bounded, then $\exists M_j > 0, j = 1,2,3,$ such that the following inequalities are valid: $|a(t)| \leq M_1$, $|b(t)| \leq M_2$, $|c(t)| \leq M_3$. Using this inequalities and (3.22), we obtain the following estimates:

$$
\begin{aligned}
&|g_n(x(\tau,t)| \le 4M\left|\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)\right|^3 + M|\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)|\times\\
&\times\left|\sum_{k=-\infty,k\neq 0}^{\infty}\left\{\left[\lambda_{2k}+\lambda_{2k-1}+\gamma_k\sin^2 x_k(\tau,t)\right]\gamma_k\cos^2 x_k(\tau,t)\right\}\right|+\\
&+M|\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)|\times\\
&\times\left|\sum_{k=-\infty,k\neq 0}^{\infty}\left\{\left[2\lambda_{2k-1}+\gamma_k\sin^2 x_k(\tau,t)\right]\gamma_k\sin^2 x_k(\tau,t)\right\}\right|+\\
&+\frac{M}{2|\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)|}\times\\
&\times\exp\left\{|C(t)|+\int_0^\tau\left|\sum_{k=-\infty,\,k\neq 0}^{\infty}(-1)^{k-1}\gamma_k\sigma_k^0(s)\sin(2x_k(s,t))f_k\left(x\left(s,t\right)\right)\right|ds\right\}+\\
&+M|\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)| \le 4M(A_1|n|)^3 + MA_1|n|\sum_{k=-\infty,k\neq 0}^{\infty}\{A_2|n|\gamma_k\}+\\
&+MA_1|n|\sum_{k=-\infty,k\neq 0}^{\infty}\{A_3|n|\gamma_k\}+\frac{M}{2A_4|n|}\exp\left\{M_1+\int_0^\tau\sum_{k=-\infty,\,k\neq 0}^{\infty}C_2\gamma_k ds\right\}+\\
&+MA_1|n| \le C_4|n|^3,
\end{aligned}
$$

where $M = max\{M_1, M_2, M_3\}$.

Now, let's estimate the function $\left|\frac{\partial g_n(x(\tau,t))}{\partial x_m}\right|$:

$$
\begin{aligned}
&\left|\frac{\partial g_n(x(\tau,t))}{\partial x_m}\right| \le |a(t)|\left|-2[\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)](\lambda_{2m-1}+\gamma_m\sin^2 x_m(\tau,t))\right|\times\\
&\times\gamma_m\sin^2 x_m(\tau,t)+\frac{|b(t)|}{2|\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)|}\times\\
&\times\exp\left\{|C(t)|+\int_0^\tau\left|\sum_{k=-\infty,\,k\neq 0}^{\infty}(-1)^{k-1}\gamma_k\sigma_k^0(s)\sin(2x_k(s,t))f_k\left(x\left(s,t\right)\right)\right|ds\right\}\times\\
&\times\left[\left|\int_0^\tau(-1)^{m-1}\gamma_m\sigma_m^0(s)\times\right.\right.\\
&\times\left.\left[2\cos(2x_m(s,t))f_m\left(x(s,t)\right)+\sin(2x_m(s,t))\frac{\partial f_m\left(x(s,t)\right)}{\partial x_m(s,t)}\right]ds\right|+\\
&\left.+\int_0^\tau\left|\sum_{k=-\infty,\,k\neq m}^{\infty}(-1)^{k-1}\gamma_k\sigma_k^0(s)\sin(2x_k(s,t))\frac{\partial f_k\left(x(s,t)\right)}{\partial x_m(s,t)}\right|ds\right]\le\\
&\le MB_1|n||m|\gamma_m+\frac{M}{2A_4|n|}\exp\left\{M_1+\int_0^\tau\sum_{k=-\infty,\,k\neq 0}^{\infty}C_2\gamma_k ds\right\}\times
\end{aligned}
$$

$$\times\left\{B_2\gamma_m|\tau| + B_3\gamma_m^2|\tau| + \gamma_m\int_0^\tau \sum_{k=-\infty,\,k\neq m}^{\infty} C_3\gamma_k ds\right\} \leq C_5\gamma_m|n||m|.$$

Lemma 3.1 is proved.

Lemma 3.2. *If* $q_0(x+\pi) = q_0(x) \in C^6(\mathbb{R})$, *then the vector function* $H(x(\tau,t))$ *satisfies the Lipschitz condition in the Banach space* $\mathbb{K}$, *i.e. there exists a constant* $\mathfrak{L} > 0$ *such that the following inequality holds for arbitrary elements* $x(\tau,t),\ y(\tau,t) \in \mathbb{K}$

$$\|H(x(\tau,t)) - H(y(\tau,t))\| \leq L\,\|x(\tau,t) - y(\tau,t)\|$$

where

$$L = C\sum_{n=-\infty,n\neq 0}^{\infty} |n|^3\left(|n|+1\right)\left(\lambda_{2n} - \lambda_{2n-1}\right) < \infty. \tag{3.27}$$

Proof. First, using Lemma 3.1, we estimate the derivative with respect to $x_m(\tau,t)$ of the function $F_n(x) = g_n(x)f_n(x), n \in \mathbb{Z}\setminus\{0\}$:

$$\left|\frac{\partial F_n(x)}{\partial x_m}\right| = \left|\frac{\partial g_n(x)}{\partial x_m}f_n(x) + \frac{\partial f_n(x)}{\partial x_m}g_n(x)\right| \leq \left|\frac{\partial f_n(x)}{\partial x_m}\right||g_n(x)| + \left|\frac{\partial g_n(x)}{\partial x_m}\right||f_n(x)| \leq$$
$$\leq C_3C_4\,|n|^3\,\gamma_m + C_2C_5\,|m|\,|n|\,\gamma_m \leq C\,|n|^3\left(|m|+1\right)\gamma_m$$

where $C = const > 0$ does not depend on m and n.

Next, using the following equality

$$H_n(x(\tau,t)) = (-1)^n\sigma_n^0(\tau)F_n(x(\tau,t)), n \in \mathbb{Z}\setminus\{0\},$$

we have

$$|H_n(x(\tau,t)) - H_n(y(\tau,t))| = |F_n(x(\tau,t)) - F_n(y(\tau,t))|.$$

Now, let's apply Lagrange Mean Value Theorem to the function

$$\varphi(t) = F_n(x + t(y-x)),$$

on the interval $t \in [0,1]$. Then we have the following equality

$$\varphi(1) - \varphi(0) = \varphi'(t^*),$$

i.e.

$$F_n(x) - F_n(y) = \sum_{m=-\infty,m\neq 0}^{\infty}\frac{\partial F_n(\theta)}{\partial x_m}(x_m - y_m),$$

where $\theta = x + t^*(y-x)$. Hence it follows that

$$|H_n\left(x(\tau,t)\right) - H_n\left(y(\tau,t)\right)| = |F_n\left(x(\tau,t)\right) - F_n\left(y(\tau,t)\right)| \leq$$
$$\leq \sum_{m=-\infty,m\neq 0}^{\infty}\left|\frac{\partial F_n(\theta)}{\partial x_m}\right|\cdot|x_m(\tau,t) - y_m(\tau,t)| \leq$$
$$\leq C\,|n|^3\sum_{m=-\infty,m\neq 0}^{\infty}\left(|m|+1\right)\cdot|\lambda_{2m} - \lambda_{2m-1}|\cdot|x_m(\tau,t) - y_m(\tau,t)| =$$
$$= C\,|n|^3\,\|x-y\|.$$

Now let's estimate the norm $\|H(x(\tau,t)) - H(y(\tau,t))\|$:

$$\|H(x) - H(y)\| = \sum_{n=-\infty,n\neq 0}^{\infty} (|n|+1)(\lambda_{2n} - \lambda_{2n-1})\,|H_n(x) - H_n(y)| \leq$$

$$\leq \sum_{n=-\infty,n\neq 0}^{\infty} C\,|n|^3 (|n|+1)(\lambda_{2n} - \lambda_{2n-1}) \cdot \|x-y\| = L\,\|x-y\|\,.$$

where

$$L = \sum_{n=-\infty,n\neq 0}^{\infty} C\,|n|^3 (|n|+1)(\lambda_{2n} - \lambda_{2n-1}) =$$

$$= C \sum_{n=-\infty,n\neq 0}^{\infty} |n|^3 (|n|+1)\,\gamma_n < \infty.$$

Thus, the Lipschitz condition is satisfied. Therefore, the solution of the Cauchy problem (3.2), (3.3) is exists and unique for all $t > 0$ and $\tau \in \mathbb{R}$.

Lemma 3.2 is proved.

Remark 3.1. Theorem 3.1 and Lemma 3.2 give a method for solving the problem (1.1)-(1.3).

Proof. To do this, first find the spectral data λ_n, $\xi_n^0(\tau)$, $\sigma_n^0(\tau) = \pm 1$, $n \in \mathbb{Z}\backslash\{0\}$, of the Dirac operator $\mathfrak{L}(\tau, 0)$. Denote the spectral data of the operator $\mathfrak{L}(\tau,t)$ by λ_n, $\xi_n(\tau,t)$, $\sigma_n(\tau,t) = \pm 1$, $n \in \mathbb{Z}\backslash\{0\}$. Then, solving the Cauchy problem (3.20), (3.3) for an arbitrary value of τ, and find $\xi_n(\tau,t)$, $\sigma_n(\tau,t)$, $n \in \mathbb{Z}\backslash\{0\}$. From the trace formula (3.17) we define the function $q_\tau(\tau,t)$, i.e. solving the problem (1.1)-(1.3).

So far, we have assumed that the Cauchy problem (1.1)-(1.3) has a solution. It is not difficult to get rid of this assumption by directly verifying that the function $q_\tau(\tau,t)$, $\tau \in \mathbb{R}$, $t > 0$ obtained in this way actually satisfies the equations (1.1).

Remark 3.2. The function $q_\tau(\tau,t)$ which is constructed using the Dubrovin system of equations (3.2), (3.3) and the trace formula (3.17) satisfies the equation (1.1).

Proof. To do this, we use the second trace formula

$$\left(\frac{1}{2}q_\tau(\tau,t)\right)^2 + \frac{1}{2}q_{\tau\tau}(\tau,t) = \sum_{k=-\infty,k\neq 0}^{\infty}\left(\frac{\lambda_{2k-1}^2 + \lambda_{2k}^2}{2} - \xi_k^2(\tau,t)\right). \qquad (3.28)$$

Differentiating the formula (3.28) with respect to t, we have

$$\frac{1}{2}q_\tau(\tau,t)\,q_{\tau t}(\tau,t) + \frac{1}{2}\left(q_{\tau t}(\tau,t)\right)_\tau = -\sum_{k=-\infty,k\neq 0}^{\infty} 2\xi_k(\tau,t)\,\frac{\partial \xi_k(\tau,t)}{\partial t}. \qquad (3.29)$$

Here, we used the equality $\left(q_\tau(\tau,t)\right)_{t\tau} = \left(q_\tau(\tau,t)\right)_{\tau t}$. Further, taking into account Dubrovin system of equations (3.2) and equality (3.29), we obtain

$$q_\tau(\tau,t)\,z(\tau,t) + z_\tau(\tau,t) = -8 \sum_{k=-\infty,k\neq 0}^{\infty} (-1)^k \sigma_k(\tau,t) h_k(\xi)\xi_k(\tau,t) \times$$

$$\times\left[a(t)\left\{4\xi_k^3(\tau,t)+\xi_k(\tau,t)\left[\frac{1}{2}q_\tau^2+q_{\tau\tau}\right]\right\}+\right.$$
$$\left.+\frac{b(t)}{2\xi_k(\tau,t)}e^q+c(t)\xi_k(\tau,t)\right],\tag{3.30}$$

where

$$z(\tau,t)=q_{\tau t}(\tau,t).\tag{3.31}$$

Now, we use Dubrovin system of differential equations corresponding to the equation (1.4),

$$\frac{\partial\xi_n(\tau,t)}{\partial\tau}=2(-1)^{n-1}\sigma_n(\tau,t)\,h_n(\xi(\tau,t))\,\xi_n(\tau,t),\tag{3.32}$$

as well as the trace formula (3.17). Then from (3.30), we have the following equality

$$q_\tau(\tau,t)\,z(\tau,t)+z_\tau(\tau,t)=4a(t)\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial\xi_k^4(\tau,t)}{\partial\tau}+$$
$$+2a(t)\left[\frac{1}{2}q_\tau^2+q_{\tau\tau}\right]\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial\xi_k^2(\tau,t)}{\partial\tau}+2b(t)e^{q(\tau,t)}q_\tau(\tau,t)+2c(t)\sum_{k=-\infty}^{\infty}\frac{\partial\xi_k^2(\tau,t)\,(\tau,t)}{\partial\tau}.\tag{3.33}$$

Next, we calculate the sums of the following functional series:

$$\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial\xi_k^4(\tau,t)}{\partial\tau},\quad\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial\xi_k^2(\tau,t)}{\partial\tau}.$$

To do this, we differentiate with respect to τ the trace formulas (3.28), then we have

$$-q_{\tau\tau\tau\tau}(\tau,t)-q_\tau(\tau,t)q_{\tau\tau\tau}(\tau,t)+q_{\tau\tau}(\tau,t)q_\tau^2(\tau,t)+\frac{1}{4}q_\tau^4(\tau,t)=$$
$$=4\sum_{k=-\infty,k\neq 0}^{\infty}\left(\frac{\lambda_{2k}^4+\lambda_{2k-1}^4}{2}-\xi_k^4(\tau,t)\right).\tag{3.34}$$

Thus,

$$q_{\tau\tau\tau\tau\tau}+q_{\tau\tau}q_{\tau\tau\tau}+q_\tau q_{\tau\tau\tau\tau}-q_{\tau\tau\tau}q_\tau^2-2q_{\tau\tau}^2q_\tau-q_\tau^3q_{\tau\tau}=4\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial\xi_k^4}{\partial\tau},$$

$$q_\tau q_{\tau\tau}+q_{\tau\tau\tau}=-2\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial\xi_k^2}{\partial\tau}.$$

Using these formulas from (3.33), we have

$$q_\tau z(\tau,t)+z_\tau(\tau,t)=$$
$$=a(t)\left(q_{\tau\tau\tau\tau\tau}+q_{\tau\tau}q_{\tau\tau\tau}+q_\tau q_{\tau\tau\tau\tau}-q_{\tau\tau\tau}q_\tau^2-2q_{\tau\tau}^2q_\tau-q_\tau^3q_{\tau\tau}\right)-$$
$$-a(t)\left[\frac{1}{2}q_\tau^2+q_{\tau\tau}\right](q_\tau q_{\tau\tau}+q_{\tau\tau\tau})+2b(t)e^qq_\tau,$$

i.e.

$$\begin{gathered} q_\tau z(\tau,t) + z_\tau(\tau,t) = \\ = a(t)\left[q_{\tau\tau\tau\tau\tau} + q_\tau q_{\tau\tau\tau\tau} - \frac{3}{2} q_{\tau\tau\tau} q_\tau^2 - 3q_{\tau\tau}^2 q_\tau - \frac{3}{2} q_\tau^3 q_{\tau\tau}\right] + \\ + 2b(t)e^{q(\tau,t)} q_\tau - c(t) q_\tau q_{\tau\tau} - c(t) q_{\tau\tau\tau}. \end{gathered} \tag{3.35}$$

Here, $q = q(\tau,t), \xi_k = \xi_k(\tau,t)$.

It is easy to check that the function

$$z(\tau,t) = a(t)\left(q_{\tau\tau\tau\tau} - \frac{3}{2} q_\tau^2 q_{\tau\tau}\right) + b(t)e^q - c(t) q_{\tau\tau} + C_1(t)e^{-q}$$

is a solution of the linear equation (3.35). Choosing $C_1(t) = 0$, we have

$$z(\tau,t) = a(t)\left(q_{\tau\tau\tau\tau} - \frac{3}{2} q_\tau^2 q_{\tau\tau}\right) + b(t)e^q - c(t) q_{\tau\tau}.$$

From here and from the notation (3.31), we get the equation (1.1):

$$q_{\tau t}(\tau,t) = a(t)\left\{q_{\tau\tau\tau\tau}(\tau,t) - \frac{3}{2} q_\tau^2(\tau,t) q_{\tau\tau}(\tau,t)\right\} + b(t)e^{q(\tau,t)} - c(t) q_{\tau\tau}(\tau,t).$$

Remark 3.3. The uniform convergence of the series in the above formulas (3.17)-(3.19) and (3.27), as well as (3.34) follow from the equality (3.24) and the estimate (3.25).

Thus, we have established the solvability of the Cauchy problem (1.1)-(1.3) in the class $C^n(\mathbb{R})$, $n \geq 6$, of periodic infinite-gap functions.

Theorem 3.2. *If the initial function $q_0(x)$ satisfies the following condition*

$$q_0(x+\pi) = q_0(x) \in C^n(\mathbb{R}), n \geq 6,$$

then there exists a solution $q'_x(x,t)$, $x \in \mathbb{R}$, $t > 0$ of the problem (1.1)-(1.3)*, which is uniquely is given by the formula* (3.17) *and belongs to the class* $C^{4,1}_{x,t}(t > 0) \cap C(t \geq 0)$.

Corollary 3.1. *Using the results of* [18], [44]*, we deduce that if the initial function $q_0(x)$ is a real analytic π periodic function, then the solution $q(x,t)$ is a real analytic function with respect to x.*

Corollary 3.2. *If the number $\frac{\pi}{2}$ is a period (antiperiod) for the initial function $q_0(x)$, then all roots of the equation $\Delta(\lambda)+2 = 0$ ($\Delta(\lambda)-2 = 0$) are double. Since the Lyapunov function corresponding to the coefficient $q(x,t)$ coincides with $\Delta(\lambda)$, according to the results of* [4], [23] *the number $\frac{\pi}{2}$ is also a period (antiperiod) for the solution $q(x,t)$ with respect to the variable x.*

4. Open Question.

Currently, the problem of the solvability of the Cauchy problem (1.1)-(1.3) in the class $C^n(\mathbb{R})$, $0 \leq n \leq 5$, is open.

References

[1] N.I. Akhiezer, Continuum analog of orthogonal polynomials on a system of intervals *Report USSR Academy of Sciences* **144** (1961), 262–266.

[2] B.A. Babazhanov, A.B. Khasanov, Integration of equation of Toda periodic chain kind *Ufa Mathematical Journal* **9** (2017), 17–24.

[3] D. Battig, B. Grebert, J.C. Guillot, T. Kappeler, Folation of phase space for the cubic non-linear Schrodinger equation *Composito Mathematica* **85** (1993), 163–199.

[4] S. Currie, T. Roth, B. Watson, Borg's periodicity theorems for first-order self-adjoint systems with complex potentials *Proceedings of the Edinburgh Mathematical Society* **60** (2017), 615–633.

[5] A.A. Danielyan, B.M. Levitan, A.B. Khasanov, Asymptotic behavior of Weyel-Titchmarsh m-function in the case of the Dirac system *Mathematical notes of the Academy of Sciences of the USSR volume* **50** (1991), 816–823.

[6] P.B. Djakov, B.S. Mityagin, Instability zones of periodic 1-dimensional Schrödinger and Dirac operators *Russian Mathematical Surveys* **61** (2006), 77–182.

[7] A.V. Domrin, Remarks on the local version of the inverse scattering method *Proceedings of the Steklov Institute of Mathematics volume* **253** (2006), 37–50.

[8] B.A. Dubrovin, S.P. Novikov, Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg-de Vries equation, *Journal of Experimental and Theoretical Physics* **67** (1974), 2131–2143.

[9] H. Flaschka, On the inverse problem for Hill's operator, *Archive for Rational Mechanics and Analysis* **59** (1975), 293–309.

[10] B. Grebert, J.C. Guillot, Gap of one dimensional periodic AKNS systems *Forum mathematicum* **5** (1993), 459–504.

[11] E.L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.

[12] A.R. Its, V.B. Matveev, Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg-de Vries equation, *Theoretical and Mathematical Physics* **23** (1975), 343–355.

[13] A.R. Its, V.P. Kotlyarov, Explicit formulas for solutions of a nonlinear Schrödinger equation, *Report USSR Academy of Sciences* **11** (1976), 965–968.

[14] A.R. Its, Inversion of hyperelliptic integrals and integration of non-linear differential equations. (Russian) *Vestnik Leningradskogo universiteta* **7** (1976), 39–46.

[15] A.V. Jiber, R.D. Murtozina, I.T. Khabibullin, A.B. Shabat, *Characteristic Lie rings and nonlinear integrable equations* (Russian), Nauka, Moscow, 2012.

[16] A.V. Jiber, N.Kh. Ibragimov, A.B. Shabat, Liouville type equations. (Russian) *Report USSR Academy of Sciences* **249** (1979), 26–29.

[17] A.B. Khasanov, U.A. Hoitmetov, On Integration of the Loaded mKdV Equation in the Class of Rapidly Decreasing Functions, *Bulletin of Irkutsk State University, Series Mathematics* **38** (2021), 19–35.

[18] A.B. Khasanov, A.M. Ibragimov, On the inverse problem for the Dirac operator with a periodic potential. (Russian) *Uzbek Mathematical Journal* **3** (2001), 48–55.

[19] A.B. Hasanov, M.M. Hasanov, Integration of the Nonlinear Schrodinger Equation with an Additional Term in the Class of Periodic Functions *Theoretical and Mathematical Physics* **199** (2019), 525–532.

[20] A.B. Khasanov, T.G. Khasanov, Integration of a Nonlinear Korteweg-de Vries Equation with a Loaded Term and a Source *Journal of Applied and Industrial Mathematics* **16** (2022), 227–239.

[21] A.B. Khasanov, Kh.N. Normurodov, U.O. Khudoyorov, Integrating the modified Korteweg–de Vries– sine-Gordon equation in the class of periodic infinite-gap functions *Theoretical and Mathematical Physics* **214** (2023), 170–182.

[22] A.B. Khasanov, A.B. Yakhshimuratov, The Almost-Periodicity of infinite-Gap Potensials of the Dirac operator *Doklady Mathematics* **54** (1996), 767–769.

[23] A.B. Khasanov, A.B. Yakhshimuratov, An analogue of G. Borg's inverse theorem for the Dirac operator. (Russian) *Uzbek Mathematical Journal* **3** (2000), 40–46.

[24] A.B. Khasanov, A.B. Yakhshimuratov, Inverse problem on the half-line for the Sturm-Liouville operator with periodic potential, *Differential equations* **51** (2015), 23–32.

[25] E. Korotayev, Inverse problem and estimates for periodic Zakharov-Shabat systems *Journal für die reine und angewandte Mathematik* **583** (2005), 87–115.

[26] E. Korotayev, D. Mokeev, Dubrovin equation for periodic Dirac operator on the half-line *Applicable Analysis* **101** (2020), 1–29.

[27] P.D. Lax, Almost Periodic Solutions of the KdV Equation *Society for Industrial and Applied Mathematics Review* **18** (1976), 351–375.

[28] B.M. Levitan, A.B. Khasanov, Estimation of the Cauchy function for finite-zone nonperiodic potentials, *Functional Analysis and Its Applications* **26** (1992), 91–98.

[29] B.M. Levitan, Almost Periodicity of infinite-zone potentials *Mathematics of the USSR-Izvestiya* **45** (1981), 291–320.

[30] A. Makin, On periodic and antiperiodic spectra of non-self-adjoint Dirac operators *Journal of Spectral Theory* **12(4)** (2022), 1567–1588.

[31] G.A. Mannonov, A.B. Khasanov, Cauchy problem for the nonlinear Hirota equation in the class of periodic infinite-zone functions, *St. Petersburg Mathematical Journal* **34** (2023), 821–845.

[32] V.B. Matveev, 30 years of finite-gap integration theory, *Philosophical Transactions of The Royal Society A Mathematical Physical and Engineering Sciences* **366** (2018), 837–875.

[33] V.B. Matveev, A.O. Smirnov, Solutions of the Ablowitz-Kaup-Newell-Segur Hierarchy Equations of the "Rogue Wave" Type: A Unified Approach, *Theoretical and Mathematical Physics* **186** (2016), 156–182.

[34] V.B. Matveev, A.O. Smirnov, Two-Phase Periodic Solutions to the AKNS Hierarchy Equations, *Journal of Mathematical Sciences* **242** (2019), 722–741.

[35] H. P. McKean Jr., E. Trubowitz, Hill's operator and hyperelliptic function theory in the presence of infinitely many branchpoints *Communications on Pure and Applied Mathematics* **29** (1976), 143–226.

[36] H. P. Mckean, E. Trubowitz, Hill's surfaces and their theta functions *Bulletin of the American Mathematical Society* **84** (1978), 1052–1085.

[37] T.V. Misura, Characteristics of the spectra of periodic and antiperiodic boundary value problems generated by the Dirac operation I. (in Russian) *Theory of function, functional analysis and their applications.* **30** (1978), 90–101.

[38] Yu.A. Mitropolskiy, N.N. Bogolyubov, A.K. Prikarpatskiy, V.G. Samoylenko, *Integrable dynamical systems: spectral and differential-geometric aspects* (in Russian), Naukova Dumka, Kiev, 1987.

[39] U.B. Muminov, A.B. Khasanov, Integration of a defocusing nonlinear Schrodinger equation with additional terms *Theoretical and Mathematical Physics* **211** (2022), 514–531.

[40] I. M. Nabiev, The inverse periodic problem for the Dirac operator *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **19** (2003), 177–180.

[41] I. M. Nabiev, Solution of the inverse quasiperiodic problem for the Dirac system *Mathematical Notes* **89(6)** (2011), 845–852.

[42] A.O. Smirnov, Elliptic solutions of the nonlinear Schrödinger equation and the modified Korteweg-de Vries equation, *Russian Academy of Sciences. Sbornik: Mathematics* **82** (1995), 461–470.

[43] I.V. Stankeevich, On a problem of spectral analysis for the Hill equation. (Russian) *Report USSR Academy of Sciences* **192** (1970), 34–37.
[44] E. Trubowtz, The inverse problem for periodic potentials *Communications on Pure and Applied Mathematics* **30** (1977), 321–337.
[45] M. Wadati, The exact solution of the modified Korteweg-de Vries equation, *Journal of the Physical Society of Japan* **32** (1972), 44–47.
[46] V.E. Zakharov, S.V. Manakov, S.P. Novikov, L.P. Pitaevskiy, *Soliton Theory: Inverse Problem Method* (Russian), Nauka, Moscow, 1980.

Aknazar Khasanov
Samarkand State University named after Sharof Rashidov, Samarkand, 140104, Uzbekistan
E-mail address: `ahasanov2002@mail.ru`

Ulughbek Khudayorov
Samarkand State Architecture and Construction University, 140147, Samarkand, Uzbekistan
E-mail address: `xudayorov.2022@bk.ru`

Temur Khasanov
Urgench State University, 220100, Urgench, Uzbekistan
E-mail address: `temur.xasanov.2018@mail.ru`

Received: August 25, 2023; Revised: February 27, 2024; Accepted: March 29, 2024