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ONE EXAMPLE OF SINGULAR REPRESENTATIONS OF REAL NUMBERS FROM THE UNIT INTERVAL

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Abstract. In this article, the operator approach for modeling numeral systems, previously introduced by the author of this research, is generalized for a certain case. An example of such numeral systems is introduced and discussed.

1. Introduction

In modern science, the modeling of various numeral systems and their investigation is widely utilized for constructing objects and techniques in applied mathematics, which are used to model real objects and phenomena in economics, physics, computer science, computer security, social sciences, and other fields. It should be remarked that the construction of new representations of real numbers is an important tool for the following areas of science: applications of non-cracking coding mechanisms in cybersecurity, encoding and decoding information, and modeling and studying of "pathological" mathematical objects (Cantor and Moran sets, singular functions, non-differentiable, or nowhere monotonic functions, etc.; the notion of "pathology" in mathematics is explained in [26]). Pathological mathematical objects are widely applied in various areas of science (see [1, 2, 3, 7, 13, 9, 22, 23, 25]).

Finally, it's worth noting that there is a significant amount of research dedicated to modeling and investigating various numeral systems and pathological mathematical objects (for example, see surveys in [2, 3, 5, 12, 6, 8, 4, 10, 21, 5, 11, 15, 16, 20], etc.).

In the present research, certain results from the paper [22] are generalized, and investigations of problems introduced in the last-mentioned paper are initiated.

An approach for modeling new representations of real numbers is based on utilizing specific types of functions through various construction methods. Pathological functions can be employed for modeling expansions of real numbers and vice versa. Additionally, another technique involves modeling Cantor series expansions with non-integer (fractional) bases (see [22]), quasy-nega-representations, etc. (see also [19]).

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The expansions introduced in this paper can be modeled using the singular Salem function and a specific operator for changing digits in the representation. Let's start with the definitions.

Let $q > 1$ be a fixed positive integer. It is well known that any number $x \in [0, 1]$ can be represented by the following form

$$\sum_{k=1}^{\infty} \frac{i_k}{q^k} := \Delta^q_{i_1 i_2 \ldots i_k \ldots} = x,$$

where $i_k \in A \equiv \{0, 1, \ldots, q-1\}$. The last-mentioned representation is called *the q-ary representation* of x.

Let η be a random variable defined by a q-ary expansion

$$\eta = \frac{\xi_1}{q} + \frac{\xi_2}{q^2} + \frac{\xi_3}{q^3} + \cdots + \frac{\xi_k}{q^k} + \cdots := \Delta^q_{\xi_1 \xi_2 \ldots \xi_k \ldots},$$

where digits ξ_k $(k = 1, 2, 3, \ldots)$ are random and taking the values $0, 1, \ldots, q-1$ with probabilities $p_0, p_1, \ldots, p_{q-1}$. That is, ξ_k are independent and $P\{\xi_k = i_k\} = p_{i_k}$, $i_k \in A$. Here $0 < p_t < 1$, $P_q = \{p_0, p_1, \ldots, p_{q-1}\}$, and $\sum_{t=0}^{q} p_t = 1$.

From the definition of distribution function and the expressions

$$\{\eta < x\} = \{\xi_1 < i_1\} \cup \{\xi_1 = i_1, \xi_2 < i_2\} \cup \cdots$$
$$\cdots \cup \{\xi_1 = i_1, \xi_2 = i_2, \ldots, \xi_{k-1} < i_{k-1}\} \cup$$
$$\cup \{\xi_1 = i_1, \xi_2 = i_2, \ldots, \xi_{k-1} = i_{k-1}, \xi_k < i_k\} \cup \ldots,$$
$$P\{\xi_1 = i_1, \xi_2 = i_2, \ldots, \xi_{k-1} = i_{k-1}, \xi_k < \varepsilon_k\} = \beta_{i_k} \prod_{r=1}^{k-1} p_{i_r},$$

it follows that the following is true: the distribution function S_η of the random variable η can be represented in the following form for $x = \Delta^q_{i_1 i_2 \ldots i_k \ldots} \in [0, 1]$:

$$S_\eta(x) = S(x) = \beta_{i_1} + \sum_{k=2}^{\infty} \left(\beta_{i_k} \prod_{r=1}^{k-1} p_{i_r} \right).$$

The last-mentioned function was introduced by Salem in [14] and is called *the Salem function.* This function is an increasing singular function.

Let us consider an analytic representation of the Salem function as an expansion of real numbers from $[0, 1]$, since this function is a defined and continuous on $[0, 1]$. That is,

$$S(x) = \beta_{i_1} + \sum_{k=2}^{\infty} \left(\beta_{i_k} \prod_{r=1}^{k-1} p_{i_r} \right) = \Delta^{P_q}_{i_1 i_2 \ldots i_k \ldots} = x \in [0, 1].$$

The notation $\Delta^{P_q}_{i_1 i_2 \ldots i_k \ldots}$ is called *the singular Salem representation (or P_q-representation) of $x \in [0, 1]$.*

Remark 1.1. It is well-known that for a q-ary expansions, numbers of the form

$$\Delta^q_{i_1 i_2 \ldots i_{m-1} i_m 00\ldots} = \Delta^q_{i_1 i_2 \ldots i_{m-1} i_m (0)} = \Delta^q_{i_1 i_2 \ldots i_{m-1} [i_m - 1][q-1][q-1]\ldots} = \Delta^q_{i_1 i_2 \ldots i_{m-1} [i_m - 1](q-1)}$$

are called *q-rational.* The rest of numbers are *q-irrational* and have the unique q-representation.

Since S is a distribution function, we get that numbers of the form

$$\Delta^{P_q}_{i_1i_2\ldots i_{m-1}i_m(0)}=\Delta^{P_q}_{i_1i_2\ldots i_{m-1}[i_m-1](q-1)}$$

are P_q*-rational.* The rest of numbers are P_q*-irrational* and have the unique P_q-representation.

Let us consider the case when $q=3$. Let θ be an operator defined on digits $\{0,1,2\}$ by the following rule: $\theta(0)=0$, $\theta(1)=2$, and $\theta(2)=1$.

In this paper, the main attention is given to the P_θ-representation of real numbers which related to the P_3-representation by the following rule:

$$\Delta^{P_\theta}_{\gamma_1\gamma_2..\gamma_k\ldots}=\Delta^{P_3}_{\theta(i_1)\theta(i_2)\ldots\theta(i_k)\ldots}:=\beta_{\theta(i_1)}+\sum^{\infty}_{k=2}\left(\beta_{\theta(i_k)}\prod^{k-1}_{r=1}p_{\theta(i_r)}\right), \tag{1.1}$$

where $i_k\in\{0,1,2\}$.

Numbers of the form

$$\Delta^{P_\theta}_{\gamma_1\gamma_2\ldots\gamma_{m-1}\gamma_m(0)}=\Delta^{P_\theta}_{\gamma_1\gamma_2\ldots\gamma_{m-1}[\gamma_m-1](2)}$$

are P_θ*-rational.* The rest of numbers are P_θ*-irrational* and have the unique P_θ-representation.

The P_θ-representation of real numbers is a generalization of the $3'$-representation ([22]) and coincides with the last representation under the condition $p_0=p_1=p_2=\frac{1}{3}$.

To investigate relationships of certain generalized positive and alternating (sign-variable) expansions of real numbers, let us consider an operator aproach and model a simple example of numeral system, the geometry of which is a generalization of the geometries of some positive and alternating expansions.

2. The basis of the metric theory

Lemma 2.1. *Each number $x\in[0,1]$ can be represented in terms of the P_θ-representation* (1.1)*. In addition, every P_θ-irrational number has the unique P_θ-representation, and every P_θ-rational number has two different representations.*

Proof. Let us consider the following function

$$f:x=\Delta^{P_3}_{i_1i_2\ldots i_k\ldots}\to\Delta^{P_3}_{\theta(i_1)\theta(i_2)\ldots\theta(i_k)\ldots}=\Delta^{P_\theta}_{\gamma_1\gamma_2\ldots\gamma_k\ldots}=f(x)=y, \tag{2.1}$$

where $\theta(0)=0$, $\theta(1)=2$, and $\theta(2)=1$.

The last function was investigated in [24], its partial cases are considered in [18, 17]. Let us recall some its properties:

- "this function is continuous at P_3-irrational points, and P_3-rational points are points of its discontinuity;
- the function is a singular nowhere monotone function;
- the graph is a fractal dust".

The statement follows from the last-mentioned properties of f. □

Suppose $c_1,c_2,\ldots,c_m$ is an ordered tuple of digits from $\{0,1,2\}$. Then *a cylinder $\Lambda^{P_\theta}_{c_1c_2\ldots c_m}$ of rank m with base $c_1c_2\ldots c_m$* is a set of the form:

$$\Lambda^{P_\theta}_{c_1c_2\ldots c_m}\equiv\left\{x:x=\Delta^{P_\theta}_{c_1c_2\ldots c_m\gamma_{m+1}\gamma_{m+2}\gamma_{m+3}\ldots},\gamma_t\in\{0,1,2\},t>m\right\},$$

Lemma 2.2. *Cylinders* $\Lambda^{P_\theta}_{c_1c_2\ldots c_m}$ *have the following properties:*

(1) *Any cylinder* $\Lambda^{P_\theta}_{c_1c_2\ldots c_m}$ *is a closed interval, as well as*

$$\Lambda^{P_\theta}_{c_1c_2\ldots c_m} := \left[\Delta^{P_\theta}_{c_1c_2\ldots c_m(0)}, \Delta^{P_\theta}_{c_1c_2\ldots c_m(2)}\right].$$

(2) *For the Lebesgue measure* $|\cdot|$ *of a set, the following holds:*

$$\left|\Lambda^{P_\theta}_{c_1c_2\ldots c_m}\right| = \prod_{r=1}^{m} p_{c_r} = \prod_{r=1}^{m} p_{\theta(d_r)},$$

where $\theta(d_r) = c_r$ *for all* $r = \overline{1,m}$.

(3)

$$\Lambda^{P_\theta}_{c_1c_2\ldots c_m c} \subset \Lambda^{P_\theta}_{c_1c_2\ldots c_m}.$$

(4)

$$\Lambda^{P_\theta}_{c_1c_2\ldots c_m} = \bigcup_{c=0}^{2} \Lambda^{P_\theta}_{c_1c_2\ldots c_m c}.$$

(5) *Cylinders* $\Lambda^{P_\theta}_{c_1c_2\ldots c_m}$ *are left-to-right situated.*

Proof. Choose a certain $x_0 \in \Lambda^{P_3}_{d_1d_2\ldots d_m}$ such that $\theta(d_r) = c_r$ for all $r = \overline{1,m}$.

For proving, one can use properties of f. Then

$$\begin{aligned} f(x_0) &= f\left(\Delta^{P_3}_{d_1d_2\ldots d_m i_{m+1} i_{m+2}\ldots}\right) \\ &= \Delta^{P_3}_{\theta(d_1)\theta(d_2)\ldots\theta(d_m)\theta(i_{m+1})\theta(i_{m+2})\ldots} \\ &= \Delta^{P_\theta}_{c_1c_2\ldots c_m \gamma_{m+1}\gamma_{m+2}\ldots} \in \Lambda^{P_\theta}_{c_1c_2\ldots c_m}. \end{aligned} \tag{2.2}$$

Also,

$$f\left(\inf \Lambda^{P_3}_{d_1d_2\ldots d_m}\right) = f\left(\Delta^{P_3}_{d_1d_2\ldots d_m(0)}\right) = \Delta^{P_\theta}_{c_1c_2\ldots c_m(0)} = \inf \Lambda^{P_\theta}_{c_1c_2\ldots c_m} \in \Lambda^{P_\theta}_{c_1c_2\ldots c_m}, \tag{2.3}$$

$$f\left(\sup \Lambda^{P_3}_{d_1d_2\ldots d_m}\right) = f\left(\Delta^{P_3}_{d_1d_2\ldots d_m(2)}\right) = \Delta^{P_\theta}_{c_1c_2\ldots c_m(1)} \in \Lambda^{P_\theta}_{c_1c_2\ldots c_m}, \tag{2.4}$$

and

$$\sup \Lambda^{P_\theta}_{c_1c_2\ldots c_m} = \Delta^{P_\theta}_{c_1c_2\ldots c_m(2)} = f\left(\Delta^{P_3}_{d_1d_2\ldots d_m(1)}\right). \tag{2.5}$$

Let us prove that a cylinder is a segment. Suppose that $x \in \Lambda^{P_\theta}_{c_1c_2\ldots c_m}$. That is,

$$x = \beta_{c_1} + \sum_{k=2}^{m}\left(\beta_{c_k}\prod_{r=1}^{k-1} p_{c_r}\right) + \left(\prod_{r=1}^{k} p_{c_r}\right)\left(\beta_{\gamma_{m+1}} + \sum_{t=m+2}^{\infty}\left(\beta_{\gamma_t}\prod_{s=m+1}^{t-1} p_{\gamma_s}\right)\right),$$

where $\gamma_t \in \{0,1,2\}$ for $t = m+1, m+2, m+3, \ldots$. Whence,

$$x^{'} = \beta_{c_1} + \sum_{k=2}^{m}\left(\beta_{c_k}\prod_{r=1}^{k-1} p_{c_r}\right) \le x \le \beta_{c_1} + \sum_{k=2}^{m}\left(\beta_{c_k}\prod_{r=1}^{k-1} p_{c_r}\right) + \left(\prod_{r=1}^{k} p_{c_r}\right) = x^{''}.$$

So $x \in \left[x^{'}, x^{''}\right] \supseteq \Lambda^{P_\theta}_{c_1c_2\ldots c_m}$. Since

$$x^{'} = \beta_{c_1} + \sum_{k=2}^{m}\left(\beta_{c_k}\prod_{r=1}^{k-1} p_{c_r}\right) + \left(\prod_{r=1}^{k} p_{c_r}\right)\inf\left(\beta_{\gamma_{m+1}} + \sum_{t=m+2}^{\infty}\left(\beta_{\gamma_t}\prod_{s=m+1}^{t-1} p_{\gamma_s}\right)\right)$$

and

$$x^{''}=\beta_{c_1}+\sum_{k=2}^{m}\left(\beta_{c_k}\prod_{r=1}^{k-1}p_{c_r}\right)+\left(\prod_{r=1}^{k}p_{c_r}\right)\sup\left(\beta_{\gamma_{m+1}}+\sum_{t=m+2}^{\infty}\left(\beta_{\gamma_t}\prod_{s=m+1}^{t-1}p_{\gamma_s}\right)\right)$$

hold for any $x\in\Lambda^{P_\theta}_{c_1c_2...c_m}$, we obtain $x,x^{'},x^{''}\in\Lambda^{P_\theta}_{c_1c_2...c_m}$.

So, *the first and second properties are proven.*

Let us prove *the third property.* Suppose $d_1,d_2\dots,d_m$ is a fixed tuple of digits from $\{0,1,2\}$ such that $\theta(d_r)=c_r$ for all $r=\overline{1,m}$. Then

$$f\left(\inf\Lambda^{P_3}_{d_1d_2...d_md}\right)=\inf\Lambda^{P_\theta}_{c_1c_2...c_nc}\in\Lambda^{P_\theta}_{c_1c_2...c_nc},$$

where $\theta(d)=c$, and

$$\inf\Lambda^{P_\theta}_{c_1c_2...c_nc}<f\left(\sup\Lambda^{P_3}_{d_1d_2...d_md}\right)<\sup\Lambda^{P_\theta}_{c_1c_2...c_nc}.$$

Hence, using the first and second properties of cylinders, as well as relationships (2.2)-(2.5), we have

$$\inf\Lambda^{P_\theta}_{c_1c_2...c_mc}\ge\inf\Lambda^{P_\theta}_{c_1c_2...c_m}$$

and

$$\sup\Lambda^{P_\theta}_{c_1c_2...c_mc}\le\sup\Lambda^{P_\theta}_{c_1c_2...c_m}.$$

So,

$$\Lambda^{P_\theta}_{c_1c_2...c_mc}\subset\Lambda^{P_\theta}_{c_1c_2...c_m}=\bigcup_{c=0}^{2}\Lambda^{P_\theta}_{c_1c_2...c_mc}.$$

Let us prove tha last property. Let us consider the differences:

$$\inf\Lambda^{P_\theta}_{c_1c_2...c_{m-1}2}-\sup\Lambda^{P_\theta}_{c_1c_2...c_{m-1}1}=\Delta^{P_\theta}_{c_1c_2...c_{m-1}2(0)}-\Delta^{P_\theta}_{c_1c_2...c_{m-1}1(2)}=0,$$

$$\inf\Lambda^{P_\theta}_{c_1c_2...c_{m-1}1}-\sup\Lambda^{P_\theta}_{c_1c_2...c_{m-1}0}=\Delta^{P_\theta}_{c_1c_2...c_{m-1}1(0)}-\Delta^{P_\theta}_{c_1c_2...c_{m-1}0(2)}=0.$$

Our Lemma is proven. □

Theorem 2.1. *The map*

$$f:x=\Delta^{P_3}_{i_1i_2...i_k...}\to\Delta^{P_3}_{\theta(i_1)\theta(i_2)...\theta(i_k)...}=\Delta^{P_\theta}_{\gamma_1\gamma_2...\gamma_k...}=y$$

does not preserve a distance between points and the Lebesgue measure of an interval (segment).

Proof. Let us prove that f does not preserve a distance. Let us choose $x_1,x_2\in[0,1]$ such that the condition $|f(x_2)-f(x_1)|\neq|x_2-x_1|$ holds. The statement follows from the existence of jump discontinuities of f. Really, for example, suppose that $p_0=\frac{1}{2}$, $p_1=\frac{1}{3}$, $p_2=\frac{1}{6}$, as well as $x_1=\Delta^{P_3}_{22(0)}$ and $x_2=\Delta^{P_3}_{21(0)}$. Then

$$|x_1-x_2|=\beta_2+\beta_2p_2-\beta_2-\beta_1p_2=p_2(\beta_2-\beta_1)=p_2(p_0+p_1-p_0)=p_1p_2=\frac{1}{18}$$

and

$$|f(x_2)-f(x_1)|=\left|\Delta^{P_\theta}_{11(0)}-\Delta^{P_\theta}_{12(0)}\right|=|\beta_1+\beta_1p_1-\beta_1-\beta_2p_1|=|p_1(\beta_1-\beta_2)|=\frac{1}{9}\neq|x_1-x_2|.$$

Let us prove that f does not preserve the Lebesgue measure. Suppose $[x_1,x_2]\subset[0,1]$ is a segment.

If $[x_1, x_2] \equiv \Lambda^{P_3}_{d_1d_2...d_m}$, then considering the last lemma, we get

$$\left|\Lambda^{P_3}_{d_1d_2...d_m}\right| = \prod_{t=1}^{m} p_{d_t}$$

and

$$\left|f\left(\Lambda^{P_3}_{d_1d_2...d_m}\right)\right| = \left|\Lambda^{P_\theta}_{c_1c_2...c_m}\right| = \prod_{t=1}^{m} p_{c_t} = \prod_{t=1}^{m} p_{\theta(d_t)}.$$

In a general case,

$$\prod_{t=1}^{m} p_{d_t} \neq \prod_{t=1}^{m} p_{\theta(d_t)}.$$

Really, suppose $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{2}$, and $p_2 = \frac{1}{4}$. For example, then

$$\left|\Lambda^{P_3}_{11122}\right| = p_1^3 p_2^2 = \frac{1}{8} \cdot \frac{1}{16} = \frac{1}{128}$$

but

$$\left|f\left(\Lambda^{P_\theta}_{22211}\right)\right| = p_2^3 p_1^2 = \frac{1}{64} \cdot \frac{1}{4} = \frac{1}{256}.$$

Let $[x_1, x_2] \subset [0, 1]$ be a segment that are not a certain cylinder $\Lambda^{P_3}_{d_1d_2...d_m}$; then there exists ε-covering by cylinders of rank k and m such that

$$\bigcup_m \Lambda^{P_3}_{d_1d_2...d_m} \subseteq [x_1, x_2] \subseteq \bigcup_k \Lambda^{P_3}_{d_1d_2...d_k}$$

and

$$\lim_{k\to\infty} \left|\bigcup_k \Lambda^{P_3}_{d_1d_2...d_k}\right| = \lim_{m\to\infty} \left|\bigcup_m \Lambda^{P_3}_{d_1d_2...d_m}\right| = |[x_1, x_2]|.$$

Whence, in this case, our results depends on the case when $[x_1, x_2]$ is a certain cylinder. □

Remark 2.1. It is useful the problem on geometry of a positive expansion of real numbers and its corresponding alternating or sign-variable expansion. This connection is useful for studying metric, dimensional and other properties of mathematical objects defined by these representations of real numbers. The last lemma give the answer on this problem.

Let $\Delta_{i_1i_2...i_k...}, i_k \in A_k$, be the representation of a number x from some interval by a positive expansion. One can model the corresponding alternating (sign-variable; here the case of alternating expansions is explained) representation $\Delta^-_{j_1j_2...j_k...}$ by the following way (relationship):

$$\Delta^-_{j_1j_2...j_k...} \equiv \Delta_{\theta'(i_1)i_2\theta'(i_3)i_4...\theta'(i_{2k-1}).i_{2k}...},$$

where cylinders of even rank are left-to-right situated and cylinders of odd rank are right-to-left situated, as well as

$$\theta'(i_{2k-1}) = \max_{i_{2k-1}\in A_{2k-1}} \{i_{2k-1}\} - i_{2k-1},$$

or

$$\Delta^-_{j_1j_2...j_k...} \equiv \Delta_{i_1\theta'(i_2)i_3\theta'(i_4)i_5...i_{2k-1}\theta'(i_{2k})...}$$

and here cylinders of odd rank are left-to-right situated but cylinders of even rank are right-to-left situated, as well as

$$\theta^{'}(i_{2k}) = \max_{i_{2k}\in A_{2k}} \{i_{2k}\} - i_{2k}.$$

Let us consider an example. For the case of P_3-representation, we obtain

$$\Delta^{-P_3}_{j_1j_2...j_k...} \equiv \Delta^{P_3}_{i_1[2-i_2]i_3[2-i_4]i_5...i_{2k-1}[2-i_{2k}]...}$$

or

$$\Delta^{-P_3}_{j_1j_2...j_k...} \equiv \Delta^{P_3}_{[2-i_1]i_2[2-i_3]i_4...i_{2k-2}[2-i_{2k-1}]...}.$$

Suppose even rank cylinders are left-to-right situated, as well as $p_0 = \frac{1}{2}, p_1 = \frac{1}{3}$, and $p_2 = \frac{1}{6}$; then

$$\left|\Lambda^{P_3}_{121200}\right| = p_0^2p_1^2p_2^2 = \frac{1}{4\cdot 9\cdot 36} = \frac{1}{1296}$$

and

$$\left|\Lambda^{-P_3}_{121220}\right| = p_0p_1^2p_2^3 = \frac{1}{2\cdot 9\cdot 216} = \frac{1}{3888}.$$

Corollary 2.1. *In a general case, the metric theories of the P_q- and quasi-nega-P_q-representations can be different.*

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