Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 50, Number 1, 2024, Pages 104–114
https://doi.org/10.30546/2409-4994.2024.50.1.104

NODAL SOLUTIONS OF SOME NONLINEAR FOURTH-ORDER BOUNDARY VALUE PROBLEMS

YAGUT N. ALIYEVA

Abstract. In the present paper, we consider nonlinear boundary value problems for fourth-order ordinary differential equations with a boundary condition containing a parameter. Using previously obtained results on the global bifurcation of solutions from zero and infinity to nonlinear fourth-order eigenvalue problems with a spectral parameter in the boundary condition, we establish the existence of nodal solutions to these nonlinear boundary value problems.

1. Introduction

In this paper, we consider the existence of nodal solutions to the following nonlinear boundary value problem for ordinary differential equations of fourth order

$$\ell(y) \equiv (p(x)y''(x))'' - (q(x)y'(x))' = \tau r(x)f(y(x)),\ x \in (0,l), \tag{1.1}$$

$$y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \tag{1.2}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{1.3}$$

$$y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \tag{1.4}$$

$$(a\lambda + b)y(l) - (c\lambda + d)Ty(l) = 0, \tag{1.5}$$

where $Ty \equiv (py'')' - qy'$, p is a positive twice continuously differentiable function on $[0,l]$, q is a non-negative continuously differentiable function on $[0,l]$, τ is a positive number, $r(x)$ is a positive continuous function on $[0,l]$, α, β, γ, a,b,c,d are real constants such that

$$\alpha,\beta,\gamma \in [0,\pi/2]$$

and

$$\sigma = bc - ad > 0.$$

The nonlinear term f is real-valued continuous function on $\mathbb{R}$ which satisfy the following conditions:

$$sf(s) > 0 \text{ for } s \in \mathbb{R}\setminus\{0\}; \tag{1.6}$$

$$\underline{f}_0,\ \overline{f_0},\ \underline{f}_\infty,\ \overline{f}_\infty \in (0,+\infty) \text{ with } \underline{f}_0 \neq \overline{f}_0,\ \underline{f}_\infty \neq \overline{f}_\infty, \tag{1.7}$$

2010 *Mathematics Subject Classification.* 34B24, 34C23, 34L15, 34L30, 47J10, 47J15.

Key words and phrases. nonlinear fourth-order problem, bifurcation from zero, bifurcation from infinity, nodal solutions, global bifurcation.

where

$$\underline{f}_0 = \liminf_{|s|\to 0} \frac{f(s)}{s}\,,\ \overline{f}_0 = \limsup_{|s|\to 0} \frac{f(s)}{s}\,, \tag{1.8}$$

$$\underline{f}_\infty = \liminf_{|s|\to +\infty} \frac{f(s)}{s}\,,\ \overline{f}_\infty = \limsup_{|s|\to +\infty} \frac{f(s)}{s}\,; \tag{1.9}$$

The purpose of this article is to determine the interval for r in which there exist solutions of problem (1.1)-(1.5) with a fixed number of simple nodal zeros. Just as when studying nodal solutions of boundary value problems for second-order ordinary and partial differential equations (see [11, 12, 14-18,], we will also use the global bifurcation technique.

Note that the global bifurcation from zero and infinity of nonlinear eigenvalue problems for second-order ordinary differential equations was intensively studied in papers [9-11, 14, 16, 19-21]. The authors of these works established the existence of global continua of solutions in $\mathbb{R} \times C^1$ bifurcating from the points and intervals of the lines $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{\infty\}$, and contained in classes of functions with the usual nodal properties in the neighborhood of these bifurcation points and intervals. Similar results for the eigenvalue problems for ordinary differential equations of fourth order (both with a spectral parameter in boundary condition, also without a spectral parameter in boundary conditions) was established in [1-5, 7, 22].

The rest of this article is structured as follows. Section 2 considers a nonlinearizable in zero and infinity eigenvalue problems for fourth-order ordinary differential equations. Here we present the results obtained in [4, 5] on the structure and behavior of global continua of solutions branching from bifurcation intervals. In Section 3, we find the interval of the parameter τ in which there are solutions to the problem (1.1)-(1.5) with a fixed number of simple nodal zeros. Here, to prove our main theorem, which consists of 4 Steps, we consider an auxiliary nonlinear eigenvalue problem. In Step 1, the bifurcation of solutions from zero to this auxiliary problem is studied. The bifurcation intervals are found, and the existence of two families of unbounded components of the set of nontrivial solutions branching from these intervals and contained in classes of functions with a fixed number of nodes is proved. In Step 2, the bifurcation of solutions from infinity to an auxiliary nonlinear eigenvalue problem is studied. Here we also find bifurcation intervals and prove the existence of two families of global components of the set of nontrivial solutions branching from these intervals and contained in classes of functions with a fixed number of nodes in the neighborhood of these intervals. In Step 3 it is proved that the global components of the set of nontrivial solutions to the auxiliary problem branching from the intervals of the line $\mathbb{R} \times \{\infty\}$ are also contained in classes with a fixed number of simple nodal zeros and intersect the bifurcation intervals of the line of trivial solutions. It is also established that the global components of nontrivial solutions branching from the line of trivial solutions coincide with the corresponding components of nontrivial solutions branching from intervals of the line $\mathbb{R} \times \{\infty\}$. In step 4, using the results of Steps 1-3, we find intervals for the parameter r in which there are solutions to the boundary value problem (1.1)-(1.5) with a fixed number of simple nodal zeros.

2. Preliminary

We denote by BC_0 and BC_λ the sets of functions that satisfy the boundary conditions (1.2)-(1.4) and (1.2)-(1.5), respectively.

Let $E = C^3[0,l] \cap BC_0$ be a Banach space which is equipped with the usual norm $||y||_3 = \sum\limits_{s=0}^{3} ||\, y^{(s)}||_\infty$, where $||y||_\infty = \max\limits_{x\in[0,\,l]} |y\,(x)|$.

Note that in [1] (see also [2]), using the Prüfer type transformation, the author constructed classes $\mathcal{S}_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+\,,\,-\}$, of functions $y \in E$, which a fixed oscillation count. The sets $\mathcal{S}_k^+$, $\mathcal{S}_k^-$ and $\mathcal{S}_k = \mathcal{S}_k^+ \cap \mathcal{S}_k^-$ are pairwise disjoint open subsets of E. Moreover, it follows from [1, Lemma 2.2] that if $y \in \partial\mathcal{S}_k^\nu\,(\partial\mathcal{S}_k)$, then y has at least one zero multiple four in the interval $(0,l)$.

We consider the following linear spectral problem

$$\begin{cases} \ell(y)(x) = \lambda r(x)y(x),\; x \in (0,l), \\ y \in BC_\lambda. \end{cases} \tag{2.1}$$

Problem (2.1) was studied in [13], where it was established that the eigenvalues of this problem are real and simple, and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^\infty$ such that $\lambda_k > 0$ for $k \geq 3$; in this case, each of the first two eigenvalues can be either positive or non-positive. Moreover, by [8, Theorems 3.3, 4.2, 5.4] and [13, Theorem 2.2] it follows from [1, Remark 2.1] and [2, § 3.1] that for each $k \in \mathbb{N}$ the eigenfunction $y_k(x)$ corresponding to the eigenvalue $\lambda_k > 0$ lies in S_k.

Remark 2.1. Throughout what follows we will assume that the first eigenvalue of problem (2.1) is positive.

From now on ν will denote an element of $\{\,+\,,\,-\,\}$ that is, either $\nu = +$ or $\nu = -\,$.

To study the existence of solutions to problem (1.1)-(1.5) with a fixed number of nodes, consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(y) = \lambda r(x)y + h(x,y,y',y'',y''',\lambda),\; x \in (0,l), \\ y \in BC_\lambda, \end{cases} \tag{2.2}$$

Here h is real-valued continuous functions on $[0,l] \times \mathbb{R}^5$ that satisfy the following conditions: there exist constants $M_0 > 0$ and sufficiently small $\delta_0 > 0$ such that

$$\left|\frac{h(x,y,s,v,w,\lambda)}{y}\right| \leq M_0,\; (x,y,s,v,w) \in [0,l] \times \mathbb{R}^4,\; |y| + |s| + |v| + |w| \;\leq \delta_0, \\ y \neq 0,\;\; \lambda \in \mathbb{R}; \tag{2.3}$$

there exist constants $M_1 > 0$ and sufficiently large $\Delta_1 > 0$ such that

$$\left|\frac{h(x,y,s,v,w,\lambda)}{y}\right| \leq M_1,\; (x,y,s,v,w) \in [0,l] \times \mathbb{R}^4,\; |y| + |s| + |v| + |w| \;\geq \Delta_1, \\ y \neq 0,\;\; \lambda \in \mathbb{R}; \tag{2.4}$$

We introduce the following notations:

$$r_0 = \min_{x\in[0,\,l]} r(x),$$

$$I_{k,\,0} = \left[\lambda_k - \frac{M_0}{r_0}, \lambda_k + \frac{M_0}{r_0}\right],\;\; I_{k,\,1} = \left[\lambda_k - \frac{M_1}{r_0}, \lambda_k + \frac{M_1}{r_0}\right].$$

Since conditions (2.3) and (2.4) are satisfied, we consider the bifurcation of nontrivial solutions of problem (2.2) simultaneously both from the line of trivial solutions $\mathbb{R} \times \{0\} = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$ and also from the line $\mathbb{R} \times \{\infty\} = \{(\lambda, \infty) : \lambda \in \mathbb{R}\}$ (should be noted that as in [7, 20, 21] we add the points of the line $\mathbb{R} \times \{\infty\}$ to our space $\mathbb{R} \times E$ and define an appropriate topology on the resulting set). It follows from [4, 5] that the following results hold.

Lemma 2.1 [5, Lemmas 3 and 4]. *For each $k \in \mathbb{N}$ and each ν the set of bifurcation points of problem (2.2) with respect to $\mathbb{R} \times \mathcal{S}_k^\nu$ is nonempty. Moreover, if $(\lambda, 0)$ is a bifurcation point of problem (2.2) with respect to $\mathbb{R} \times \mathcal{S}_k^\nu$, then $\lambda \in I_{k,0}$.*

Let $\mathcal{D}$ be the set of nontrivial solutions of the nonlinear eigenvalue problem (2.2).

For each $k \in \mathbb{N}$ and each ν by $\mathcal{D}_{k,0}^\nu$ we denote the union of all the components of the set $\mathcal{D}$ which meet the interval $I_{k,0} \times \{0\}$ with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$. Let $\tilde{\mathcal{D}}_{k,0}^\nu = \mathcal{D}_{k,0}^\nu \cup (I_{k,0} \times \{0\})$. Note that the set $\mathcal{D}_k^\nu$ may not be connected in $\mathbb{R} \times E$, but the set $\tilde{\mathcal{D}}_k^\nu$ is connected in $\mathbb{R} \times E$.

Theorem 2.1 [5, Theorem 3]. *For each $k \in \mathbb{N}$ and each ν the set D_k^ν is nonempty, is unbounded in $\mathbb{R} \times E$ and lies in $\mathbb{R} \times S_k^\nu$.*

Lemma 2.2 [4, Theorem 3.1]. *For each $k \in \mathbb{N}$ and each ν the set of bifurcation points from $\mathbb{R} \times \{\infty\}$ of problem (2.2) with respect to the set $\mathbb{R} \times S_k^\nu$ is nonempty. Moreover, if (λ, ∞) is a bifurcation point at $\mathbb{R} \times \{\infty\}$ of problem (2.2) with respect to $\mathbb{R} \times \mathcal{S}_k^\nu$, then $\lambda \in I_{k,1}$.*

For each $k \in \mathbb{N}$ and each ν let $\tilde{\mathcal{D}}_{k,1}^\nu$ be the union of all the components of the set $\mathcal{D}$ which meet the interval $I_{k,1} \times \{\infty\}$ with respect to the set $\mathbb{R} \times S_k^\nu$. Let $\tilde{\mathcal{D}}_{k,1}^\nu = \mathcal{D}_{k,1}^\nu \cup (I_{k,1} \times \{\infty\})$. Note that $\mathcal{D}_{k,1}^\nu$ may not be connected in $\mathbb{R} \times E$, but $\tilde{\mathcal{D}}_k^\nu$ is connected $\mathbb{R} \times E$.

Theorem 2.2 [4, Theorem 4.1]. *For each $k \in \mathbb{N}$ and each ν the set $\mathcal{D}_{k,1}^\nu$ is nonempty and either*

(i) *the set $\mathcal{D}_{k,1}^\nu$ meets $I_{k',1} \times \{\infty\}$ with respect to $\mathbb{R} \times \mathcal{S}_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$, or*

(ii) *the set $\mathcal{D}_{k,1}^\nu$ meets $\mathbb{R} \times \{0\}$ for some $\lambda \in \mathbb{R}$, or*

(iii) *the natural projection of $\mathcal{D}_{k,1}^\nu$ on $\mathbb{R} \times \{0\}$ is unbounded.*

In addition, if cases (ii) and (iii) are not satisfied for the union $\mathcal{D}_{k,1} = \mathcal{D}_{k,1}^+ \cup \mathcal{D}_{k,1}^-$, then case (i) is satisfied for it with $k' \neq k$.

3. Existence of solutions to problem (1.1)-(1.5) with fixed number of nodal simple zeros

In this section we will determine the interval of τ, in which there exist nodal solutions of problem (1.1)-(1.5).

Theorem 3.1. *Suppose that for some $k \in \mathbb{N}$ one of the following conditions is satisfied:*

$$\frac{\lambda_k}{\underline{f}_0} < \tau < \frac{\lambda_k}{\overline{f}_\infty}; \ \frac{\lambda_k}{\underline{f}_\infty} < \tau < \frac{\lambda_k}{\overline{f}_0}.$$

Then problem (1.1)-(1.5) has two solutions y_k^+ and y_k^- such that $y_k^+ \in \mathcal{S}_k^+$ and $y_k^- \in \mathcal{S}_k^-$, and consequently, y_k^+ has $k-1$ or $k-2$ simple zeros in $(0, l)$ and is *positive near $x = 0$, and y_k^- has either $k-1$ or $k-2$ simple zeros in $(0,l)$ and is negative near $x = 0$.*

Proof. To prove the theorem, consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda \tau r(x) y(x) + \tau r(x)(f(y(x)) - y(x)),\ x \in (0,l), \\ y \in (b.c.)_\lambda. \end{cases} \tag{3.1}$$

The proof will be carried out in 4 steps.

Step 1. Let $\varrho_0 > 0$ be an arbitrary fixed sufficiently small number. Then according to (1.6)-(1.7) there is a sufficiently small $\varkappa_0 \in (0, \delta_0)$ such that for any $s \in \mathbb{R}$ satisfying the condition $|s| \in (0, \varkappa_0)$ the following relation holds:

$$\underline{f}_0 - \varrho_0 < \frac{f(s)}{s} < \overline{f}_0 + \varrho_0. \tag{3.2}$$

Let

$$F(s) = f(s) - s,\ \ s \in \mathbb{R}.$$

Then it follows from (3.2) that

$$\underline{f}_0 - 1 - \varrho_0 < \frac{F(s)}{s} < \overline{f}_0 - 1 + \varrho_0,\ \ |s| \in (0, \varkappa_0). \tag{3.3}$$

Hence by (3.3) for any $s \in \mathbb{R}$, $|s| \in (0, \varkappa_0)$, we get

$$\left| \frac{F(s)}{s} \right| \leq M_0^*, \tag{3.4}$$

where

$$M_0^* = \max \left\{ \left| \underline{f}_0 - 1 - \varrho_0 \right|, \left| \overline{f}_0 - 1 + \varrho_0 \right| \right\} > 0.$$

Consequently, by relation (3.4) Lemma 2.1 implies that for each $k \in \mathbb{N}$ and each ν the set of bifurcation points from the line of trivial solutions to nonlinear eigenvalue problem (3.1) with respect to $\mathbb{R} \times \mathcal{S}_k^\nu$ is nonempty. Note that if $(\tilde{\lambda}, 0)$ is a bifurcation point to problem (3.1) with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$, then there exists a sequence $\{(\tilde{\lambda}_n, \tilde{y}_n)\}_{n=1}^\infty$ of solutions of problem (3.1) such that

$$(\tilde{\lambda}_n, \tilde{y}_n) \in \mathbb{R} \times \mathcal{S}_k^\nu,$$

and

$$(\tilde{\lambda}_n, \tilde{y}_n) \to (\tilde{\lambda}, 0)\ \text{ as } n \to \infty. \tag{3.5}$$

We define the function $\tilde{\psi}_n(x)$, $x \in [0,1]$, as follows:

$$\tilde{\psi}_n(x) = \begin{cases} -\frac{F(\tilde{y}_n(x))}{\tilde{y}_n(x)} & \text{if } \tilde{y}_n(x) \neq 0, \\ 0 & \text{if } \tilde{y}_n(x) = 0. \end{cases} \tag{3.6}$$

It is clear that $(\tilde{\lambda}_n, \tilde{y}_n)$ for each $n \in \mathbb{N}$ solves the linear spectral problem

$$\begin{cases} \frac{1}{\tau r(x)} \ell(y)(x) + \tilde{\psi}_n(x) y(x) = \lambda y(x), x \in (0,l). \\ y \in (b.c.)_\lambda. \end{cases} \tag{3.7}$$

By [6, Remark 4.2 and Theorem 4.3] for each fixed $n \in \mathbb{N}$ the eigenvalues of the linear problem (3.7) are real and simple, and form an unboundedly increasing sequence $\{\tilde{\lambda}_{k,n}\}_{k=1}^\infty$. In addition, for each $k \in \mathbb{N}$ the eigenfunction $\tilde{y}_{k,n}(x)$

corresponding to the eigenvalue $\tilde{\lambda}_{k,\,n}$ is contained in the set $\mathcal{S}_k$. Therefore, for each fixed $k \in \mathbb{N}$, we have

$$\tilde{\lambda}_n = \tilde{\lambda}_{k,\,n},\ \tilde{y}_n = \tilde{y}_{k,\,n}.$$

Then, by (3.3) and (3.6) it follows from [1, Lemma 4.2, formula (21)] that

$$\tilde{\lambda}_k - \overline{f}_0 + 1 - \varrho_0 \leq \tilde{\lambda}_n \leq \tilde{\lambda}_k - \underline{f}_0 + 1 + \varrho_0, \tag{3.8}$$

where $\tilde{\lambda}_k$, $k \in \mathbb{N}$, is a kth eigenvalue of the linear spectral problem

$$\begin{cases} \ell(y)(x) = \lambda \tau r(x) y(x),\ x \in (0,l), \\ y \in (b.c.)_\lambda. \end{cases} \tag{3.9}$$

From (2.1) and (3.9) we see that $\lambda_k = \tau \tilde{\lambda}_k$ for each $k \in \mathbb{N}$. Hence by (3.8) we get

$$\frac{\lambda_k}{\tau} - \overline{f}_0 + 1 - \varrho_0 \leq \tilde{\lambda}_n \leq \frac{\lambda_k}{\tau} - \underline{f}_0 + 1 + \varrho_0,$$

which implies that

$$\frac{\lambda_k}{\tau} - \overline{f}_0 + 1 - \varrho_0 \leq \tilde{\lambda} \leq \frac{\lambda_k}{\tau} - \underline{f}_0 + 1 + \varrho_0. \tag{3.10}$$

Since ϱ_0 is sufficiently small, from (3.10) we can conclude that

$$\frac{\lambda_k}{\tau} - \overline{f}_0 + 1 \leq \tilde{\lambda} \leq \frac{\lambda_k}{\tau} - \underline{f}_0 + 1. \tag{3.11}$$

By (3.11) the bifurcation points of problem (3.1) with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$ are contained in the interval $I_k^0 \times \{0\}$, where

$$I_k^0 = \left[\frac{\lambda_k}{\tau} - \overline{f}_0 + 1, \frac{\lambda_k}{\tau} - \underline{f}_0 + 1\right].$$

Let $\mathfrak{D}$ be the set of nontrivial solutions of the nonlinear eigenvalue problem (3.1).

For each $k \in \mathbb{N}$ and each ν by $\mathfrak{D}_{k,\,0}^\nu$ we denote the union of all the components of the set $\mathfrak{D}$ which meet the interval $I_k^0 \times \{0\}$ with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$. Then, it follows from Theorem 2.1 that for each $k \in \mathbb{N}$ and each ν the set $\mathfrak{D}_{k,\,0}^\nu$ is unbounded in $\mathbb{R} \times E$ and is contained in $(\mathbb{R} \times \mathcal{S}_k^\nu) \cup (I_k^0 \times \{0\})$.

Step 2. We assume that $\varrho_1 > 0$ is an arbitrary fixed sufficiently small number. Then by conditions (1.6), (1.7) and (1.9) there exists a sufficiently large $\varkappa_1 \in (\Delta_1, +\infty)$ such that for any $s \in \mathbb{R}$ satisfying the condition $|s| \in (\varkappa_1, +\infty)$ we have the following relation

$$\underline{f}_\infty - \varrho_1 < \frac{f(s)}{s} < \overline{f}_\infty + \varrho_1. \tag{3.12}$$

Then (3.12) implies that

$$\underline{f}_\infty - 1 - \varrho_1 < \frac{F(s)}{s} < \overline{f}_\infty - 1 + \varrho_1,\ \ |s| \in (\varkappa_1, +\infty). \tag{3.13}$$

Let

$$M_1^* = \max\left\{\left|\underline{f}_\infty - 1 - \varrho_1\right|, \left|\overline{f}_\infty - 1 + \varrho_1\right|\right\} > 0.$$

Then by (3.13) we have

$$\left|\frac{F(s)}{s}\right| \leq M_1^*, \ \ |s| \in (\varkappa_1, +\infty). \tag{3.14}$$

In view of (3.14) it follows from Lemma 2.2 that for each $k \in \mathbb{N}$ and each ν the set of bifurcation points from the line $\mathbb{R} \times \{\infty\}$ of nonlinear eigenvalue problem (3.1) with respect to $\mathbb{R} \times \mathcal{S}_k^\nu$ is nonempty. Moreover, if $(\hat{\lambda}, \infty)$ is a bifurcation point of problem (3.1) with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$, then there exists a sequence $\{(\hat{\lambda}_n, \hat{y}_n)\}_{n=1}^\infty$ of solutions of problem (3.1) such that

$$(\hat{\lambda}_n, \hat{y}_n) \in \mathbb{R} \times \mathcal{S}_k^\nu$$

and

$$\hat{\lambda}_n \to \hat{\lambda}, \ \ ||\hat{y}_n||_3 \to +\infty \text{ as } n \to \infty. \tag{3.15}$$

Let $\hat{\psi}_n(x)$, $x \in [0,1]$, be the function defined by

$$\hat{\psi}_n(x) = \begin{cases} -\frac{F(\hat{y}_n(x))}{\hat{y}_n(x)} & \text{if } \ \hat{y}_n(x) \neq 0, \\ 0 & \text{if } \ \hat{y}_n(x) = 0. \end{cases} \tag{3.16}$$

It is obvious that $(\hat{\lambda}_n, \hat{y}_n)$, $n \in \mathbb{N}$, solves the linear eigenvalue problem

$$\begin{cases} \frac{1}{\tau r(x)}\, \ell(y)(x) + \hat{\psi}_n(x) y(x) = \lambda y(x), x \in (0,l), \\ y \in (b.c.)_\lambda, \end{cases} \tag{3.17}$$

the eigenvalues of which are real and simple, and form an unboundedly increasing sequence $\{\hat{\lambda}_{k,\,n}\}_{k=1}^\infty$; moreover, for each $k \in \mathbb{N}$ the eigenfunction $\hat{y}_{k,\,n}(x)$ corresponding to the eigenvalue $\hat{\lambda}_{k,\,n}$ is contained in $\mathcal{S}_k$ (see [6]). Then for each fixed $k \in \mathbb{N}$, we have

$$\hat{\lambda}_n = \hat{\lambda}_{k,\,n}, \ \ \hat{y}_n = \hat{y}_{k,\,n}.$$

Hence, in view of (3.13) and (3.16) by Lemma 4.2 of [1] we get

$$\tilde{\lambda}_k - \overline{f}_\infty + 1 - \varrho_1 \leq \hat{\lambda}_n \leq \tilde{\lambda}_k - \underline{f}_\infty + 1 + \varrho_1, \tag{3.18}$$

and consequently,

$$\frac{\lambda_k}{\tau} - \overline{f}_\infty + 1 - \varrho_1 \leq \hat{\lambda}_n \leq \frac{\lambda_k}{\tau} - \underline{f}_\infty + 1 + \varrho_1.$$

Next it follows from (3.18) that

$$\frac{\lambda_k}{\tau} - \overline{f}_\infty + 1 - \varrho_1 \leq \hat{\lambda} \leq \frac{\lambda_k}{\tau} - \underline{f}_\infty + 1 + \varrho_1. \tag{3.19}$$

As can be seen from (3.19), the arbitrariness of the number ϱ_1 leads to the conclusion that

$$\frac{\lambda_k}{\tau} - \overline{f}_\infty + 1 \leq \hat{\lambda} \leq \frac{\lambda_k}{\tau} - \underline{f}_\infty + 1. \tag{3.20}$$

The relation (3.20) shows that the bifurcation points from $\mathbb{R} \times \{\infty\}$ of problem (3.1) with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$ lies in the interval $I_k^1 \times \{\infty\}$, where

$$I_k^1 = \left[\frac{\lambda_k}{\tau} - \overline{f}_\infty + 1, \frac{\lambda_k}{\tau} - \underline{f}_\infty + 1\right].$$

For each $k \in \mathbb{N}$ and each ν by $\mathfrak{D}_{k,\,1}^\nu$ we denote the union of all the components of the set $\mathfrak{D}$ which meet the interval $I_k^1 \times \{\infty\}$ with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$.

Then, it follows from Theorem 2.2 that for each $k \in \mathbb{N}$ and each ν the set $\mathfrak{D}_{k,1}^{\nu}$ is nonempty and either

(i) the set $\mathfrak{D}_{k,1}^{\nu}$ meets $I_{k',1} \times \{\infty\}$ with respect to $\mathbb{R} \times \mathcal{S}_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$, or

(ii) the set $\mathfrak{D}_{k,1}^{\nu}$ meets $\mathbb{R} \times \{0\}$ for some $\lambda \in \mathbb{R}$, or

(iii) the natural projection $\mathcal{P}_{\mathbb{R}}(\mathfrak{D}_{k,1}^{\nu})$ of $\mathfrak{D}_{k,1}^{\nu}$ on $\mathbb{R} \times \{0\}$ is unbounded.

In addition, if cases (ii) and (iii) are not satisfied for the union $\mathfrak{D}_{k,1} = \mathfrak{D}_{k,1}^{+} \cup \mathfrak{D}_{k,1}^{-}$, then case (i) is satisfied for it with $k' \neq k$.

Step 3. It follows from [2, Lemma 1.1] (see also beginning of the proof of Theorem 3.1 in p. 1818 of [1]) that if $(\lambda, y) \in \mathbb{R} \times E$ is a solution of problem (3.1) such that $y \in \partial \mathcal{S}_k^{\nu}$, then $y \equiv 0$. Then we have the following relation

$$\mathfrak{D} \cap (\mathbb{R} \times \partial \mathcal{S}_k^{\nu}) = \emptyset.$$

Consequently, the sets $\mathfrak{D} \cap (\mathbb{R} \times \mathcal{S}_k^{\nu})$ and $\mathfrak{D} \backslash (\mathbb{R} \times \mathcal{S}_k^{\nu})$ are mutually separated in our space $\mathbb{R} \times E$ (regarding the definition of mutually separable two sets see [23, Definition 26.4]). Therefore, by [23, Corollary 26.6] any component of $\mathfrak{D}$ must be a subset either of the set $\mathfrak{D} \cap (\mathbb{R} \times \mathcal{S}_k^{\nu})$ or $\mathfrak{D} \backslash (\mathbb{R} \times \mathcal{S}_k^{\nu})$. Since the set $\mathfrak{D}_{k,1}^{\nu}$ is the union of all components of $\mathfrak{D}$ that intersect $\mathbb{R} \times \mathcal{S}_k^{\nu}$, each of these components must be a subset of $\mathbb{R} \times \mathcal{S}_k^{\nu}$, which implies that $\mathfrak{D}_{k,1}^{\nu} \subset \mathbb{R} \times \mathcal{S}_k^{\nu}$. From this we conclude that alternative (i) above for the set $\mathfrak{D}_{k,1}^{\nu}$ cannot be satisfied.

In other hand, since $f(s) \in C(\mathbb{R})$ it follows that there exists positive constant M_2 such that

$$\left| \frac{f(s)}{s} \right| \leq M_2 \text{ for any } s \in \mathbb{R} \text{ such that } \varkappa_0 \leq |s| \leq \varkappa_1,$$

whence implies that

$$\left| \frac{F(s)}{s} \right| \leq M_2^*, \ \ \varkappa_0 \leq |s| \leq \varkappa_1, \tag{3.21}$$

where

$$M_2^* = M_2 + 1.$$

Let

$$M^* = \max\{M_0^*, \, M_1^*, \, M_2^*\}.$$

Then by relations (3.4), (3.14) and (3.21) we get

$$\left| \frac{F(s)}{s} \right| \leq M^*, \ \ s \in \mathbb{R}, \, s \neq 0. \tag{3.22}$$

Let $(\lambda^*, y^*) \in \mathbb{R} \times E$ be a solution of problem (3.1) such that $y^* \in \mathcal{S}_k^{\nu}$. The function $\psi(x)$, $x \in [0, 1]$, we defined by

$$\phi^*(x) = \begin{cases} -\frac{F(y^*(x))}{y^*(x)} & \text{if } \ \hat{y}^*(x) \neq 0, \\ \quad 0 & \text{if } \ y^*(x) = 0. \end{cases} \tag{3.23}$$

Then λ^* is a kth eigenvalue of the linear eigenvalue problem

$$\begin{cases} \frac{1}{\tau r(x)} \, \ell(y)(x) + \, \phi^*(x) y(x) = \lambda y(x), x \in (0, l). \\ y \in (b.c.)_{\lambda}. \end{cases} \tag{3.24}$$

Hence by (3.22) it follows from [1, formula (21)] that

$$\tilde{\lambda}_k - M^* \leq \lambda^* \leq \tilde{\lambda}_k + M^*, \tag{3.25}$$

which shows that alternative (iii) above for the set $\mathfrak{D}_{k,1}^{\nu}$ cannot be satisfied. Consequently, the set $\mathfrak{D}_{k,1}^{\nu}$ meets $\mathbb{R} \times \{0\}$ for some $\lambda \in \mathbb{R}$.

Thus by arguments of Step 1 and Step 2, and the above arguments we conclude that for each $k \in \mathbb{N}$ and each ν the relation holds:

$$\mathfrak{D}_{k,0}^{\nu}, \mathfrak{D}_{k,1}^{\nu} \subset I_k^* \times \mathcal{S}_k^{\nu},$$

where

$$I_k^* = \left[\frac{\lambda_k}{\tau} - M^*, \frac{\lambda_k}{\tau} + M^*\right].$$

Consequently, by Step 2 the set $\mathfrak{D}_{k,0}^{\nu}$ can only meet (λ, ∞) for $\lambda \in I_k^1$ and by Step 1 the set $\mathfrak{D}_{k,1}^{\nu}$ can only meet $(\lambda, 0)$ for $\lambda \in I_k^0$. Therefore, for each $k \in \mathbb{N}$ and each ν we have the following relation

$$\mathfrak{D}_{k,0}^{\nu} = \mathfrak{D}_{k,1}^{\nu}. \tag{3.26}$$

Step 4. Note that every solution of the nonlinear eigenvalue problem (3.1) of the form $(1, y)$ gives a solution y of the nonlinear problem (1.1)-(1.5). Hence, according to (3.26), it is obvious that if on the real axis $\mathbb{R}$ the interval I_k^0, $k \in \mathbb{N}$, lies to the left of 1 and the interval I_k^1 lies to the right of 1, or the interval I_k^0 lies to the right of 1, and the interval I_k^1 lies to the left of 1, then for each ν problem (1.1)-(1.5) has a solution y_k^{ν} which is contained in $\mathcal{S}_k^{\nu}$.

Let condition

$$\overline{f}_{\infty} < \frac{\lambda_k}{\tau} < \underline{f}_0$$

be satisfied. Then we have

$$\frac{\lambda_k}{\tau} - \underline{f}_0 + 1 < 1 \text{ and } \frac{\lambda_k}{\tau} - \overline{f}_{\infty} + 1 > 1,$$

i.e. the right end of the interval I_k^0 is to the left of 1 and the left end of the interval I_k^1 is to the right of 1.

If the condition

$$\frac{\lambda_k}{\underline{f}_{\infty}} < \tau < \frac{\lambda_k}{\overline{f}_0}$$

is satisfied, then we get

$$\frac{\lambda_k}{\tau} - \underline{f}_{\infty} + 1 < 1 \text{ and } < \frac{\lambda_k}{\tau} - \overline{f}_0 + 1 > 1,$$

i.e. the right end of the interval I_k^1 is to the left of 1 and the left end of the interval I_k^0 is to the right of 1 which completes the proof of this theorem.

References

[1] Z.S. Aliyev, Some global results for nonlinear fourth order eigenvalue problems, *Cent. Eur. J. Math.* **12** (2014), no. 12, 1811-1828.

[2] Z.S. Aliyev, Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order, *Sb. Math.* **207** (2016), no. 12, 1625-1649.

[3] Z.S. Aliyev, Y.N. Aliyeva, Global bifurcation results for some fourth-order nonlinear eigenvalue problem with a spectral parameter in the boundary condition, *Math. Methods Appl. Sci.* **46** (2023), no. 1, 1282-1294.

[4] Z.S. Aliyev, Y.N. Aliyeva, Global properties of continua of solutions bifurcating from infinity of some nonlinear eigenvalue problem fourth order, *Bull. Malays. Math. Sci. Soc.* (2024), to appear.

[5] Z.S. Aliyev, X.A. Asadov, Global bifurcation from zero in some fourth-order nonlinear eigenvalue problems. *Bull. Malays. Math. Sci. Soc.* **44** (2021), no. 2, 981-992.

[6] Z.S. Aliyev, S.B. Guliyeva, Spectral properties for the equation of vibrating beam, *Proc. Ins. Math. Mech. Nat. Acad. Sci. of Azerbaijan* **41** (2015), no. 1, 135-145.

[7] Z.S. Aliyev, N.A. Mustafayeva, Bifurcation of solutions from infinity for certain nonlinear eigenvalue problems of fourth-order ordinary differential equations, *Electron. J. Differ. Equ.* **2018** (2018), No. 98, 1-19.

[8] D.O. Banks, G.J. Kurowski, A Prüfer transformation for the equation of a vibrating beam subject to axial forces, *J. Differential Equations* **24** (1977), no. 1, 57-74.

[9] H. Berestycki, On some nonlinear Sturm-Liouville problems, *J. Differential Equations* **26** (1977), no. 3, 375-390.

[10] P.A. Binding, P.J. Browne, B.A. Watson, Spectral problem for nonlinear Sturm-Liouville equations with eigenparameter dependent boundary conditions, *Canad. J. Math.* **52** (2000), no. 2, 248-264.

[11] G. Dai, R. Ma, Bifurcation from intervals for Sturm-Liouville problems and its applications, *Electron. J. Differ. Equ.* 2014 (2014), no. 3, 1-10.

[12] J. Henderson, H. Wang, Positive solutions for nonlinear eigenvalue problems, *J. Math. Anal. Appl.* **208** (1997), no. 1, 252-259.

[13] N.B. Kerimov, Z.S. Aliev, On the basis property of the system of eigenfunctions of a spectral problem with spectral parameter in a boundary condition, *Diff. Equ.*, **43** (2007), no. 7, 905-915.

[14] R. Ma and G. Dai, Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity, *J. Funct. Anal.* **265** (2013), no. 8, 1443-1459.

[15] R. Ma, B. Thompson, Nodal solutions for nonlinear eigenvalue problems, *Nonlinear Anal.* **59** (2004), no. 5, 707-718.

[16] R. Ma, B. Thompson, A note on bifurcation from an interval, *Nonlinear Anal.* **62** (2005), no. 4, 743-749.

[17] S. Padhi, J.R. Graef, A. Kanaujiya. Positive solutions for nonlinear elliptic equations depending on a parameter with Dirichlet boundary conditions, *Differ. Equ. Dyn. Syst.* **31** (2023), no. 1, 319-336.

[18] Y. Naito, S. Tanaka, On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations, *Nonlinear Anal.* **56** (2004), no. 4, 919-935

[19] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7** (1971), no. 3, 487-513.

[20] P. H. Rabinowitz, On bifurcation from infinity, *J. Differential Equations* **14** (1973), no. 3, 462-475.

[21] B.P. Rynne, Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable, *J. Math. Anal. Appl.* **228** (1998), no. 1, 141-156.

[22] B.P. Rynne, Infinitely many solutions of superlinear fourth-order boundary value problems, *Topol.Methods Nonlinear Anal.* **19** (2002), no. 2, 303-312.

[23] S. Willard, *General Topology*, Addison-Wesley, Reading, MA, 1970

Yagut N. Aliyeva
Institute of Mathematics and Mechanics, Baku AZ1141, Azerbaijan
French-Azerbaijan University, Baku AZ1000, Azerbaijan
E-mail address: `yaqutaliyeva12@gmail.com`

Received: February 12, 2024; Accepted: April 15, 2024