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THE REPRESENTATION PROBLEM FOR A DIFFUSION EQUATION AND FRACTAL R-L LADDER NETWORKS

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Abstract. The representation problem is to prove that a discretization in space of the Fourier transform of a diffusion equation with a constant diffusion coefficient can be realized explicitly by an infinite fractal R-L ladder networks. We prove a rigidity theorem: a solution to the representation problem exists if and only if the space discretization is a geometric space scale and the fractal ladder networks is a Oustaloup one. In this case, the resistance and inertance of the ladder are explicitly determined up to a constant.

1. Introduction

We consider a classical diffusion equation of the form

$$\frac{\partial^2 \mathscr{I}}{\partial z^2} = \mathscr{D} \frac{\partial \mathscr{I}}{\partial t}, \tag{1.1}$$

where $\mathscr{I}(z,t)$ is defined for $z \in \mathbb{R}^+$, $t \in \mathbb{R}$ and $\mathscr{D}$ is a constant real number.

Denoting by $\mathscr{F}[g]$ the Fourier transform of a function g defined by $\mathscr{F}[g](\omega) = \int_{\mathbb{R}} e^{-2j\pi\omega s} g(s) ds$ and by $I(z,\omega)$ the function defined by

$$\mathscr{F}[\mathscr{I}(z,\cdot)](\omega) = I(z,\omega) \tag{1.2}$$

and using the fact that $\mathscr{F}(f')(\omega) = 2\pi j\omega \mathscr{F}(f)$, equation (1.1) gives

$$\frac{\partial^2 I}{\partial z^2} = j\mathscr{D}\omega I, \tag{1.3}$$

A discretization of equation (1.3) in the spatial variable z, for $z \geq 0$, over a (discrete) space scale $\mathbf{T}_{z,h}$ (or simply $\mathbf{T}_z$) defined by

$$\mathbf{T}_{z,h} = \{z_0 = 0,\ z_1 = h,\ z_{n+1} > z_n, n \geq 1\}, \tag{1.4}$$

where $h > 0$, is given by

$$\Delta_{-,z} \circ \Delta_{+,z}[I](z_n,\omega) = j\mathscr{D} I(z_n,\omega), \tag{1.5}$$

for all $n \geq 1$ where $\Delta_{\pm,z}$ are the backward and forward difference operators on $\mathbf{T}_z$ defined for all functions $f : \mathbb{R} \to \mathbb{C}$ by $\Delta_{+,z}[f](z_k) = (f(z_{k+1}) - f(z_k))/(z_{k+1} - z_k)$

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and $\Delta_{-,z}[f](z_k) = (f(z_k) - f(z_{k-1}))/(z_k - z_{k-1})$.

A classical result in engineering [6, 8] says that the discretization of equation (1.3) over the space scale T_z given by (1.5) can be represented by a ladder network made of resistance R_n, $n \geq 1$ and inertance L_n, $n \geq 1$ [5, 6, 8, 10, 9] with very special characteristics as long as the space scale $\mathbf{T}_z$ has a special "geometric" form. The word "represented" is not clearly defined in the previous references but relies on the fact that the constitutive equations of the ladder coincide in some sense with equation (1.5). However, it is not easy to understand from the computations presented in [6, 8] if we have some freedom in choosing the space-scale and the characteristic of the ladder network in order to provide a correspondance.

In order to formulate more precisely what we call the *representation problem*, let us first introduce the family of ladder networks under considerations : we consider R-L ladder network defined by

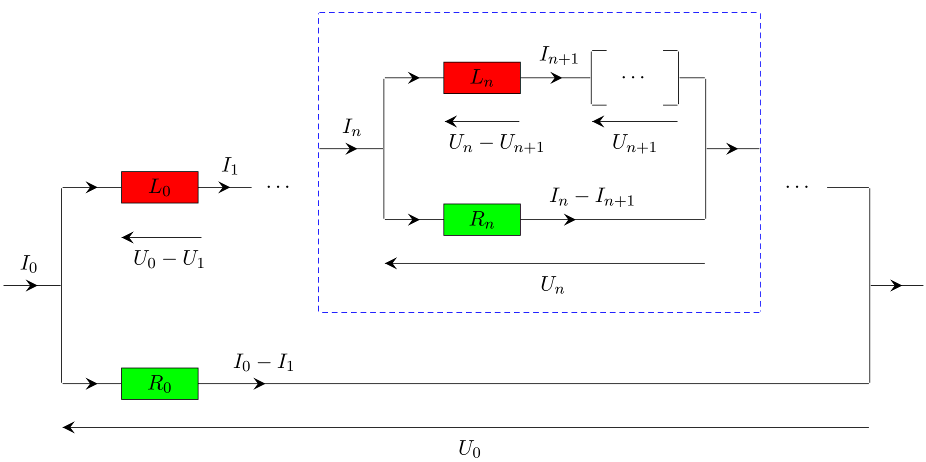


FIGURE 1. Structure of R-L ladder networks

where U_n and I_n stand for voltage and current. The *constitutive equations* of the ladder network are given by (see [8] and Section 3) :

$$\begin{aligned} U_n - U_{n-1} &= -jL_{n-1}\omega I_n, \\ I_{n+1} - I_n &= -\frac{1}{R_n} U_n. \end{aligned} \tag{1.6}$$

Introducing the discrete functions $U : \mathbb{N} \times \mathbb{R} \mapsto \mathbb{R}$, $I : \mathbb{N} \times \mathbb{R} \mapsto \mathbb{R}$, $R : \mathbb{N} \to \mathbb{R}$ and $L : \mathbb{N} \to \mathbb{C}$ and denoting by Δ_+ (resp. Δ_-) the classical forward (resp. backward) difference operator on $\mathbb{N}$ defined for all $f : \mathbb{N} \to \mathbb{C}$ by $\Delta_+[f] = f(n+1) - f(n)$ (resp. $\Delta_-[f] = f(n) - f(n-1)$) the constitutive equations for a R-L ladder network take the form

$$\Delta_-[R \circ \Delta_+[I]](n, \omega) = -jL\omega I(n, \omega), \tag{1.7}$$

for all $n \geq 1$.

The previous equation explains why a R-L ladder network are considered. Indeed, in order to reproduce the complex term $j\mathscr{D}$ one has to take an inertance whose impedance depends on j.

Let us denote by $\Phi : \mathbb{N} \mapsto \mathbf{T}_z$ the mapping defined by $\Phi(n) = z_n$ and for all $f : \mathbb{N} \to \mathbb{C}$ we denote by f_h the function $f_h = f \circ \Phi^{-1}$. We have the following relation between the finite differences operators defined for function on $\mathbb{N}$ and the associated one on $\mathbf{T}_z$:

$$(z_{n+1}-z_n)\Delta_{+,z}[f_h](z_n) = \Delta_+[f](n), \quad \text{(resp. } (z_n-z_{n-1})\Delta_{-,z}[f_h](z_n) = \Delta_-[f](n). \tag{1.8}$$

Denoting by $\mu^+ : \mathbf{T}_z \mapsto \mathbf{T}_z$ defined by $\mu_+(z_k) = z_{k+1} - z_k$, $\mu_-(z_k) = z_k - z_{k-1}$, equation (1.7) on R and I can be rewritten for R_h, L_h and I_h as

$$\mu^-(z_n)\Delta_{-,z}[\mu^+(z)R_h \circ \Delta_{+,z}[I_h]](z_n) = -jL_h\omega I_h(z_n), \tag{1.9}$$

for all $n \geq 1$.

The $R-L$*-ladder-network representation problem for the diffusion equation* (1.1) can be formulated as follows: *Can we define a space scale* $\mathbf{T}_{z,h}$ *and two functions* R_h *and* L_h*, continuous in* h*, defined on* $\mathbb{R}$ *such that equation* (2.2) *reduces to* (1.5) *for all* $h > 0$ *?*

In this article, we solve the $R-L$-ladder network representation problem for the diffusion equation (1.1) using a special class of ladder network called fractal and more precisely *Oustaloup ladder networks*: A fractal ladder network is such that there exists two constants μ and ρ such that $R_{n+1} = \mu R_n$ and $L_{n+1} = \rho L_n$. Oustaloup ladder networks (see [5]) are fractal ladder networks such that $\mu = \rho^{-1}$.

The aim of this article is to prove the following result :

Theorem 1.1. *A diffusion equation can be represented by a fractal ladder network with scaling factor* (μ, ρ) *if and only if the space scale of discretization* $\mathbf{T}_z$ *is of the form*

$$\mathbf{T}_{z,h} = \{z_0 = 0,\ z_1 = h,\ z_{n+1} - z_n = \rho(z_n - z_{n-1}),\ n \geq 1\}, \tag{1.10}$$

where $h > 0$ *is a constant and* $\rho > 1$ *and the R-L ladder network has the following characteristic :*

- *It is a Oustaloup ladder network, i.e. that the scaling factors satisfy* $\mu = \rho^{-1}$
- *The function* R_h *and* L_h *are defined by*

$$R_h(z) = \frac{c}{h + (\rho-1)z}, \quad L_h(z) = \mathscr{D}c\frac{h}{\rho}(h + (\rho-1)z), \tag{1.11}$$

  where c *is a positive constant.*
- *The resistance and inertance are of the form*

$$R_n = \frac{c}{\rho^n h}, \quad L_n = \mathscr{D}c\rho^{n-1}h. \tag{1.12}$$

The proof of this Theorem is done in several step:

- The representation implies that the function R_h and L_h have a special form, namely $L_h(z_n) = L_n = c_L(z_n - z_{n-1})$ and $R_h(z_n) = R_n = c_R/(z_{n+1} - z_n)$ where c_L and c_R are constants satisfying $c_L = \mathscr{D} c_R$.
- Fractal ladder networks have impedance which satisfies an invariance functional equation. Using this functional, we can prove that a fractal ladder network lead to a fractional behavior of order $-1/2$ if and only if $\rho = \mu^{-1}$.
- The fractal character of the ladder implies that the space scale $\mathbf{T}_z$ has a special form, namely (1.10).

In other words, we have no freedom on the choice of the space scale of discretization as well as the form of the characteristic of the ladder networks as long as fractal ladder networks are considered.

The specific choice of fractal ladder network is due to the fact that we can easily put conditions ensuring a fractional behavior of order $-1/2$. An open question is to characterize general ladder networks admitting a fractional behavior of order $-1/2$.

The structure of the article is as follows: in Section 2, we precise our formulation of the representation problem and we derive the conditions for a ladder network to satisfy it on a given space scale $\mathbf{T}_z$. In Section 3, we derive the constitutive equations of a ladder networks and we give explicit formulae for the impedance using a representation of R-L ladder networks as colored labeled forest. We then derive an invariance functional relation satisfied by the impedance. This enable use to precise under which condition a fractional behavior of order $-1/2$ can be realized by the ladder. Section 4, we prove that in the context of fractal ladder network, the space scale has to be given by (1.10) and we give the characteristic of the associated fractal ladder network. Section 5 gives some perspective of this work.

2. The $R - L$ ladder network representation problem for diffusion

In order to solve the representation problem, we impose the following set of constraints :

Representation conditions: Let $\mathbf{T}_z$ be a discrete space scale. We assume that there exists two constants c_L and c_R such that

$$\frac{L(z)}{\mu^-(z)} = c_L, \quad \mu^+(z)R(z) = c_R. \tag{2.1}$$

Under these conditions, equation (2.2) reduces to

$$\Delta_{-,z} \circ \Delta_{+,z}[I](z_n) = -j\frac{c_L}{c_R}\omega I(z_n), \tag{2.2}$$

for all $n \geq 1$ and we have to impose the compatibility relation

$$\frac{c_L}{c_R} = \mathscr{D}. \tag{2.3}$$

The representation conditions are by itself not sufficient to constrain the function L and R. However, we can look for more specific properties of the classical diffusion equation in order to select more appropriate R-L ladder network.

The main idea is the following: It can be proved that the Fourier transform of a solution of the diffusion equation on a semi-infinite domain $z \geq 0$ with boundary conditions $I(z,0) = 0$ and $\lim_{z\to+\infty} I(z,\omega)$ behaves as $\sqrt{\omega}$, i.e. has a fractional behavior of order $1/2$.

We then look for the constitutive equation of a R-L ladder network and characterize whose admit a fractional behavior of order 1/2. This characterization is then used to constrain the functions L and R of the representation problem.

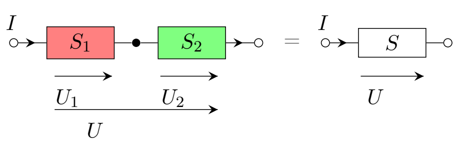
3. Ladder network and constitutive equations

3.1. Classical relations on electronic circuits and colored labeled forest. We follow the presentation made by R. Feynman et al. in [2] about ladder networks, in particular ([2], 22-6 p. 22-12 to 22-15).

Let us denote by Z and Y the impedance and admittance, respectively. We have $Z = \dfrac{1}{Y}$. In the following, by S we denote an equivalent to one of any component of electronic circuit: resistance (R), inductance (L), capacitance (C).

We introduce two kinds of the basic relations of the impedance with the current I and the voltage U.

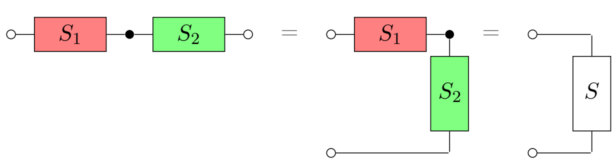
- series type relation:



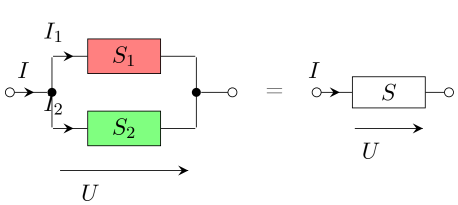
$U = U_1 + U_2 = Z(S)I$

$Y(S) = \dfrac{1}{Y[S_1]^{-1} + Y[S_2]^{-1}}$

Another representation of the above relation can be introduce as follows



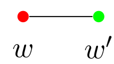
- parallel type relation:

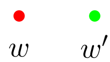


$U = Z(S)(I_1 + I_2) = Z(S)I$

$Y[S] = Y[S_1] + Y[S_2]$

From a mathematical point of view an electric circuit is a graph obtained in the following way. Let F be a family of basic electronic devices, then

- each node of the graph represents a specific element of F;
- each element of F is represented by a color and posses a weight which is a real number;
- two elements of F: $\bullet_w$ and $\bullet_{w'}$ in serie is represented as 

and in parallel as 

.

Using these rules, every series-parallel networks [3, 7] can be represented by a collection of colored labeled trees, i.e. colored labeled forest.

As an example, we have the following correspondence between electrical diagrams of figure 2 and colored labeled forest given in figure 3.

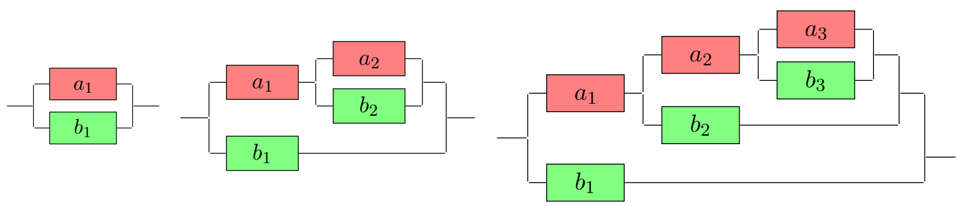


FIGURE 2. Sample electrical diagrams.

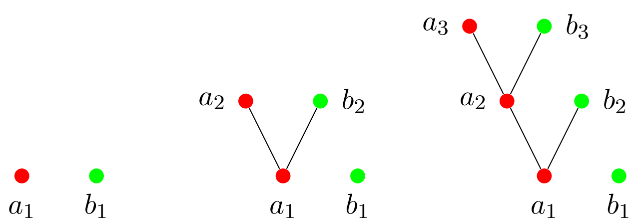


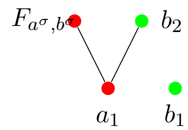
FIGURE 3. Colored labeled forest

3.2. General properties of admittance for electronic diagrams. Let us consider a parallel-serie electronic diagram encoded by a colored labeled forest F. We are interested in the explicit computation of the associated admittance denoted by $Y(F)$. For R-L ladder network, we have two colors: green for resistance and red for inertance. The admittance of such nodes are given by

$$Y[\bullet_R](s) = R^{-1}, \quad Y[\bullet_L](s) = \frac{1}{sjC} \text{ with } j^2 = -1. \tag{3.1}$$

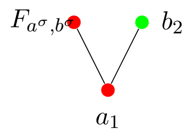
Let $F_{a,b}$ be an infinite colored labeled forest constructed recursively with two colors and labels $a = a_1, \dots$ and $b = b_1, \dots$. associated to the structure of $R - L$ ladder networks given in Figure 1. We have :

$$Y(F_{a,b}) = Y\left(\right.$$



$$\left.\right)$$

$$= Y\left(\right.$$



$$\left.\right) + Y({\color{green}\bullet}_{b_1}) \qquad (3.2)$$

$$= \frac{1}{Y({\color{red}\bullet}_{a_1})^{-1} + Y(F_{a^\sigma,b^\sigma})^{-1}} + Y({\color{green}\bullet}_{b_1}),$$

where $a^\sigma = a_2, \dots$ and $b^\sigma = b_2, \dots$.

Using the classical notation for **continued fractions** given by

$$[\alpha_1, \alpha_2, \dots] = \alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\ddots}}, \qquad (3.3)$$

we can prove by induction the following Lemma :

Lemma 3.1. *For any recursive diagram $F_{a,b}$ with labels $a = a_1, \dots$ and $b = b_1, \dots$ we have*

$$Y(F_{a,b})(\omega) = [Y({\color{green}\bullet}_{b_1}), Y({\color{red}\bullet}_{a_1}(\omega)^{-1}, Y({\color{green}\bullet}_{b_2}), Y({\color{red}\bullet}_{a_2})(\omega)^{-1}, \dots, Y({\color{green}\bullet}_{b_n}), Y({\color{red}\bullet}_{a_n})(\omega)^{-1}, \dots]. \qquad (3.4)$$

In order to go further we will focus on R-L ladder networks which possess a special structure called fractal.

3.3. Admittance for fractal R-L networks. Let $F_{a,b}$ be a ladder network. We say that $F_{a,b}$ is fractal if there exists two (non zero) constants μ and ρ such that

$$a_{n+1} = \mu a_n, \quad b_{n+1} = \rho b_n, \qquad (3.5)$$

for all $n \geq 1$.

A fractal ladder network is then determine by two constants a_1 and b_1 and the two scaling constants μ and ρ. We denote by $F_{a_1,b_1}(\mu,\rho)$ such a network and $Y^{\mu,\rho}_{a_1,b_1}$ its admittance.

Remembering that we have

$$Y({\color{red}\bullet}_a)(\delta\omega) = \delta^{-1} Y({\color{red}\bullet}_a)(\omega) \qquad (3.6)$$

and the fact that for all $\alpha \neq 0$, we have

$$\alpha[a_1, a_2, \dots] = [\alpha a_1, \alpha^{-1} a_2, \dots] \qquad (3.7)$$

we deduce the following Lemma:

Lemma 3.2. *Let $F_{a_1,b_1}(\mu,\rho)$ be a fractal ladder network then*

$$Y^{\sigma,\rho}_{a_1,b_1}(\omega) = [Y({\color{green}\bullet}_{b_1}), (Y({\color{red}\bullet}_{a_1})(\omega))^{-1}, \rho^{-1} Y^{\sigma,\rho}_{a_1,b_1}(\mu\rho^{-1}\omega)]. \tag{3.8}$$

We then have an invariance relation satisfied by the impedance. This relation can be simplified under some conditions.

We assume that

$$\lim_{\omega \to 0} Y(\mu\rho^{-1}\omega) = +\infty, \tag{3.9}$$

and

$$\lim_{\omega \to 0} \frac{Y(\mu\rho^{-1}\omega)}{Y({\color{red}\bullet}_{a_1})(\omega)} = 0. \tag{3.10}$$

Under these assumptions one obtain for ω sufficiently close to zero the simplified invariance relation

$$Y^{\sigma,\rho}_{a_1,b_1}(\omega) = \rho^{-1} Y^{\sigma,\rho}_{a_1,b_1}(\mu\rho^{-1}\omega). \tag{3.11}$$

Assuming that the impedance possesses a fractional behavior of the

$$Y(\omega) = c\omega^{\nu}, \tag{3.12}$$

where c is a constant.

Replacing directly in relation (3.11) we obtain the following Lemma :

Lemma 3.3. *A function $Y(\omega)$ satisfies the simplified invariance relation* (3.11) *if and only if*

$$\nu = \frac{\ln(\rho)}{\ln(\mu\rho^{-1})}. \tag{3.13}$$

As our ladder network has to reproduce the fractional behavior of order $-1/2$ expected for the Fourier transform of the diffusion equation, we impose the relation

$$\nu = \frac{\ln(\rho)}{\ln(\mu\rho^{-1})} = -1/2, \tag{3.14}$$

which leads to

$$\rho = \mu^{-1}. \tag{3.15}$$

We resume these computations as follows:

Lemma 3.4. *A fractal ladder network has an impedance with a fractional behavior of order $-1/2$ if and only if $\rho = \mu^{-1}$.*

Fractal ladder networks with $\rho = \mu^{-1}$ where already used by A. Oustaloup in [5]. We then call this particular kind of fractal ladder networks a *Oustaloup ladder networks*.

4. Solving the representation problem

Lemma 3.4 imposes serious constraints on the space scale $\mathbf{T}_z$ which can be used to discretize the diffusion equation if one want to obtain a representation by a fractal ladder network with a compatible fractional behavior. Indeed, we have:

Lemma 4.1. *A diffusion equation can be represented by a fractal ladder network if and only if the space scale of discretization* $\mathbf{T}_z$ *is of the form*

$$\mathbf{T}_{z,h} = \{z_0 = 0,\ z_1 = h,\ z_{n+1} - z_n = \rho(z_n - z_{n-1}),\ n \geq 1\}, \tag{4.1}$$

where $h > 0$ *is a constant.*

Proof. This comes directly from the scale dependance of $R(z)$ and $L(z)$ for a fractal ladder network. Indeed, we must have $R(z_{n+1}) = \rho^{-1}R(z_n)$ which implies due to equations (2.1) that $\mu^+(z_{n+1}) = \rho\mu^+(z_n)$ or more explicitly

$$z_{n+1} - z_n = \rho(z_n - z_{n-1}), \tag{4.2}$$

for all $n \geq 1$. As we are considering a diffusion equation on the semi-infinite space $z \geq 0$, we take $z_0 = 0$ and $z_1 = h$. The previous relation fixes the space scale. □

It must be noted that this space scale of discretization is usually fixed at the beginning in the engineering literature (see [6, 8]) but follows from the constraints of the representation problem in our case.

A direct consequence of Lemma 4.1 is that one can explicit the characteristic of the associated fractal ladder network. Indeed, by definition of the space scale $\mathbf{T}_z$, we have for all $n \geq 2$

$$z_n = (\rho^{n-1} + \cdots + \rho + 1)z_1 = \frac{\rho^n - 1}{\rho - 1}h, \tag{4.3}$$

and for $n \geq 0$

$$\mu^+(z_n) = z_{n+1} - z_n = \rho^n h. \tag{4.4}$$

As a consequence, we obtain using the equations (2.1) that for all $z_n \in \mathbf{T}_z$, the function R is given by

$$R(z_n) = \frac{c_R}{\rho^n h}. \tag{4.5}$$

Using equation (4.3), we can express $\rho^n h$ as a function of z_n :

$$\rho^n h = h + (\rho - 1)z_n. \tag{4.6}$$

As a consequence, we obtain

$$R(z_n) = \frac{c_R}{h + (\rho - 1)z_n}. \tag{4.7}$$

In the same way, we have

$$L(z_n) = \mathscr{D}c\frac{h}{\rho}(h + (\rho - 1)z_n). \tag{4.8}$$

This concludes the proof of the Theorem.

5. Conclusion and perspectives

We have proved that in order to obtain a representation for a given discretization of the Fourier transform of a diffusion equation by a fractal R-L ladder network, a specific geometric space scale of discretization is necessary and moreover that in this case, the form of the resistance and inertance of the ladder are uniquely determined up to a constant. We then justify the classical choice made in engineering [6, 8] in order to connect the diffusion equation with R-L ladder networks.

Another representation of the diffusion equation can also be obtained using fractional calculus as proved by K.B. Oldham and J. Spanier [4]. The representation by R-L ladder networks appears as an alternative to this approach (see for example [6, 8]).

The interest for all these different approach is to obtain finite and accurate representation of the behavior of a diffusion with a small number of parameters which can be identified experimentally. Works in this direction are needed.

Recently, a related statement was proved for more general diffusion equations with spatial dependent coefficients (see [10, 9]). The computations are very intricated and it is difficult to determine which freedom we have on the construction of the associated ladder network. A mathematical justification of these results is in progress [1].

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