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# DETERMINATION OF A SPACEWISE DEPENDENT HEAT SOURCE IN BIHARMONIC HEAT EQUATION FROM FINAL TEMPERATURE MEASUREMENTS

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**Abstract**. The paper deals with the inverse problem associated with the biharmonic heat equation, aiming to recover a space-dependent heat source and the temperature distribution based on measurements of the final temperature. The study establishes both the existence and uniqueness of the classical solution, as well as the existence and uniqueness of the generalized solution. These results are attained through the application of the method of series expansion in terms of eigenfunctions for the Dirichlet bi-Laplacian.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$ be bounded open set with piecewise smooth boundary $\partial\Omega$ in an $d$-dimensional Euclidean space $\mathbb{R}^d$. A multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ is p- tuple of non-negative integer numbers. Its length is defined as $|\alpha|$, namely $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_d}^{\alpha_d}$, where $\partial_{x_i}^{\alpha_i} = \dfrac{\partial^{\alpha_i}}{\partial_{x_j}^{\alpha_i}}$.

Consider the following problem:

$$\partial_t u + \Delta^2 u = f(x)g(t), \quad \text{in } \Omega_T, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

$$u = \partial_\nu u \ = 0 \qquad \text{on } \partial\Omega \times (0,T], \tag{1.3}$$

where $u_0$ and $g$ are known functions, $\nu$ denotes the outward unit normal vector field of the boundary $\partial\Omega$ and $\Omega_T = \Omega \times (0,T]$. Here $\Delta = \sum_{j=1}^d \partial_{x_j}^2$ be the Laplace operator in $\mathbb{R}^d$. The derivative $\partial_\nu = \sum_{j=1}^p \nu_j \partial_{x_j}$ denotes the derivative at the direction of the exterior unit normal vector $\nu = (\nu_1, ..., \nu_d)$ to the surface $\partial\Omega$. If $\partial\Omega \in C^1$ then the first order normal derivatives $\partial_\nu$ are defined near $\partial\Omega$.

We consider the inverse problem of finding the pair of function $\langle u, f\rangle$ from (1.1)-(1.3) and from final time observation measurement

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$$u(x,T) = u_f(x), \quad x \in \Omega, \tag{1.4}$$

where $u_f(x)$ is known function defined on $\mathbb{R}$. The relevant literature on the usual heat equation includes works on the one-dimensional case, such as those by [5, 8, 9]. For the multidimensional case, studies by [2, 3, 4, 7, 6, 11] contribute to the understanding of the inverse problem involving the identification of a space-dependent source term.

The remaining chapters of the paper are organized as follows. The chapter on existence and uniqueness consists of two sub-chapters. In the first sub-chapter, we establish the admissible class of initial and final data by utilizing Weyl's asymptotic formula for the Dirichlet bi-Laplacian. The existence and uniqueness of the classical solution are derived through the application of the Fourier method to the orthonormal eigenfunctions of the Dirichlet bi-Laplacian. Moving on to the second sub-chapter, we prove the existence and uniqueness of the generalized solution for the minimal class of initial and final data. Finally, in the conclusion, we delve into the future perspectives of the inverse coefficient problem for the biharmonic heat equation.

## 2. Existence and Uniqueness of the Solution

**2.1. Clamped plate problem.** We consider the Dirichlet eigenvalue problem of the biharmonic operator, the so-called clamped plate problem, which describes vibrations of a clamped plate. The Dirichlet eigenvalues are found by solving the following problem for an unknown function $y \neq 0$ and eigenvalue $\lambda$:

$$\begin{cases} \Delta^2 y - \lambda y = 0, \text{in } \Omega, \\ y = \partial_\nu y = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Delta^2$ is the biharmonic operator in $\mathbb{R}^d$ and $\partial_\nu$ denotes the outward normal derivative on boundary $\partial\Omega$.

By using the spectral theorem for compact self-adjoint operators, it can be shown that the eigenspaces are finite-dimensional and that the Dirichlet eigenvalues $\lambda$ are real, positive, and have no limit point. Thus, they can be arranged in increasing order:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty,$$

where each eigenvalue is counted according to its geometric multiplicity. It follows from [1, 10] that Weyl's asymptotic formula for Dirichlet bi-Laplacian is

$$\lambda_n \sim (2\pi)^4 \left(\frac{n}{\omega_d |\Omega|}\right)^{\frac{4}{d}}, \ n \to \infty, \tag{2.2}$$

where $\omega_d$ is volume of unite ball in $\mathbb{R}^d$ and $|\Omega|$ is the volume of a domain $\Omega \subset \mathbb{R}^d$.

The eigenspaces are orthogonal in the space of square-integrable functions, and consist of smooth functions. In fact, the system of eigenfunctions $y_n$, $n = 1, 2, 3, ...$ are complete orthonormal system in $L_2(\Omega)$.

Suppose that the domain $\Omega \subset \mathbb{R}^d$ has a sufficiently smooth boundary $\partial\Omega$, and that corresponding orthonormal Dirichlet eigenvalues satisfy the following condition:

$$|\partial_x^\alpha y_n(x)| = \mathcal{O}\left(|\lambda_n|^{\frac{|\alpha|}{4}}\right), \quad x \in \bar{\Omega} \tag{2.3}$$

for the multi-indices $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_p)$ with $|\alpha| \leq 4$. It is easy to verify that this is true for $d = 1$ and also for rectangular (piecewise smooth) domain in $d \geq 2$ dimensions.

**2.2. Existence and uniqueness of the classical solution.** We define a classical solution to the problem (1.1)-(1.4) as a pair of functions $u(x,t) \in C^{4,1}(\Omega_T) \cap C^{1,0}(\tilde{\Omega}_T) \cap C(\bar{\Omega})$ and $f(x) \in C(\bar{\Omega})$ that, when applied, transform the problem into an identity, where $\tilde{\Omega}_T = \bar{\Omega} \times (0,T]$. The consistency conditions are $u_0(x)$, $u_f(x) \in D(\Delta^2)$ where $D(\Delta^2) \equiv \left\{\varphi \in C^1(\bar{\Omega}) : \varphi = \partial_\nu \varphi = 0 \text{ on } \partial\Omega\right\}$.

The following lemma is applicable for the classical solution of biharmonic heat equation.

**Lemma 2.1.** *Let $k > \frac{d}{8} + 1$ be an integer number. If $\varphi \in C^{4k}(\bar{\Omega})$ and $\Delta^{2m}\varphi = 0$, $\partial_\nu \Delta^{2m}\varphi = 0$, $m = 0, 1, ..., k-1$ on $x \in \partial\Omega$ , then we have*

$$\sum_{n=1}^{\infty} \lambda_n |\varphi_n| \leq c \left\| \Delta^{2k}\varphi \right\|_{L_2},$$

*where* $c = \left[\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2(k-1)}}\right]^{\frac{1}{2}}$.

*Proof.* From the second Green's identity for the Laplacian we get

$$\varphi_n = \frac{1}{\lambda_n^k}\left(\Delta^{2k}\varphi\right)_n,$$

where $\left(\Delta^{2k}\varphi\right)_n = \left(\Delta^{2k}\varphi, y_n\right)$ . We have the estimate

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n |\varphi_n| &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{k-1}} \left|\left(\Delta^{2k}\varphi\right)_n\right| \leq \left[\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2(k-1)}}\right]^{\frac{1}{2}} \left[\sum_{n=1}^{\infty} \left|\left(\Delta^{2k}\varphi\right)_n\right|^2\right]^{\frac{1}{2}} \\ &\leq \left[\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2(k-1)}}\right]^{\frac{1}{2}} \left\|\Delta^{2k}\varphi\right\|_{L_2} \end{aligned}$$

by the Cauchy-Schwartz and Bessel inequalities. The series $c^2 = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2(k-1)}}$ is convergent by Weyl's asymptotic formula (2.2), since $k > \frac{d}{8} + 1$. □

The formal solution of (1.1)-(1.4) is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) y_n(x), \;\; f(x) = \sum_{n=1}^{\infty} f_n y_n(x) \tag{2.4}$$

with

$$\begin{aligned} u_n'(t) + \lambda_n u_n(t) &= f_n g(t), \\ u_n(0) &= u_{0,n}, \;\; u_n(T) = u_{f,n}, \end{aligned}$$

where $u_{n,0} = (u_0, y_n), \; u_{f,n} = (u_f, y_n)$.

The solution of last problem is

$$u_n(t) = u_{0,n}e^{-\lambda_n t} + f_n \int_0^t g(\tau)e^{-\lambda_n(t-\tau)}d\tau.$$

where

$$f_n = \frac{u_{f,n} - u_{0,n}e^{-\lambda_n T}}{\int_0^T g(\tau)e^{-\lambda_n(T-\tau)}d\tau}.$$

It is easy to show that

$$\int_0^T g(\tau)e^{-\lambda_n(T-\tau)}d\tau \geq \frac{g_0}{\lambda_n}(1 - e^{-\lambda_n T}),$$

if we suppose that $g(t) \geq g_0 > 0,\ 0 \leq t \leq T$.

The series

$$f(x) \sim \sum_{n=1}^{\infty} f_n y_n(x),\ \ |f_n| \leq C\lambda_n \left(|u_{f,n}| + |u_{0,n}|\right)$$

$$u(x,t) \sim \sum_{n=1}^{\infty} u_n(t) y_n(x),\ \ |u_n| \leq C\lambda_n \left(|u_{f,n}| + |u_{0,n}|\right) \tag{2.5}$$

are uniformly convergent in $\bar{\Omega}_T$ and in $\bar{\Omega}_T$, respectively, since the majorant series $M\sum_{n=1}^{\infty} \lambda_n |u_{0,n}|$ and $M\sum_{n=1}^{\infty} \lambda_n |u_{f,n}|$ are convergent. By the estimate (2.3) and Lemma 2.1, where $|y_n(x)| \leq M$. It means that $f \in C(\bar{\Omega})$ and $u \in C(\bar{\Omega}_T)$ with $u = 0$ on $\partial\Omega \times [0,T]$ and $u(x,0) = u_0(x)$ on $\bar{\Omega}$. The derivatives by time and spatial variables

$$u_t(x,t) \sim \sum_{n=1}^{\infty} u'(t) y_n(x),\ \ \left|u_n'(t)\right| \leq C\lambda_n \left(|u_{f,n}| + |u_{0,n}|\right), \tag{2.6}$$

$$\Delta^2 u \sim \sum_{n=1}^{\infty} \lambda_n u_n(t) y_n(x) \tag{2.7}$$

is uniformly convergent in $\bar{\Omega}_{T,\varepsilon} \equiv \{(x,t)\ :\ x \in \bar{\Omega}, \varepsilon \leq t \leq T\}$ for $\forall\varepsilon \in (0,T)$, since the majorant series $M\sum_{n=1}^{\infty} \lambda_n^{1+\frac{|\alpha|}{4}} e^{-\lambda_n\varepsilon} |u_{0,n}|$ and $M\sum_{n=1}^{\infty} \lambda_n^{1+\frac{|\alpha|}{4}} e^{-\lambda_n\varepsilon} |u_{f,n}|$ are convergent. It means that $u \in C^{4,1}(\Omega_T)$ and satisfies the equation (1.1).

In addition,

$$\partial_x^\alpha u(x,t) = \sum_{n=1}^{\infty} u_n(t)\partial_x^\alpha y_n(x),\ |\alpha| = 1$$

are continuous in $\bar{\Omega}_{T,\varepsilon}$ for $\forall\varepsilon \in (0,T)$ and satisfies $\partial_\nu u = 0$ on $\partial\Omega \times [\varepsilon, T]$. It is clear to conclude that $\partial_\nu u = 0$ on $\partial\Omega \times (0,T]$.

Each of the solution of the problem (1.1)-(1.4) can be represented in form of (2.4) given that the system of eigenfunctions $y_n$, $n = 1,2,3,...$ forms an orthonormal basis in the space $L_2(\Omega)$. The uniqueness of the solution to the inverse problem is guaranteed by the fact that the solution to problem (1.1)-(1.4) is uniquely expressible in the form of (2.4).

Let us introduce the sets:

$C_0^k(\Delta^2) = \{\varphi \in C^{4k}(\bar{\Omega}) : \ \Delta^{2m}\varphi = 0, \partial_\nu \Delta^{2m}\varphi = 0, \ m = 0, 1, ..., k-1 \text{ on } \ \partial\Omega\}$ for some integer $k > \frac{d}{8} + 1$.

The following theorem establishes the existence and uniqueness of the classical solution of inverse problem:

**Theorem 2.1.** *Let the domain $\Omega \subset \mathbb{R}^d$ has a sufficiently smooth boundary, and that the corresponding orthonormal eigenvalues satisfies the condition (2.3), the initial and final functions are $u_0$, $u_f \in C_0^k(\Delta^2)$ for some integer $k > \frac{d}{8} + 1$ and $g(t) \in C[0,T]$ with $g(t) \geq g_0 > 0$, $\forall t \in (0,T]$ ($g_0$ is positive constant).Then there exists an unique classical solution pair $\langle u, f \rangle$ for the inverse problem (1.1)- (1.4).*

**Remark:** The classical solution to the problem (1.1)-(1.4), as a pair of funcitons $\langle u, f \rangle$ from a more convenient class where $u(x,t) \in C^{4,1}(\Omega_T) \cap C^{1,0}(\bar{\Omega}_T)$ and $f(x) \in C(\bar{\Omega})$ can be obtained with the over-regularity of the initial and final data as $u_0, u_f \in C_0^k(\Delta^2)$ for some integer $k > \frac{d}{8} + 2$. This is because the majorant series $\sum_{n=1}^{\infty} \lambda_n^{1+\frac{1}{4}} |u_{0,n}|$ or $\sum_{n=1}^{\infty} \lambda_n^{1+\frac{1}{4}} |u_{f,n}|$ must be convergent.

**2.3. Existence and uniqueness of the generalized solution.** The generalized solution for the problem (1.1)- (1.4) is comprehended as a pair of functions $u(x,t) \in H^{4,1}(\Omega_T)$ and $f(x) \in L_2(\Omega)$ such that they render the problem identically satisfied almost everywhere.

Since the functions $u_0$ and $u_f$ belongs to class $H_0^4(\Omega)$ then satisfy the boundary condition (1.3) and they can be extended the Fourier series on eigenfunctions of problem (2.1):

$$\Delta^2 u_0(x) = \sum_{n=1}^{\infty} \lambda_n u_{0,n} y_n(x), \ \Delta^2 u_f(x) = \sum_{n=1}^{\infty} \lambda_n u_{f,n} y_n(x). \tag{2.8}$$

The system of eigenfunctions $y_n$, $n = 1, 2, 3, ...$ forms a orthonormal system in the space $L_2(\Omega)$, the Bessel inequality holds:

$$\sum_{n=1}^{\infty} |\lambda_n u_{0,n}|^2 \leq \left\|\Delta^2 u_0\right\|_{L_2}, \ \sum_{n=1}^{\infty} |\lambda_n u_{f,n}|^2 \leq \left\|\Delta^2 u_0\right\|_{L_2}.$$

Since the series (2.8) are absolute convergent and the inequalities (2.5)-(2.7) implies the convergence of series (2.4) which are solutions of (1.1)- (1.4) from the classes $u(x,t) \in H^{4,1}(\Omega_T)$, $f(x) \in L_2(\Omega)$.

Each of the solution of the problem (1.1)- (1.4) can be represented in form of (2.4) given that the system of eigenfunctions $y_n$, $n = 1, 2, 3, ...$ forms an orthonormal basis in the space $L_2(\Omega)$. The uniqueness of the solution to the inverse problem is guaranteed by the fact that the solution to problem (1.1)- (1.4) is uniquely expressible in the form of (2.4).

The following theorem establishes the existence and uniqueness of the generalized solution of inverse problem:

**Theorem 2.2.** *Let the domain $\Omega \subset \mathbb{R}^d$ has a sufficiently smooth boundary, and that the corresponding orthonormal eigenvalues satisfies the condition (2.3) the initial and final functions are $u_0(x)$, $u_f(x) \in H_0^4(\Omega)$ and $g(t) \in L_2(0,T)$ with $g(t) \geq g_0 > 0$ a.e., then there exists a unique generalized solution $u(x,t) \in H^{4,1}(\Omega_T)$, $f(x) \in L^2(\Omega)$ of the problem (1.1)- (1.4).*

## 3. Conclusion

The present paper addresses the existence and uniqueness of both classical and generalized solutions for the inverse problem involving the determination of a space-dependent source term, alongside the temperature distribution based on final temperature measurements. The paper introduces a natural class of functions pertaining to initial and final data, which is sufficient for the generalized solution but falls short for the classical solution. The admissible class of initial and final data is precisely determined using Weyl's asymptotic formula for the Dirichlet bi-Laplacian.

The obtained results for existence and uniqueness are achieved through the application of the Fourier method to the orthonormal eigenfunctions of the Dirichlet bi-Laplacian. There are two potential directions for future improvement of the presented work. The first involves extending the problems to include time-dependent diffusion, potential, or source coefficient identification for the bilharmonic heat equation. The second direction focuses on exploring inverse problems associated with the poliharmonic heat equation.

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