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OPTIMAL CONTROL PROBLEM FOR THE SECOND ORDER UNSTABLE HYPERBOLIC EQUATION WITH A NONLOCAL BOUNDARY CONDITION

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Abstract. In the paper we consider the optimal control problem for the second order unstable hyperbolic equation with a nonlocal boundary condition. The theorem on the existence of the optimal pair is proved and necessary condition for optimality in the form of variational inequality is derived.

1. Introduction

Recently, nonlocal problems for partial differential equations have been intensively studied [1, 2, 6,7, 12]. It is due to the fact that there are processes in which directly measuring the value of certain quantities sometimes becomes technically impossible. Therefore, their mean values are measured, and naturally there appear boundary conditions that relate the values of these quantities at the boundary and inside the domain under consideration.

That is why, investigation of optimal control problems for processes described by boundary value problems is more interesting [3, 4,5,11,13,14].

Furthermore, if the equation contains a nonlinear term, then additional difficulties arise in studying the solvability of boundary value problems. Note that in the problems of control of flexible structures, transfer of electrical energy and the shape of the plasma, the equations of state represent such features as discontinuity, unstability, and so on. In such systems, the given control may not correspond to any state at all, or there will be infinitely many states, or the state will be the only one, but unstable. Therefore, the study of optimal control problems in these processes is of scientific and practical interest [9]. In this work an unstable problem with a nonlocal boundary condition for hyperbolic equation of the second order was considered for the first time.

Based on the above, in this work a theorem on the existence of the optimal pair is proved and a necessary condition for optimality in the form of variational inequality is derived in the optimal control problem for unstable hyperbolic equation with a nonlocal boundary condition.

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2. Problem statement

In the open set $Q = \Omega \times (0,T), \Omega \subset R^n, n = 2$ or 3, we consider the pair (υ, u), where υ is a control, u is a state satisfying the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u - u^3 = \upsilon(x,t), (x,t) \in Q = \Omega \times (0,T), \tag{2.1}$$

with initial conditions

$$u(x,0) = \varphi^0(x), \frac{\partial u(x,0)}{\partial t} = \varphi^1(x), x \in \Omega, \tag{2.2}$$

and the boundary condition

$$\left.\frac{\partial u}{\partial \nu}\right|_S = \int_\Omega K(x,y)u(y,t)dy, (x,t) \in S, \tag{2.3}$$

where $S = \partial\Omega \times (0,T)$ is a lateral surface of Q, $\partial\Omega$ is a regular boundary of the domain Ω, $\varphi^0 \in W_2^1(\Omega), \varphi^1 \in L_2(\Omega)$, $K(x,y) \in L_\infty(\Omega\times\Omega)$ are the given functions, $\frac{\partial K(x,\ y)}{\partial x} \in L_2(\Omega \times \Omega)$, $K(x,y) = K(y,x)$, and ν is an external normal to the boundary S.

Definition 2.1. We call the pair (υ, u) admissible if

$$\upsilon \in V \subset L_2(Q),\ u \in L_6(Q), \tag{2.4}$$

satisfy (2.1)-(2.3), where $V \neq \emptyset$ is a closed convex set.

Assume that the set of admissible pairs is not empty, i.e $\{\upsilon, u\} \neq \emptyset$.

Let us define the functional

$$I(\upsilon,u) = \frac{1}{6}\|u - u_d\|^6_{L_6(Q)} + \frac{\alpha}{2}\|\upsilon - \omega\|^2_{L_2(Q)}, \tag{2.5}$$

where $u_d \in L_6(Q)$, $\omega \in L_2(Q)$, are a given functions, $\alpha > 0$ is a given number and consider the optimal control problem $infI\ (\upsilon, u)$, where (υ, u) is from the set of admissible pairs.

3. Existence of the optimal pair and necessary conditions for optimality

Theorem 3.1. *Under the above conditions imposed on the data of the problem* (2.1)-(2.3),(2.4),(2.5) *there exists the optimal pair* (υ_0, u_0), *i.e.*

$$I(\upsilon_0, u_0) = infI(\upsilon, u), \tag{3.1}$$

where (υ, u) *are admissible pairs.*

Proof. Let (υ_k, u_k) be a minimizing sequence, i.e.

$$\lim_{k\to\infty} I(\upsilon_k, u_k) = infI(\upsilon, u), \tag{3.2}$$

where (υ, u) are admissible pairs. From the definition of $I(\upsilon, u)$ it follows that

$$\|\upsilon_k\|_{L_2(Q)} + \|u_k\|_{L_6(Q)} \leq c. \tag{3.3}$$

Here and in the sequel, by c we denote the constants independent of estimated quantities and controls.

But then the sequence in the right side of the equation satisfies

$$\frac{\partial^2 u_k}{\partial t^2} - \Delta u_k = \upsilon_k + u_k^3 \subset L_2(Q) \tag{3.4}$$

in bounded in $L_2(Q)$. Hence we obtain the following uniform estimate for solution of this equation with initial conditions (2.2) and boundary condition (2.3)

$$\|u_k\|_{L_\infty(0,T,W_2^1(\Omega))} + \left\|\frac{\partial u_k}{\partial t}\right\|_{L_\infty(0,T,L_2(\Omega))} \leq c. \tag{3.5}$$

Let us right-hand side of equation (3.4) denote by $f_k(x,t) = \vartheta_k + u_k^3$.

For solvability of boundary value problem for equation (3.4) we use the Galerkin method. Let $\{\varphi_k(x)\}$ be a fundamental system in $W_2^1(\Omega)$ and the following orthonormality property be fulfilled:

$$(\varphi_k, \varphi_l) = \int_\Omega \varphi_k(x)\varphi_l(x)dx = \delta_k^l = \begin{cases} 1, l = k \\ 0, l \neq k \end{cases}.$$

We search the approximate solution $u^N(x,t)$ of the problem (3.4),(2.2),(2.3) in the form

$$u^n(x,t) = \sum_{k=1}^{N} C_k^N(t)\varphi_k(x)$$

from the following relations:

$$\int_\Omega \frac{\partial^2 u_k^N(x,t)}{\partial t^2}\varphi_l dx + \int_\Omega \sum_{i=1}^{n} \frac{\partial u_k^N(x,t)}{\partial x_i}\frac{\partial \varphi_l}{\partial x_i}dx -$$

$$-\int_{\partial\Omega} \varphi_l(x)\int_\Omega K(x,y)u_k^N(y,t)dyds = \int_\Omega f_k(x,t)\varphi_l dx,\ l = 1,2,...,N, \tag{3.6}$$

$$C_k^N(0) = \alpha_k^N, \quad \left.\frac{dC_k^N(t)}{dt}\right|_{t=0} = \beta_k^N, \tag{3.7}$$

where α_k^N and β_k^N are the coefficients of the sums $\varphi_0^N(x) = \sum\limits_{k=1}^{N} \alpha_k^N(t)\varphi_k(x)$ and $\varphi_1^N(x) = \sum\limits_{k=1}^{N} \beta_k^N(t)\varphi_k(x)$ approximating as $N \to \infty$ the functions $\varphi_0(x)$and $\varphi_1(x)$ in the norms $W_2^1(\Omega)$ and $L_2(\Omega)$, respectively.

The system (3.6) is a system of second order ordinary differential equations with respect to t for the unknowns functions $C_k^N(t)$, $k = 1,2,...,N$ solved with respect to $\frac{dC_k^N}{dt^2}$. Then for $\forall N$ system (3.6) is uniquely solvable under initial conditions (3.7) ([7,8]), moreover $\frac{dC_k^N}{dt^2} \in L_2(0,T)$.

Multiplying each of the equalities of (3.6) by its $\frac{dC_l^N}{dt}$, and summing over l, we get the equality

$$\int_\Omega \frac{\partial^2 u_k^N(x,t)}{\partial t^2}\frac{\partial u_k^N(x,t)}{\partial t}dx + \int_\Omega \sum_{i=1}^{n} \frac{\partial u_k^N(x,t)}{\partial x_i}\frac{\partial^2 u_k^N(x,t)}{\partial t \partial x_i}dx -$$

$$-\int_{\partial\Omega} \frac{\partial u_k^N(x,t)}{\partial t}\int_\Omega K(x,y)u_k^N(y,t)dyds = \int_\Omega f_k(x,t)\frac{\partial u_k^N(x,t)}{\partial t}dx,$$

Integrating the last with respect to t from 0 to $t, t \in [0,T]$, we have

$$\int_{\Omega}\left(\left(\frac{\partial u_k^N(x,t)}{\partial t}\right)^2+\sum_{i=1}^{n}\left(\frac{\partial u_k^N(x,t)}{\partial x_i}\right)^2\right)dx-$$

$$-2\int_0^t\int_{\partial\Omega}\frac{\partial u_k^N(x,t)}{\partial t}\int_{\Omega}K(x,y)u_k^N(y,t)dydsdt=$$

$$=\int_{\Omega}\left(\left(\frac{\partial u_k^N(x,0)}{\partial t}\right)^2+\sum_{i=1}^{n}\left(\frac{\partial u_k^N(x,0)}{\partial x_i}\right)^2\right)dx+$$

$$+2\int_0^t\int_{\Omega}f_k(x,t)\frac{\partial u_k^N(x,t)}{\partial t}dxdt. \tag{3.8}$$

Assuming

$$y_k^N(t)=\int_{\Omega}\left(\left(\frac{\partial u_k^N(x,t)}{\partial t}\right)^2+\sum_{i=1}^{n}\left(\frac{\partial u_k^N(x,t)}{\partial x_i}\right)^2\right)dx,$$

from (3.8) we derive

$$y_k^N(t)=y_k^N(0)+2\int_0^t\int_{\partial\Omega}\frac{\partial u_k^N(x,t)}{\partial t}\int_{\Omega}K(x,y)u_k^N(y,t)dydsdt+$$

$$+2\int_0^t\int_{\Omega}f_k(x,t)\frac{\partial u_k^N(x,t)}{\partial t}dxdt. \tag{3.9}$$

We transform the integral along the lateral surface of the cylinder $S_t=\Omega\times(0,t)$ as follows:

$$\int_0^t\int_{\partial\Omega}\frac{\partial u_k^N(x,t)}{\partial t}\int_{\Omega}K(x,y)u_k^N(y,t)dydsdt=$$

$$=\int_{\partial\Omega}\int_0^t\frac{\partial u_k^N(x,t)}{\partial t}\int_{\Omega}K(x,y)u_k^N(y,t)dydtds=$$

$$=-\int_{\partial\Omega}\int_0^t u_k^N(x,t)\int_{\Omega}K(x,y)\frac{\partial u_k^N(y,t)}{\partial t}dydtds+$$

$$+\int_{\partial\Omega}u_k^N(x,t)\int_{\Omega}K(x,y)u_k^N(y,t)dyds-$$

$$-\int_{\partial\Omega}u_k^N(x,0)\int_{\Omega}K(x,y)u_k^N(y,0)dyds=i_1+i_2+i_3,$$

where

$$i_1=-\int_0^t\int_{\partial\Omega}u_k^N(x,t\int_{\Omega}K(x,y)\frac{\partial u_k^N(y,t)}{\partial t}dydsdt$$

$$i_2=\int_{\partial\Omega}u_k^N(x,t)\int_{\Omega}K(x,y)u_k^N(y,t)dyds,$$

$$i_3=-\int_{\partial\Omega}u_k^N(x,0)\int_{\Omega}K(x,y)u_k^N(y,0)dyds.$$

Using the known inequality ([8])

$$\int_{\partial\Omega}|W(x)|\,ds\le\alpha\int_{\Omega}(|W(x)|+|\nabla W(x)|)\,dx\ \forall W(x)\in W_1^1(\Omega),$$

where

$$\nabla W = \left(\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \dots, \frac{\partial W}{\partial x_n} \right)$$

and then the Cauchy-Bunyakovsky inequality, we obtain

$$|i_1| = \left| - \int_0^t \int_{\partial\Omega} u_k^N(x,t \int_\Omega K(x,y) \frac{\partial u_k^N(y,t)}{\partial t} dydsdt \right| \le$$

$$\le c \int_0^t \int_\Omega \left(\left(u_k^N(x,t)\right)^2 + \left|\nabla u_k^N(x,t)\right|^2 + \left(\frac{\partial u_k^N(x,t)}{\partial t} \right)^2 \right) dxdt, \tag{3.10}$$

$$|i_2| \le c \left(\int_\Omega \left(\left(u_k^N(x,t)\right)^2 + \left|\nabla u_k^N(x,t)\right|^2 \right) dx \right)^{\frac{1}{2}} \left(\int_\Omega \left(u_k^N(x,t)\right)^2 dx \right)^{\frac{1}{2}}. \tag{3.11}$$

Introduce the denotation

$$Z_k^N(t) = \int_\Omega \left(\left(u_k^N(x,t)\right)^2 + \left|\nabla u_k^N(x,t)\right|^2 + \left(\frac{\partial u_k^N(x,t)}{\partial t} \right)^2 \right) dx.$$

It is clear that

$$\int_\Omega \left(u_k^N(x,t)\right)^2 dx \le 2 \int_\Omega \left(u_k^N(x,0)\right)^2 dx + 2t \int_0^t y_k^N(t) dt. \tag{3.12}$$

Then, from (3.11) we obtain

$$|i_2| \le c \left(Z_k^N(t)\right)^{\frac{1}{2}} \left(2Z_k^N(0) + 2t \int_0^t Z_k^N(t) dt \right)^{\frac{1}{2}}. \tag{3.13}$$

By means of (3.13) we can estimate i_3 as well

$$|i_3| = \left| - \int_{\partial\Omega} u_k^N(x,0) \int_\Omega K(x,y) u_k^N(y,0) dyds \right| \le$$

$$\le c \left(Z_k^N(0)\right)^{\frac{1}{2}} \left(Z_k^N(0)\right)^{\frac{1}{2}} = cZ_k^N(0).$$

Now, adding (3.9) and (3.12), we obtain

$$y_k^N(t) + \int_\Omega \left(u_k^N(x,t)\right)^2 dx \le y_k^N(0) + 2 \int_\Omega \left(u_k^N(x,0)\right)^2 dx +$$

$$+2t \int_0^t y_k^N(t) dt + 2 \int_0^t \int_{\partial\Omega} \frac{\partial u_k^N(x,t)}{\partial t} \int_\Omega K(x,y) u_k^N(y,t) dydsdt +$$

$$+2 \int_0^t \int_\Omega f_k(x,t) \frac{\partial u_k^N(x,t)}{\partial t} dxdt.$$

Hence, for i_1, i_2, i_3 we have

$$Z_k^N(t) \le c, \; t \in [0,T].$$

Hence it follows that as $N \to \infty$

$$\int_\Omega \left((u_k(x,t))^2 + |\nabla u_k(x,t)|^2 + \left(\frac{\partial u_k(x,t)}{\partial t} \right)^2 \right) dx \le c. \tag{3.14}$$

It follows from (3.14) that

$$\|u_k\|_{L_\infty(0,T,L_6(O))} \le c \tag{3.15}$$

and

$$\{u_k\} \subset K \subset L_\lambda(Q) \tag{3.16}$$

for $\lambda < 6$ at $n = 3$ and for any finite λ for $n = 2$ (K is a relatively compact set). So, we can select a subsequence again denoted by $\{\upsilon_k, u_k\}$ so that

$$\begin{aligned} \upsilon_k &\to \upsilon_0 \text{ in } L_2(Q) \text{ weakly}, \\ u_k &\to u_0 \text{ in } L_\infty(0,T,W_2^1(O)) * \text{weakly}, \\ \frac{\partial u_k}{\partial t} &\to \frac{\partial u_0}{\partial t} \text{ in } L_\infty(0,T,L_2(O)) * \text{weakly}, \\ u_k &\to u_0 \text{ in } L_\lambda(Q) \text{ strongly and a.e. in } Q(\lambda < 6). \end{aligned} \tag{3.17}$$

It is clear that for the sequence $\{\upsilon_k, u_k\}$ we have the following integral identity

$$\int_Q \left(-\frac{\partial u_k}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i=1}^n \frac{\partial u_k}{\partial x_i}\frac{\partial \eta}{\partial x_i} - u_k^3 \eta \right) dxdt -$$

$$-\int_0^T \int_{\partial\Omega} \eta\,(x,t) \int_\Omega K\,(x,y) u_k\,(y,t)\,dydsdt -$$

$$\int_\Omega u^1(x)\eta\,(x,0)\,dx = \int_Q \upsilon_k(x,t)\eta(x,t)dxdt, \eta \in W_2^1\,(Q)\,, \eta\,(x,T) = 0 \tag{3.18}$$

and the equality

$$u_k\,(x,0) = u^0(x). \tag{3.19}$$

Taking into account relation (3.17), we pass to the limit in (3.18) and (3.19). Then we obtain that (υ_0, u_0) is an admissible pair and it follows from the form of the functional $I(\vartheta, u)$ that

$$\underline{\lim}_{k\to\infty} I(\vartheta_k, u_k) \geq I(\vartheta_0, u_0).$$

From this and relation (3.2) it follows that (υ_0, u_0) is an optimal pair. □

Theorem 3.2. *Under the conditions imposed on the data of problem* (2.1)-(2.3),(2.4),(2.5), *for the pair* (υ_0, u_0) *to be optimal, it is necessary that there exists a function* $\psi(x,t)$ *for which the following relations holds true*

$$\frac{\partial^2 u_0}{\partial t^2} - \Delta u_0 - u_0^3 = \upsilon_0 \text{ in } Q,$$

$$\frac{\partial^2 \psi}{\partial t^2} - \Delta\psi - 3u_0^2\psi - \int_{\partial\Omega} K(\xi,x)\psi(\xi,t)d\xi = (u_0 - u_d)^5 \ \text{ in } \ Q,$$

$$u_0(x,0) = u^0(x), \frac{\partial u_0(x,0)}{\partial t} = u^1(x), \ x \in \Omega,$$

$$\psi(x,T) = 0, \frac{\partial \psi(x,T)}{\partial t} = 0, \ x \in O,$$

$$\left.\frac{\partial u_0}{\partial \nu}\right|_S = \int_\Omega K(x,y)u_0(y,t)dy, \quad (x,t) \in S,$$

$$\left.\frac{\partial \psi}{\partial \nu}\right|_S = 0, (x,t) \in S.$$

Moreover, $u_0 \in L_\infty(0,T;W_2^1(\Omega))$, $\frac{\partial u_0}{\partial t} \in L_\infty(0,T;L_2(\Omega))$, $\psi \in L_\infty(0,T;L_2(\Omega))$,

$$\frac{\partial \psi}{\partial t} \in L_\infty\left(0,T;\left(W_2^1(\Omega)\right)^*\right)$$

and

$$\int_Q (\psi + \alpha(\upsilon_0 - \omega))(\upsilon - \upsilon_0)\,dxdt \ge 0 \;\forall \upsilon \in V.$$

Proof. To prove Theorem 3.2, following [9], we introduce an adaptive functional

$$I_\varepsilon^a(\upsilon,u) = \frac{1}{6}\|u-u_d\|_{L_6(Q)}^6 + \frac{\alpha}{2}\|\upsilon-\omega\|_{L_2(Q)}^2 + \frac{1}{2\varepsilon}\left\|\frac{\partial^2 u}{\partial t^2} - \Delta u - u^3 - \upsilon\right\|_{L_2(Q)}^2 +$$

$$+\frac{1}{2}\|u-u_0\|_{L_2(Q)}^2 + \frac{1}{2}\|\upsilon-\upsilon_0\|_{L_2(Q)}^2, \tag{3.20}$$

where

$$\upsilon \in V, u \in L_6(Q), \frac{\partial^2 u}{\partial t^2} - \Delta u \in L_2(Q),$$

$$u(x,0) = u^0(x), \frac{\partial u(x,0)}{\partial t} = u^1(x), \tag{3.21}$$

$\frac{\partial u}{\partial \nu}\big|_S = \int_\Omega K(x,y)u(y,t)dy$, $(x,t) \in S, \varepsilon > 0$ is a penalty parameter.

As in Theorem 3.1, we can prove that in the problem of minimization of the functional (3.20) under the constraints (3.21) for each $\varepsilon > 0$ there exists the optimal pair $(\upsilon_\varepsilon, u_\varepsilon)$.

Prove that as $\varepsilon \to 0$ $u_\varepsilon \to u_0$ strongly in $L_6(Q)$ and $\upsilon_\varepsilon \to \upsilon_0$ strongly in $L_2(Q)$. We have

$$I_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon) = inf I_\varepsilon^a(\upsilon,u) \le I_\varepsilon^a(\upsilon_0,u_0) = I(\upsilon_0,u_0). \tag{3.22}$$

Hence, by definition of the functional we obtain

$$\|\upsilon_\varepsilon\|_{L_2(Q)} + \|u_\varepsilon\|_{L_6(Q)} \le c, \tag{3.23}$$

where c are various constants independent of ε, and also

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} - \triangle u_\varepsilon - u_\varepsilon = \upsilon_\varepsilon + \sqrt{\varepsilon} f_\varepsilon,$$

$$u_\varepsilon(x,0) = u^0(x), \; \frac{\partial u_\varepsilon}{\partial \nu}\bigg|_S = \int_\Omega K(x,y)\,u_\varepsilon(y,t)\,dy, \tag{3.24}$$

where $f_\varepsilon(x,t)$ such that, $\|f_\varepsilon(x,t)\|_{L_2(Q)} \le c$.

It follows from (3.23) and (3.24) that

$$\|u_\varepsilon\|_{W_2^1(Q)} \le c. \tag{3.25}$$

Consequently, from $(\upsilon_\varepsilon, u_\varepsilon)$ we can extract a subsequence again denoted by $(\upsilon_\varepsilon, u_\varepsilon)$, that $\upsilon_\varepsilon \to \overline{\upsilon}$ in $L_2(Q)$ weakly, $u_e \to \overline{u}$ in $W_2^1(Q)$ weakly and in $L_6(Q)$ weakly, strongly in $L_2(Q)$ and a.e. in Q.

Then by lemma 1.3 [10, p.25] it follows that $u_e^3 \to \overline{u}^3$ in $L_2(Q)$ weakly.

Therefore, in the integral identity

$$\int_Q \left(-\frac{\partial u_\varepsilon}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i=1}^n \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial \eta}{\partial x_i} - u_\varepsilon^3 \eta\right) dxdt -$$

$$-\int_0^T \int_{\partial\Omega} \eta(x,t) \int_\Omega K(x,y) u_\varepsilon(y,t)\,dydsdt-$$

$$\int_\Omega u^1(x)\eta(x,0)\,dx = \int_Q \upsilon_\varepsilon \eta dxdt, \eta \in W_2^1(Q), \eta(x,T)=0$$

we can pass to the limit as $\varepsilon \to 0$ and obtain that $\overline{u}(x,t)$ is a generalized solution to problem (2.1)-(2.3), i.e. of the problem:

$$\frac{\partial^2 \overline{u}}{\partial t^2} - \triangle \overline{u} - \overline{u^3} = \overline{\upsilon},$$

$$\overline{u}(x,0) = u^0(x), \quad \frac{\partial \overline{u}(x,0)}{\partial t} = u^1(x), \left.\frac{\partial \overline{u}}{\partial \nu}\right|_S = \int_\Omega K(x,y)\,\overline{u}(y,t)\,dy.$$

So, the inequality

$$I_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon) \geq I(\upsilon_\varepsilon, u_\varepsilon) + \frac{1}{2}\|u_0 - \overline{u}\|^2_{L_2(Q)} + \frac{1}{2}\|\upsilon_0 - \overline{\upsilon}\|^2_{L_2(Q)}$$

leads to the inequality

$$\underline{\lim}_{\varepsilon\to 0} I\ (\upsilon_\varepsilon, u_\varepsilon) \geq I(\overline{\upsilon}, \overline{u}) + \frac{1}{2}\|u_0 - \overline{u}\|^2_{L_2(Q)} + \frac{1}{2}\|\upsilon_0 - \overline{\upsilon}\|^2_{L_2(Q)}.$$

And since by (3.22) $\underline{\lim}_{\varepsilon\to 0} I(\upsilon_\varepsilon, u_\varepsilon) \leq\ I(\upsilon_0, u_0)$, then it follows that $I(\overline{\upsilon}, \overline{u}) \leq I(\upsilon_0, u_0)$, and that is, $I(\overline{\upsilon}, \overline{u}) = I(\upsilon_0, u_0)$. Then

$$\frac{1}{2}\|u_0 - \overline{u}\|^2_{L_2(Q)} + \frac{1}{2}\|\upsilon_0 - \overline{\upsilon}\|^2_{L_2(Q)} = 0,$$

so that $\overline{u} = u_0$, $\overline{\upsilon} = \upsilon_0$, and consequently, we obtain (weak) convergence not extracting a subsequence (as the limit is unique).

Since $I_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon) \geq I(\upsilon_\varepsilon, u_\varepsilon)$ and $\underline{\lim}_{\varepsilon\to 0} I(\upsilon_\varepsilon, u_\varepsilon) \geq I(\upsilon_0, u_0)$, then obviously $I(\upsilon_\varepsilon, u_\varepsilon) \to I(\upsilon_0, u_0)$. From this by definition of $I(\upsilon, u)$ we obtain that as as $\varepsilon \to 0$ $u_\varepsilon \to u_0$ in $L_6(Q)$, $\upsilon_\varepsilon \to \upsilon_0$ in $L_2(Q)$.

Then we derive a necessary condition for optimality. We write a necessary condition for $(\upsilon_\varepsilon, u_\varepsilon)$ to be a solution of the problem

$$I_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon) = inf I_\varepsilon^a(\upsilon, u):$$

$$\left.\frac{d}{d\lambda} I_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon + \lambda\xi)\right|_{\lambda=0} = 0, \tag{3.26}$$

$$\forall \xi \in C^2(\overline{Q}), \xi(x,0) = 0, \frac{\partial \xi(x,0)}{\partial t} = 0, \left.\frac{\partial \xi}{\partial \nu}\right|_S = \int_\Omega K(x,y)\xi(y,t)dy, (x,t) \in S, \tag{3.27}$$

$$\left.\frac{d}{d\lambda} I_\varepsilon^a(\upsilon_\varepsilon + \lambda(\upsilon - \upsilon_\varepsilon), u_\varepsilon)\right|_{\lambda=0} \geq 0\ \forall \upsilon \in V, \upsilon_\varepsilon \in V, \tag{3.28}$$

where the derivatives in formulas (3.26), (3.28) are understood in the Gato sense.

Calculating the derivative from (3.26) and equating it to zero, we have

$$-\int_Q \psi_\varepsilon(\frac{\partial^2 \xi}{\partial t^2} - \Delta\xi - 3u_\varepsilon^2\xi)dxdt + \int_Q (u_\varepsilon - u_d)^5\,\xi dxdt + \int_Q (u_\varepsilon - u_0)\xi dxdt = 0,$$

$$\forall \xi \in C^2(\overline{Q}), \xi(x,0) = 0, \frac{\partial \xi(x,0)}{\partial t} = 0, \left.\frac{\partial \xi}{\partial \nu}\right|_S = \int_\Omega K(x,y)\xi(y,t)dy, \quad (x,t) \in S,$$

where

$$\psi_\varepsilon(x,t) = -\frac{1}{\varepsilon}\left(\frac{\partial^2 u_\varepsilon}{\partial t^2} - \Delta u_\varepsilon - u_\varepsilon^3 - \upsilon_\varepsilon\right).$$

Then $\psi_\varepsilon(x,t)$ will be a weak solution of the problem

$$\frac{\partial^2 \psi_\varepsilon}{\partial t^2} - \Delta\psi_\varepsilon - 3u_\varepsilon^2 - \int_{\partial\Omega} K(\xi,x)\psi_\varepsilon(\xi,t)d\xi = (u_\varepsilon - u_d)^5 + (u_\varepsilon - u_0),$$

$$(x,t) \in Q,$$

$$\psi_\varepsilon(x,T) = 0, \frac{\partial \psi_\varepsilon(x,T)}{\partial t} = 0, x \in \Omega, \tag{3.29}$$

$$\left.\frac{\partial \psi_\varepsilon}{\partial \nu}\right|_S = 0, (x,t) \in S.$$

Following [9], this problem has a solution $\psi_\varepsilon(x,t)$ satisfying $\psi_\varepsilon(x,t) \in L_\infty(0,T;L_2(\Omega))$, $\frac{\partial\psi_\varepsilon(x,t)}{\partial t} \in L_\infty\left(0,T;\left(W_2^1(\Omega)\right)^*\right)$

The condition (3.28) yields

$$\int_Q (\psi_\varepsilon + \alpha(\upsilon_\varepsilon - \omega))(\upsilon - \upsilon_\varepsilon)\,dxdt + \int_Q (\upsilon_\varepsilon - \upsilon_0)(\upsilon - \upsilon_\varepsilon)\,dxdt \geq 0 \ \forall \upsilon \in V. \tag{3.30}$$

As $u_\varepsilon \to u_0$ in $L_6(Q)$ from (3.29) we obtain that the limit function $\psi(x,t)$ of the functions $\psi_\varepsilon(x,t)$ will be a weak solution to the following problem

$$\frac{\partial^2 \psi}{\partial t^2} - \Delta\psi - 3u_0^2\psi - \int_{\partial\Omega} K(\xi,x)\psi(\xi,t)d\xi = (u_0 - u_d)^5,$$

$$\psi(x,T) = 0, \frac{\partial\psi(x,T)}{\partial t} = 0, \ x \in \Omega,$$

$$\left.\frac{\partial\psi}{\partial\nu}\right|_S = 0, (x,t) \in S,$$

as $\upsilon_\varepsilon \to \upsilon_0$ in $L_2(Q)$ from the condition (3.30) we obtain

$$\int_Q (\psi + \alpha(\upsilon_0 - \omega))(\upsilon - \upsilon_0)dxdt \geq 0 \ \ \forall \upsilon \in V.$$

□

Example. Suppose that in the considered problem $\varphi^0 = 0$, $\varphi^1 = 0$, $u_d = \frac{t^2}{2}$, $\omega = 1 - \frac{t^6}{8}$, and $K(x,y)$ such that, $\int_\Omega K(x,y)dy = 0$. If in the example we take $\upsilon_0 = 1 - \frac{t^6}{8}$, $u_0 = \frac{t^2}{2}$, then $u_0 - u_d = 0$, $\upsilon_0 - \omega = 0$ and pair (υ_0, u_0) give minimum value to the functional $I(\upsilon,u)$. In this case solution of adjoint problem while the $\psi(x,t) = 0$ and optimality conditions are satisfied automatically.

4. Conclusion

In this work we prove the theorem on the existence of the optimal pair and obtain a necessary condition for optimality in the form of variational inequality in the optimal control problem for the unstable hyperbolic equation of the second order with a nonlocal boundary condition. The results may be applied to the solution of problems for the wave and vibration processes and developed for the different equations of mathematical physics.

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