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# INERTIAL TSENG METHOD WITH NONDECREASING ADAPTIVE STEPSIZE FOR VARIATIONAL INEQUALITY ON HADAMARD MANIFOLDS

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**Abstract**. In this article, we propose an inertial and a viscosity iterative method for solving variational inequality problem on Hadamard manifolds. The iterative algorithm is inspired by Tseng's extragradient method with a self-adaptive procedure which generates dynamic stepsizes converging to a positive constant. The proposed method does not require the knowledge of the Lipschitz constant as well as the sequential weak continuity of the corresponding operator. Under a pseudomonotone assumption on the underlying vector field, we establish a convergence result for solving a pseudomonotone variational inequality and fixed point problems of nonexpansive mapping under some mild assumptions. Finally, we present some fundamental experiment to illustrate the numerical behavior of our proposed method. The result discussed in this article extends and complements many related results in the literature.

## 1. Introduction

Variational inequality problems (in short, VIP) were initially studied by Stampacchia [29] in 1964. Since its inception, many kinds of variational inequalities have been studied and generalized in several directions using novel and innovative techniques ( see [8, 35, 36] and the references therein). The theory of variational inequalities has been studied quite extensively and has emerged as an important tool in the study of a wide class of problems from mechanics, optimization, engineering, science and social sciences. Many problems in applied fields can be formulated as variational inequalities or boundary value problems on manifolds, which are nonlinear in general. It is well-known that the generalization of optimization methods from Euclidean spaces to Riemannian manifolds has some important advantages. For instance, constrained optimization problems can be seen as unconstrained ones from the Riemannian geometry point of view. Another important advantage of doing this is that optimization problems with non convex objective functions become convex through the introduction of an appropriate Riemannian metric (see [10, 26]). Therefore, the study and approximation of solutions to variational inequalities on the Riemannian manifolds is natural.

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In 2003, Németh [21] introduced the variational inequality problem on Hadamard manifolds: find $u \in C$ such that

$$\langle F(u), \exp_u^{-1} v\rangle \geq 0, \ \forall\, v \in C, \tag{1.1}$$

where $C$ is a nonempty, closed and convex subset of Hadamard manifold $\mathbb{M}$, $F : C \to T\mathbb{M}$ is a vector field, that is, $F(u) \in T_u\mathbb{M}$ for some $u \in C$, and $\exp^{-1}$ is the inverse of exponential map. If $\mathbb{M} = \mathbb{R}_n$, then the vector field F reduces to the operator $F : C \to \mathbb{R}_n$ and problem (1.1) reduces to the one introduced by [29] defined by: find $u \in C$ such that

$$\langle F(u), v - u\rangle \geq 0, \ \forall\, v \in C.$$

We denote the solution set of (1.1) by $VIP(C, F)$.
The extragradient method developed by Korpelevich [17] in 1976 is one of the most widely used techniques for resolving VIP (1.1). It is important to say that the extragradient method is not efficient in the case where the feasible set does not have a closed form expression, which makes projection onto it very difficult. It is also important to note that the mapping in the extragradient method requires knowledge of the Lipschitz constant. Lipschitz constants are regrettably, frequently unknown or challenging to accurately estimate. Many researchers have paid close attention to extragradient method and have greatly improved it in various ways.
Recently, Tseng [33] introduced a single projection extragradient method for solving variational inequalities in real Hilbert spaces. A typical disadvantage of Tseng's algorithm and many other algorithm is the assumption that the Lipschitz constant of the monotone operator can be estimated. Recently, Thong and Vuong [32] proposed a modified Tseng extragradient method in which the operator is pseudomonotone and there is no requirement for a prior estimate of the Lipschitz constant of the cost operator. In the setting of Hadamard manifolds, Tang [31] introduced the Korpelevich's method for solving variational inequality problem. Using the idea in [31], Chen *et al.* proposed the following Tseng extragradient method with new step size which does not require the knowledge of the Lipschitz constant for solving pseudomonotone variational inequality problem as follows:

**Algorithm 1.1.** Modified Tseng's extragradient method.

**Initialization:** Choose $\lambda_0 > 0$, $\mu, \theta \in (0, 1)$ and let $x_0 \in \mathbb{M}$ be arbitrary starting points.

**Step 1**: Given the current iterate $x_n$, compute

$$y_n = P_C(\exp_{x_n}(-\lambda_n F(x_n)), \tag{1.2}$$

If $x_n = y_n$, then stop: $x_n$ is a solution. Otherwise

**Step 2**: Compute

$$x_{n+1} = \exp_{y_n} \lambda_n(P_{y_n,x_n} F(x_n) - F(y_n)) \tag{1.3}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu d(x_n, y_n)}{\|P_{y_n,x_n} F(x_n) - F(y_n)\|}\right\}, \ \lambda_n & \text{if } F(x_n) \neq F(y_n), \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{1.4}$$

**Stopping criterion** Set $n := n + 1$ and return to **step 1**.

They proved that the sequence generated by their method converges to a solution to variational inequality problem. We observed that very little research have been carried out on variational inequality problem in the settings of Hadamard manifolds. Due to this, we introduce a new iterative method for approximating the solution of VIP((1.5)) in the setting of Hadamard manifolds using an inertial and viscosity method.

One of the best ways to speed up the convergence rate of iterative algorithms is to combine the iterative scheme with the inertial term. This term is represented by $\theta_n(x_n - x_{n-1})$ and is a remarkable tool for improving the performance of algorithms and it is known to have some nice convergence characteristics. For growing interests in this direction (see [1, 2, 4, 22]). The idea of inertial extrapolation method was first introduced by Polyak [25] and was inspired by an implicit discretization of a second-order-in-time dissipative dynamical system, so -called "Heavy Ball with Friction"

$$v''(t) + \gamma v'(t) + \nabla f(v(t)) = 0, \tag{1.5}$$

where $\gamma > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. System (1.5) is discretized so that, having the terms $x_{n-1}$ and $x_n$, the next term $x_{n+1}$ can be determined using

$$\frac{x_{n-1} - 2x_n + x_{n-1}}{j^2} + \gamma \frac{x_n - x_{n-1}}{j} + \nabla f(x_n) = 0,\ n \geq 1, \tag{1.6}$$

where $j$ is the step-size. Equation (1.6) yields the following iterative algorithm:

$$x_{n+1} = x_n + \beta(x_n - x_{n-1}) - \alpha \nabla f(x_n),\ n \geq 1, \tag{1.7}$$

where $\beta = 1 - \gamma_j, \alpha = j^2$ and $\beta(x_n - x_{n-1})$ is called the inertial extrapolation term which is intended to speed up the convergence of the sequence generated by (1.7).

Motivated by the result of Korpelevich method of [17], Tseng's extragradient method of [33] and viscosity method of [11], we introduce an Inertial Tseng extragradient method for solving variational inequality and fixed point problem. While we still require the operator to be Lipschitz continuous, the prior knowledge of the Lipschitz constant is not necessary. Moreover, we introduce a self-adaptive procedure which generates a sequence of stepsizes converging monotonically to a constant. We establish that the sequence generated by our proposed method converges to a common solution of pseudomonotone variational inequality and fixed point of a nonexpansive mapping.

We highlight some of the contributions of our result as follows:

(i) We employ the inertial method as introduced by Polyak [25], which is quite different from the ones in [1, 3] as this lacks essential direction to move objects to their possible destination (see [13]).

(ii) We were able to dispense with the condition $\sum_{n=1}^{\infty} \theta_n d(x_n - x_{n-1}) < \infty$ a strong condition which has been used for instance (see [15]).

(iii) The method in this article requires a self-adaptive procedure which generates dynamic step-sizes and is allowed to increase from iteration to iteration unlike the method of [31] which requires the knowledge of Lipschitz constant to be imposed on the operator.

(iv) The result discussed in this article extends and generalizes the results of [1, 7, 9, 18, 22, 23, 24] from linear to nonlinear spaces.

## 2. Preliminaries

Let $\mathbb{M}$ be an $m$-dimensional manifold, let $x \in \mathbb{M}$ and let $T_x\mathbb{M}$ be the tangent space of $\mathbb{M}$ at $x \in \mathbb{M}$. We denote by $T\mathbb{M} = \bigcup_{x\in\mathbb{M}} T_x\mathbb{M}$ the tangent bundle of $\mathbb{M}$. An inner product $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian metric on $\mathbb{M}$ if $\langle\cdot,\cdot\rangle_x : T_x\mathbb{M}\times T_x\mathbb{M} \to \mathbb{R}$ is an inner product for all $x \in \mathbb{M}$. The corresponding norm induced by the inner product $\mathcal{R}_x\langle\cdot,\cdot\rangle$ on $T_x\mathbb{M}$ is denoted by $\|\cdot\|_x$. We will drop the subscript $x$ and adopt $\|\cdot\|$ for the corresponding norm induced by the inner product. A differentiable manifold $\mathbb{M}$ endowed with a Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian manifold. In what follows, we denote the Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle$ when no confusion arises. Given a piecewise smooth curve $\gamma : [a,b] \to \mathbb{M}$ joining $x$ to $y$ (that is, $\gamma(a) = x$ and $\gamma(b) = y$), we define the length $l(\gamma)$ of $\gamma$ by $l(\gamma) := \int_a^b \|\gamma'(t)\|dt$. The Riemannian distance $d(x,y)$ is the minimal length over the set of all such curves joining $x$ to $y$. The metric topology induced by $d$ coincides with the original topology on $\mathbb{M}$. We denote by $\nabla$ the Levi-Civita connection associated with the Riemannian metric [28].

Let $\gamma$ be a smooth curve in $\mathbb{M}$. A vector field $X$ along $\gamma$ is said to be parallel if $\nabla_{\gamma'}X = \mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector. If $\gamma'$ itself is parallel along $\gamma$, then we say that $\gamma$ is a geodesic and $\|\gamma'\|$ is a constant. If $\|\gamma'\| = 1$, then the geodesic $\gamma$ is said to be normalized. A geodesic joining $x$ to $y$ in $\mathbb{M}$ is called a minimizing geodesic if its length equals $d(x,y)$. A Riemannian manifold $\mathbb{M}$ equipped with a Riemannian distance $d$ is a metric space $(\mathbb{M},d)$. A Riemannian manifold $\mathbb{M}$ is said to be complete if for all $x \in \mathbb{M}$, all geodesics emanating from $x$ are defined for all $t \in \mathbb{R}$. The Hopf-Rinow theorem [28], posits that if $\mathbb{M}$ is complete, then any pair of points in $\mathbb{M}$ can be joined by a minimizing geodesic. Moreover, if $(\mathbb{M},d)$ is a complete metric space, then every bounded and closed subset of $\mathbb{M}$ is compact. If $\mathbb{M}$ is a complete Riemannian manifold, then the exponential map $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ at $x \in \mathbb{M}$ is defined by

$$\exp_x v := \gamma_v(1,x) \ \forall\ v \in T_x\mathbb{M},$$

where $\gamma_v(\cdot,x)$ is the geodesic starting from $x$ with velocity $v$ (that is, $\gamma_v(0,x) = x$ and $\gamma_v'(0,x) = v$). Then, for any $t$, we have $\exp_x tv = \gamma_v(t,x)$ and $\exp_x \mathbf{0} = \gamma_v(0,x) = x$. Note that the mapping $\exp_x$ is differentiable on $T_x\mathbb{M}$ for every $x \in \mathbb{M}$. The exponential map $\exp_x$ has an inverse $\exp_x^{-1} : \mathbb{M} \to T_x\mathbb{M}$. For any $x,y \in \mathbb{M}$, we have $d(x,y) = \|\exp_y^{-1} x\| = \|\exp_x^{-1} y\|$ (see [28] for more details). The parallel transport $P_{\gamma,\gamma(b),\gamma(a)} : T_{\gamma(a)}\mathbb{M} \to T_{\gamma(b)}\mathbb{M}$ on the tangent bundle $T\mathbb{M}$ along $\gamma : [a,b] \to \mathbb{R}$ with respect to $\nabla$ is defined by

$$P_{\gamma,\gamma(b),\gamma(a)}v = F(\gamma(b)),\ \forall\ a,b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)}\mathbb{M},$$

where $F$ is the unique vector field such that $\nabla_{\gamma'(t)}v = \mathbf{0}$ for all $t \in [a,b]$ and $F(\gamma(a)) = v$. If $\gamma$ is a minimizing geodesic joining $x$ to $y$, then we write $P_{y,x}$ instead of $P_{\gamma,y,x}$. Note that for every $a,b,r,s \in \mathbb{R}$, we have

$$P_{\gamma(s),\gamma(r)} \circ P_{\gamma(r),\gamma(a)} = P_{\gamma(s),\gamma(a)} \text{ and } P^{-1}_{\gamma(b),\gamma(a)} = P_{\gamma(a),\gamma(b)}.$$

Also, $P_{\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}\mathbb{M}$ to $T_{\gamma(b)}\mathbb{M}$, that is, the parallel transport preserves the inner product

$$\langle P_{\gamma(b),\gamma(a)}(u), P_{\gamma(b),\gamma(a)}(v)\rangle_{\gamma(b)} = \langle u, v\rangle_{\gamma(a)},\ \forall\ u, v \in T_{\gamma(a)}\mathbb{M}. \tag{2.1}$$

We now give some examples of Hadamard manifolds.

**Space 1:** Let $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{M} = (\mathbb{R}_{++}, \langle\cdot,\cdot\rangle)$ be the Riemannian manifold equipped with the inner product $\langle x, y\rangle = xy\ \forall\ x, y \in \mathbb{R}$. Since the sectional curvature of $\mathbb{M}$ is zero [5], $\mathbb{M}$ is an Hadamard manifold. Let $x, y \in \mathbb{M}$ and $v \in T_x\mathbb{M}$ with $\|v\|_2 = 1$. Then $d(x, y) = |\ln x - \ln y|$, $\exp_x tv = xe^{\frac{vx}{t}}$, $t \in (0, +\infty)$, and $\exp_x^{-1} y = x\ln y - x\ln x$.

**Space 2:** Let $\mathbb{R}^m_{++}$ be the product space $\mathbb{R}^m_{++} := \{(x_1, x_2, \cdots, x_m) : x_i \in \mathbb{R}_{++},\ i = 1, 2, \cdots, m\}$. Let $\mathbb{M} = ((R)_+{+}, \langle\cdot,\cdot\rangle)$ be the $m$-dimensional Hadamard manifold with the Riemannian metric $\langle p, q\rangle = p^T q$ and the distance $d(x, y) = |\ln\frac{x}{y}| = |\ln\sum\limits_{i=1}^{m}\frac{x_i}{y_i}|$, where $x, y \in \mathbb{M}$ with $x = \{x_i\}_{i=1}^m$ and $y = \{y_i\}_{i=1}^m$.

A subset $K \subset \mathbb{M}$ is said to be convex if for any two points $x, y \in K$, the geodesic $\gamma$ joining $x$ to $y$ is contained in $K$. That is, if $\gamma : [a, b] \to \mathbb{M}$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1 - t)a + tb) \in K$ for all $t \in [0, 1]$. A complete simply connected Riemannian manifold of non-positive sectional curvature is called an Hadamard manifold. We denote by $\mathbb{M}$ a finite dimensional Hadamard manifold. Henceforth, unless otherwise stated, we represent by $K$ a nonempty, closed and convex subset of $\mathbb{M}$.

**Definition 2.1.** Let $X(\mathbb{M})$ be the set of all single-valued vector fields $V : \mathbb{M} \to T\mathbb{M}$ such that $V(x) \in T_x\mathbb{M}$ for each $x \in \mathbb{M}$ and the domain $\mathbb{D}(V)$ of $V$ be defined by $\mathbb{D}(V) = \{x \in \mathbb{M} : V(x) \neq \emptyset\}$. Let $V \in X(\mathbb{M})$. We say that $V$ is

(i) pseudomonotone, if for any $x, y \in \mathbb{D}(V)$,

$$\langle V(x), \exp_x^{-1} y\rangle \geq 0 \Rightarrow \langle V(y), \exp_y^{-1} x\rangle \leq 0.$$

(ii) Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|P_{y,x}V(x) - V(y)\| \leq Ld(x, y),\ \forall\ x, y \in \mathbb{M}.$$

**Definition 2.2.** A mapping $S : K \to K$ is said to be

(i) contractive, if there exits a constant $k \in (0, 1)$ such that

$$d(Sx, Sy) \leq kd(x, y), \forall\ x, y \in K. \tag{2.2}$$

If $k = 1$ in (2.2), then $S$ is said to be nonexpansive.

We now collect some results and definitions which we shall use in the next section.

**Proposition 2.1.** [28]. Let $x \in \mathbb{M}$. The exponential mapping $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ is a diffeomorphism. For any two points $x, y \in \mathbb{M}$, there exists a unique normalized geodesic joining $x$ to $y$, which is given by

$$\gamma(t) = \exp_x t\exp_x^{-1} y\ \forall\ t \in [0, 1].$$

A geodesic triangle $\Delta(p, q, r)$ of a Riemannian manifold $\mathbb{M}$ is a set containing three points $p$, $q$, $r$ and three minimizing geodesics joining these points.

**Proposition 2.2.** [28]. Let $\Delta(p, q, r)$ be a geodesic triangle in $\mathbb{M}$. Then

$$d^2(p, q) + d^2(q, r) - 2\langle\exp_q^{-1} p, \exp_q^{-1} r\rangle \leq d^2(r, q) \tag{2.3}$$

and

$$d^2(p,q) \leq \langle \exp_p^{-1} r, \exp_p^{-1} q\rangle + \langle \exp_q^{-1} r, \exp_q^{-1} p\rangle. \tag{2.4}$$

Moreover, if $\theta$ is the angle at $p$, then we have

$$\langle \exp_p^{-1} q, \exp_p^{-1} r\rangle = d(q,p)d(p,r)\cos\theta. \tag{2.5}$$

Also,

$$\| \exp_p^{-1} q\|^2 = \langle \exp_p^{-1} q, \exp_p^{-1} q\rangle = d^2(p,q). \tag{2.6}$$

*Remark* 2.1. [19] If $x, y \in \mathbb{M}$ and $v \in T_y\mathbb{M}$, then

$$\langle v, -\exp_y^{-1} x\rangle = \langle v, P_{y,x}\exp_x^{-1} y\rangle = \langle P_{x,y}v, \exp_x^{-1} y\rangle. \tag{2.7}$$

*Remark* 2.2. From (2.4) and Remark 2.1, let $v \in T_p\mathbb{M}$, we have

$$\langle v, \exp_p^{-1} q\rangle \leq \langle v, \exp_p^{-1} r\rangle + \langle v, P_{p,r}\exp_r^{-1} q\rangle. \tag{2.8}$$

For any $x \in \mathbb{M}$ and $K \subset \mathbb{M}$, there exists a unique point $y \in K$ such that $d(x,y) \leq d(x,z)$ for all $z \in K$. This unique point $y$ is called the nearest point projection of $x$ onto the closed and convex set $K$ and is denoted $P_K(x)$.

**Lemma 2.1.** [34]. *For any $x \in \mathbb{M}$, there exists a unique nearest point projection $y = P_K(x)$. Furthermore, the following inequality holds:*

$$\langle \exp_y^{-1} x, \exp_y^{-1} z\rangle \leq 0 \ \forall \ z \in K.$$

**Lemma 2.2.** [19] *Let $x_0 \in \mathbb{M}$ and $\{x_n\} \subset \mathbb{M}$ with $x_n \to x_0$. Then the following assertions hold:*

(i) *For any $y \in \mathbb{M}$, we have $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} x_n$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$,*
(ii) *If $v_n \in T_{x_n}\mathbb{M}$ and $v_n \to v_0$, then $v_0 \in T_{x_0}\mathbb{M}$,*
(iii) *Given $u_n, v_n \in T_{x_n}\mathbb{M}$ and $u_0, v_0 \in T_{x_0}\mathbb{M}$, if $u_n \to u_0$, then $\langle u_n, v_n\rangle \to \langle u_0, v_0\rangle$,*
(iv) *For any $u \in T_{x_0}\mathbb{M}$, the function $F : \mathbb{M} \to T\mathbb{M}$, defined by $F(x) = P_{x,x_0}u$ for each $x \in \mathbb{M}$ is continuous on $\mathbb{M}$.*

The next lemma presents the relationship between triangles in $\mathbb{R}^2$ and geodesic triangles in Riemannian manifolds (see [6]).

**Lemma 2.3.** [6]. *Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle in $\mathbb{M}$. Then there exists a triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ corresponding to $\Delta(x_1, x_2, x_3)$ such that $d(x_i, x_{i+1}) = \|\bar{x}_i - \bar{x}_{i+1}\|$ with the indices taken modulo 3. This triangle is unique up to isometries of $\mathbb{R}^2$.*

The triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in Lemma 2.3 is said to be the comparison triangle for $\Delta(x_1, x_2, x_3) \subset \mathbb{M}$. The points $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are called comparison points to the points $x_1, x_2$ and $x_3$ in $\mathbb{M}$.
A function $h : \mathbb{M} \to \mathbb{R}$ is said to be geodesic if for any geodesic $\gamma \in \mathbb{M}$, the composition $h \circ \gamma : [u, v] \to \mathbb{R}$ is convex, that is,

$$h \circ \gamma(\lambda u + (1-\lambda)v) \leq \lambda h \circ \gamma(u) + (1-\lambda)h \circ \gamma(v), \ u, v \in \mathbb{R}, \ \lambda \in [0,1].$$

**Lemma 2.4.** [19] *Let $\Delta(p, q, r)$ be a geodesic triangle in a Hadamard manifold $\mathbb{M}$ and $\Delta(p', q', r')$ be its comparison triangle.*

(i) *Let $\alpha, \beta, \gamma$ (resp. $\alpha^{'}, \beta^{'}, \gamma^{'}$) be the angles of $\Delta(p, q, r)$ (resp. $\Delta(p^{'}, q^{'}, r^{'})$) at the vertices p,q,r (resp. $p^{'}, q^{'}, r^{'}$). Then, the following inequalities hold:*

$$\alpha^{'} \geq \alpha, \ \beta^{'} \geq \beta, \ \gamma^{'} \geq \gamma,$$

(ii) *Let $z$ be a point in the geodesic joining $p$ to $q$ and $z^{'}$ its comparison point in the interval $[p^{'}, q^{'}]$. Suppose that $d(z, p) = \|z^{'} - p^{'}\|$ and $d(z^{'}, q^{'}) = \|z^{'} - q^{'}\|$. Then the following inequality holds:*

$$d(z, r) \leq \|z^{'} - r^{'}\|.$$

**Lemma 2.5.** [19] *Let $x_0 \in \mathbb{M}$ and $\{x_n\} \subset \mathbb{M}$ be such that $x_n \to x_0$. Then, for any $y \in \mathbb{M}$, we have $\exp^{-1}_{x_n} y \to \exp^{-1}_{x_0} y$ and $\exp^{-1}_{y} x_n \to \exp^{-1}_{y} x_0$;*

The following propositions (see [12]) are very useful in our convergence analysis:
**Proposition 2.3.** Let $M$ be an Hadamard manifold and $d : M \times M :\to \mathbb{R}$ be the distance function. Then the function $d$ is convex with respect to the product Riemannian metric. In other words, given any pair of geodesics $\gamma_1 : [0, 1] \to M$ and $\gamma_2 : [0, 1] \to M$, then for all $t \in [0, 1]$, we have

$$d(\gamma_1(t), \gamma_2(t)) \leq (1 - t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $y \in M$, the function $d(\cdot, y) : M \to \mathbb{R}$ is a convex function.
**Proposition 2.4.** Let $\mathbb{M}$ be a Hadamard manifold and $x \in \mathbb{M}$. The map $\Phi_x = d^2(x, y)$ satisfying the following:

(1) $\Phi_x$ is convex. Indeed, for any geodesic $\gamma : [0, 1] \to \mathbb{M}$, the following inequality holds for all $t \in [0, 1]$ :

$$d^2(x, \gamma(t)) \leq (1 - t)d^2(x, \gamma(0)) + td^2(x, \gamma(1)) - t(1 - t)d^2(\gamma(0), \gamma(1)).$$

(2) $\Phi_x$ is smooth. Moreover, $\partial\Phi_x(y) = -2 \exp^{-1}_y x$.

**Proposition 2.5.** Let $M$ be an Hadamard manifold and $x \in M$. Let $\rho_x(y) = \frac{1}{2}d^2(x, y)$. Then $\rho_x(y)$ is strictly convex and its gradient at $y$ is given by

$$\partial\rho_x(y) = -\exp^{-1}_y x.$$

**Lemma 2.6.** [30] *Let $u, v \in K$ and $\lambda \in [0, 1]$. Then the following relations hold on $K$.*

(i) $\|\lambda u + (1 - \lambda)v\|^2 = \lambda\|u\|^2 + (1 - \lambda)\|v\|^2 - \lambda(1 - \lambda)\|u - v\|^2$;
(ii) $\|u \pm v\|^2 = \|u\|^2 \pm 2\langle u, v\rangle + \|v\|^2$;
(iii) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle$.

**Lemma 2.7.** [27] *Let $\{u_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\{v_n\}$ be a sequence of real numbers. Assume that*

$$u_{n+1} \leq (1 - \alpha_n)u_n + \alpha_n v_n \ \forall \ n \geq 1.$$

*If $\limsup\limits_{k\to\infty} v_{n_k} \leq 0$ for every subsequence $\{u_{n_k}\}$ of $\{u_n\}$ satisfying the condition*

$$\liminf_{k\to\infty}(u_{n_k+1} - u_{n_k}) \geq 0,$$

*then $\lim\limits_{n\to\infty} u_n = 0$.*

## 3. Main result

In this section, we propose a viscosity iterative method for solving pseudomonotone variational inequality problem and fixed point problem of nonexpansive mapping in Hadamard manifolds which is based on Tseng's extragradient method. We give the following assumptions:

**Assumption 3.1.**

- (B1) Let C be a nonempty, closed and convex subset of a Hadamard manifold $\mathbb{M}$.
- (B2) Let $f : C \to C$ be a contraction mapping with constant $k \in (0,1)$ and $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$.
- (B3) The mapping $F : C \to T\mathbb{M}$ is pseudomonotone and $L$-Lipschitz continuous. However, the execution of our method does not require the knowledge of Lipschitz constant.
- (B4) The solution set $\Omega := F(S) \cap VI(C,F)$ is nonempty.

**Assumption 3.2.**

- (D1) $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\beta_n)$, that is, $\lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0$,
- (D2) Let $\beta_n \in (0,1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$,
- (D3) $\delta_n \in (0,1)$ and $0 < \liminf \delta_n \leq \limsup \delta_n < 1$,
- (D4) $\{\eta_n\}$ is a nonnegative real numbers sequence such that $\sum_{n=1}^{\infty} \eta_n < \infty$.

**Algorithm 3.1.** Modified Tseng's method for solving solving VIP with nondecreasing stepsize.

**Initialization:** Choose $\alpha_0 > 0$, $\mu, \theta \in (0,1)$ and let $x_0, x_1 \in C$ be arbitrary starting points.

**Iterative step:** Given $x_{n-1}$, $x_n$, and $\alpha_n$, choose $\theta_n \in [0,\theta]$ where

$$\theta_n = \begin{cases} \min\left\{ \dfrac{\epsilon_n}{d(x_n, x_{n-1})}, \theta \right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Calculate $x_{n+1}$ and $\alpha_{n+1}$ for each $n \geq 1$ as follows:

**Step 1**: Compute

$$\begin{cases} w_n = \exp_{x_n}(-\theta_n \exp_{x_n}^{-1} x_{n-1}) \\ y_n = P_C(\exp_{w_n}(-\alpha_n F(w_n)), \end{cases} \tag{3.2}$$

**Step 2**: Calculate

$$z_n = \exp_{y_n} \alpha_n (P_{y_n,w_n} F(w_n) - F(y_n)) \tag{3.3}$$

**Step 3**: Calculate

$$u_n = \exp_{z_n}(1-\delta_n) \exp_{z_n}^{-1} T(z_n) \tag{3.4}$$

**Step 4**: Calculate $x_{n+1}$ and $\alpha_{n+1}$ by

$$x_{n+1} = \exp_{f(x_n)}(1-\beta_n) \exp_{f(x_n)}^{-1} u_n. \tag{3.5}$$

and

$$\alpha_{n+1} = \begin{cases} \min\left\{\dfrac{\mu d(w_n, y_n)}{\|P_{y_n,w_n}F(w_n) - F(y_n)\|}\right\}, \ \alpha_n + \eta_n & \text{if } F(w_n) \neq F(y_n), \\ \alpha_n + \eta_n, & \text{otherwise.} \end{cases} \tag{3.6}$$

**Stopping criterion** If $x_{n+1} = u_n = w_n = y_n$ for some $n \geq 1$ then stop. Otherwise set $n := n + 1$ and return to **Iterative step 1**.

---

**Lemma 3.1.** *Let $\{\alpha_n\}$ be the sequence generated by Algorithm 3.1. Then we have that $\lim\limits_{n\to\infty} \alpha_n = \alpha$ and $\alpha \in \left[\min\left\{\frac{\mu}{L}, \ \alpha_0\right\}, \alpha_0 + \eta\right]$, where $\eta = \sum\limits_{n=0}^{\infty} \eta_n$.*

*Proof.* Since $F$ is Lipschitz-continuous with constant $L > 0$, then in the case of $P_{y_n,w_n}(F(w_n) - F(y_n)) \neq 0$, we get

$$\frac{\mu d(w_n, y_n)}{\|P_{y_n,w_n}F(w_n) - F(y_n)\|} \geq \frac{\mu d(w_n, y_n)}{L d(w_n, y_n)} = \frac{\mu}{L}. \tag{3.7}$$

By the definition of $\alpha_{n+1}$ in Algorithm 3.1 and mathematical induction, we have that the sequence $\{\alpha_n\}$ has upper bound of $\alpha_0 + \eta$ and lower bound $\min\left\{\frac{\mu}{L}, \alpha_0\right\}$. The rest of the proof is similar to Lemma 3.1 in [20], so we omit it. □

*Remark* 3.1. It is obvious that the stepsize in Algorithm 3.1 is allowed to increase from iteration to iteration and so Algorithm 3.1 reduces the dependence on the initial stepsize $\alpha_0$. Also, since $\{\eta_n\}$ is summable, we obtain $\lim\limits_{n\to\infty} \eta_n = 0$. Thus the stepsize $\alpha_n$ may be non-increasing when $n$ is large. If $\eta_n \equiv 0$, the step size in Algorithm 3.1 reduces to the one in [9].

**Lemma 3.2.** *Let $\{z_n\}$, $\{w_n\}$ and $\{y_n\}$ be the sequences generated by Algorithm 3.1, then*

$$d^2(z_n, p) \leq d^2(w_n, p) - (1 - \alpha_n^2 \frac{\mu^2}{\alpha_{n+1}^2}) d^2(y_n, w_n).$$

*Proof.* Let $p \in \Omega$, then by applying Lemma 2.1, step 1 of Algorithm 3.1 and $y_n = P_C(\exp_{w_n}(-\alpha_n F(w_n)))$, we have

$$\langle \exp_{y_n}^{-1} \exp_{w_n}(-\alpha_n F(w_n)), \exp_{y_n}^{-1} p\rangle = \langle \exp_{y_n}^{-1} w_n - \alpha_n P_{y_n,w_n} F(w_n), \exp_{y_n}^{-1} p\rangle \leq 0,$$

that is,

$$\langle \exp_{y_n}^{-1} w_n, \exp_{y_n}^{-1} p\rangle \leq \alpha_n \langle P_{y_n,w_n} F(w_n), \exp_{y_n}^{-1} p\rangle. \tag{3.8}$$

Since $p \in \Omega$, we obtain that $\langle F(p), \exp_p^{-1} y_n\rangle \geq 0$. Using the pseudomonotone property of $F$, we obtain that $\langle F(y_n), \exp_{y_n}^{-1} p\rangle \leq 0$. Thus,

$$\begin{aligned} \langle F(y_n) - P_{y_n,w_n} F(w_n), \exp_{y_n}^{-1} p\rangle &= \langle F(y_n), \exp_{y_n}^{-1} p\rangle - \langle P_{y_n,w_n} F(w_n), \exp_{y_n}^{-1} p\rangle \\ &\leq -\langle P_{y_n,w_n} F(w_n), \exp_{y_n}^{-1} p\rangle. \end{aligned} \tag{3.9}$$

By considering the geodesic triangle $\Delta(w_n, y_n, p)$ and its comparison triangle $\Delta(w_n', y_n', p')$. It follows from Lemma 2.3, that $d(w_n, p) = \|w_n' - p'\|, d(y_n, p) = \|y_n' - p'\|$ and $d(w_n, y_n) = \|w_n' - y_n'\|$. From Algorithm 3.1, we have $z_n = \exp_{y_n} \alpha_n(P_{y_n,w_n}F(w_n) - F(y_n))$. Thus the comparison point of $z_n$ is $z_n' = y_n' +$

$\alpha_n(F(w_n^{'}) - F(y_n^{'})) \in \mathbb{R}^2$. Suppose that $d(z_n, y_n) = \|z_n^{'} - y_n^{'}\|$, then applying the diffeomorphism of the exp map, we obtain $\|F(y_n) - P_{y_n,w_n}F(w_n)\| = (\frac{1}{\alpha_n})d(z_n, y_n) = (\frac{1}{\alpha_n})\|z_n^{'} - y_n^{'}\| = \|F(y_n^{'}) - F(w_n^{'})\|$.
By applying Lemma 2.4 (ii), we have

$$\begin{aligned}
d^2(z_n, p) &\leq \|z_n^{'} - p^{'}\| \\
&= \|y_n^{'} + \alpha_n(F(w_n^{'}) - F(y_n^{'})) - p\| \\
&= \|y_n^{'} - p^{'}\|^2 + \alpha_n^2\|F(w_n^{'}) - F(y_n^{'})\|^2 + 2\alpha_n\langle F(w_n^{'}) - F(y_n^{'}), y_n^{'} - p^{'}\rangle \\
&= \|y_n^{'} - w_n^{'}\|^2 + \|w_n^{'} - p^{'}\|^2 + 2\langle y_n^{'} - w_n^{'}, w_n^{'} - p^{'}\rangle + \alpha_n^2\|F(w_n^{'}) - F(y_n^{'})\|^2 \\
&+ 2\alpha_n\langle F(w_n^{'}) - F(y_n^{'}), y_n^{'} - p^{'}\rangle \\
&= \|w_n^{'} - p^{'}\|^2 + \|y_n^{'} - w_n^{'}\|^2 - 2\langle y_n^{'} - w_n^{'}, y_n^{'} - w_n^{'}\rangle + 2\langle y_n^{'} - w_n^{'}, y_n^{'} - p^{'}\rangle \\
&+ \alpha_n^2\|F(w_n^{'}) - F(y_n^{'})\|^2 + 2\alpha_n\langle F(w_n^{'}) - F(y_n^{'}), y_n^{'} - p^{'}\rangle \\
&= \|w_n^{'} - p^{'}\|^2 - \|y_n^{'} - w_n^{'}\|^2 + \alpha_n^2\|F(y_n^{'}) - F(w_n^{'})\|^2 + \langle 2y_n^{'} - 2w_n^{'} \\
&+ 2\alpha_n F(w_n^{'}) - 2\alpha_n F(y_n^{'}), y_n^{'} - p^{'}\rangle \\
&= d(w_n, p) - d^2(y_n, w_n) + \alpha_n^2\|F(y_n) - P_{y_n,w_n}F(w_n)\|^2 \\
&+ \langle 2w_n^{'} - 2y_n^{'} + 2\alpha_n F(y_n^{'}) - 2\alpha_n F(w_n^{'}), p^{'} - y_n^{'}\rangle. \qquad (3.10)
\end{aligned}$$

Set $\chi = 2\exp_{y_n}^{-1} w_n + 2\alpha_n(F(y_n) - P_{y_n,w_n}F(w_n)) \in T_y\mathbb{M}$. Let $v = \exp_{y_n}\chi$, hence the comparison point of $v$ is $v^{'} = 2w_n^{'} - y_n^{'} + 2\alpha_n F(y_n^{'}) - 2\alpha_n F(w_n^{'})$. Now consider the geodesic triangle $\Delta(b, p, y_n)$ and its comparison triangle $\Delta(b^{'}, p^{'}, y_n^{'})$. Let $\vartheta$, $\vartheta^{'}$ be the angles of the vertices $y_n$ and $y_n^{'}$ respectively. By Lemma 2.4 (i), we get $\vartheta^{'} \geq \vartheta$. Therefore, we obtain from Lemma 2.3 and (2.5), we have

$$\begin{aligned}
\langle v^{'} - y_n^{'}, p^{'} - y_n^{'}\rangle &= \|v^{'} - y_n^{'}\|\ \|p^{'} - y_n^{'}\| \cos\vartheta^{'} \\
&= d(v, y_n)d(p, y_n)\cos\vartheta^{'} \\
&\leq d(v, y_n)d(p, y_n)\cos\vartheta \\
&= \langle \exp_{y_n}^{-1} v, \exp_{y_n}^{-1} p\rangle.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\langle 2w_n^{'} - 2y_n^{'} + 2\alpha_n F(y_n^{'}) - 2\alpha_n F(w_n^{'}), p^{'} - y_n^{'}\rangle \leq \langle 2\exp_{y_n}^{-1} w_n + 2\alpha_n(F(y_n) \\
- P_{y_n,w_n}F(w_n)), \exp_{y_n}^{-1} p\rangle. \qquad (3.11)
\end{aligned}$$

It follows from (3.10) and (3.11) that

$$\begin{aligned}
d^2(z_n, p) &\leq d^2(w_n, p) - d^2(y_n, w_n) + \alpha_n^2\|F(y_n) - P_{y_n,w_n}F(w_n)\|^2 \\
&+ \langle 2w_n^{'} - 2y_n^{'} + 2\alpha_n F(y_n^{'}) - 2\alpha_n F(w_n^{'}), p^{'} - y_n^{'}\rangle \\
&\leq d^2(w_n, p) - d^2(y_n, w_n) + \alpha_n^2\|F(y_n) - P_{y_n,w_n}F(w_n)\|^2 \\
&+ \langle 2\exp_{y_n}^{-1} w_n + 2\alpha_n(F(y_n) - P_{y_n,w_n}F(w_n)), \exp_{y_n}^{-1} p\rangle \\
&= d^2(w_n, p) - d^2(y_n, w_n) + \alpha_n^2\|F(y_n) - P_{y_n,w_n}F(w_n)\|^2 \\
&+ 2\langle \exp_{y_n}^{-1} w_n, \exp_{y_n}^{-1} p\rangle \\
&+ 2\alpha_n\langle F(y_n) - P_{y_n,w_n}F(w_n), \exp_{y_n}^{-1} p\rangle. \qquad (3.12)
\end{aligned}$$

On substituting (3.8) and (3.9) into (3.12), we obtain

$$\begin{aligned} d^2(z_n,p) &\leq d^2(w_n,p) - d^2(y_n,w_n) + \alpha_n^2\|F(y_n) - P_{y_n,w_n}F(w_n)\|^2 \\ &\quad + 2\alpha_n\langle P_{y_n,w_n}F(w_n), \exp_{y_n}^{-1} p\rangle - 2\alpha_n\langle P_{y_n,w_n}F(w_n), \exp_{y_n}^{-1} p\rangle \\ &= d^2(w_n,p) - d^2(y_n,w_n) + \alpha_n^2\|P_{y_n,w_n}F(w_n) - F(y_n)\|^2. \end{aligned} \tag{3.13}$$

By applying (3.6) in (3.13), we get

$$\begin{aligned} d^2(z_n,p) &\leq d^2(w_n,p) + \alpha_n^2\frac{\mu^2}{\alpha_{n+1}^2}d^2(y_n,w_n) - d^2(y_n,w_n) \\ &\leq d^2(w_n,p) - (1-\alpha_n^2\frac{\mu^2}{\alpha_{n+1}^2})d^2(y_n,w_n) &(3.14) \\ &\leq d(w_n,p). &(3.15) \end{aligned}$$

Hence, the proof complete. □

**Lemma 3.3.** *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1, then the sequence $\{x_n\}$ is bounded.*

*Proof.* Let $p \in \Omega$, $\gamma_n^1 : [0,1] \to \mathbb{M}$ and $\gamma_n^2 : [0,1] \to \mathbb{M}$ be geodesic spaces such that $\gamma_n^1(0) = f(x_n), \gamma_n^1(1) = u_n$ and $\gamma_n^2(0) = z_n, \gamma_n^2(1) = Tz_n$. Then, we have from Algorithm 3.1 that

$$\begin{aligned} d(u_n,p) &= d(\gamma_n^2(1-\delta_n),p) \\ &\leq (1-\delta_n)d(\gamma_n^2(0),p) + \delta_n d(\gamma_n^2(1),p) \\ &\leq (1-\delta_n)d(z_n,p) + \delta_n d(T(z_n),Tp) \\ &\leq (1-\delta_n)d(z_n,p) + \delta_n d(z_n,p) \\ &= d(z_n,p). \end{aligned} \tag{3.16}$$

Similarly, since $x_{n+1} = \gamma_n^1(1-\beta_n)$, we get

$$\begin{aligned} d(x_{n+1},p) &= d(\gamma_n^1(1-\beta_n),p) \\ &\leq \beta_n d(\gamma_n^1(0),p) + (1-\beta_n)d(\gamma_n^1(1),p) \\ &\leq d(f(x_n),p) + (1-\beta_n)d(u_n,p) \\ &\leq \beta_n\big[d(f(x_n),f(p)) + d(f(p),p)\big] + (1-\beta_n)d(z_n,p) \\ &\leq \beta_n\big[kd(x_n,p) + d(f(p),p)\big] + (1-\beta_n)d(w_n,p). \end{aligned} \tag{3.17}$$

By considering the geodesic triangles $\Delta(w_n,x_n,p)$ and $\Delta(x_n,x_{n-1},p)$ with their respective comparison triangle $\Delta(w_n^{'},x_n^{'},p^{'}) \subseteq \mathbb{R}^2$. Then by Lemma 2.3, we have $d(w_n,x_n) = \|w_n^{'} - x_n^{'}\|$, $d(w_n,p) = \|w_n^{'} - p^{'}\|$ and $d(x_n,x_{n-1}) = \|x_n^{'} - x_{n-1}^{'}\|$. Now, by applying step 1 of Algorithm 3.1, we have

$$\begin{aligned} d(w_n,p) &= \|w_n^{'} - p^{'}\| \\ &= \|x_n^{'} + \theta_n(x_n^{'} - x_{n-1}^{'}) - p^{'}\| \\ &\leq \|x_n^{'} - p^{'}\| + \theta_n\|x_n^{'} - x_{n-1}^{'}\| \\ &= \|x_n^{'} - p^{'}\| + \beta_n \cdot \frac{\theta_n}{\beta_n}\|x_n^{'} - x_{n-1}^{'}\|. \end{aligned} \tag{3.18}$$

Since $\frac{\theta_n}{\beta_n}\|x_n^{'}-x_{n-1}^{'}\| = \frac{\theta_n}{\beta_n}d(x_n,x_{n-1}) \to 0$ as $n\to\infty$, then there exists a constant $N_1>0$ such that $\frac{\theta_n}{\beta_n}d(x_n,x_{n-1}) \le N_1$. Thus, we obtain from (3.18) that

$$d(w_n,p) \le d(x_n,p) + \beta_n N_1. \tag{3.19}$$

Observe that,

$$\begin{aligned}
d^2(w_n,p) &= \|w_n^{'} - p^{'}\|^2\\
&\le \|x_n^{'} - p^{'}\| + \theta_n\|x_n^{'} - x_{n-1}^{'}\|\\
&= \|x_n^{'} - p^{'}\|^2 + 2\theta_n\|x_n^{'} - p^{'}\|\ \|x_n^{'} - x_{n-1}^{'}\| + \theta_n^2\|x_n^{'} - x_{n-1}^{'}\|^2\\
&= \|x_n^{'} - p^{'}\|^2 + \theta_n\|x_n^{'} - x_{n-1}^{'}\|\big[2\|x_n^{'} - p^{'}\| + \theta_n\|x_n^{'} - x_{n-1}^{'}\|\big].
\end{aligned} \tag{3.20}$$

It then follows that $2\|x_n^{'}-p^{'}\|+\theta_n\|x_n^{'}-x_{n-1}^{'}\| = 2d(x_n,p)+\theta_n d(x_n,x_{n-1}) \le N_2$ for some constant $N_2>0$. Thus, we obtain from (3.20), that

$$d^2(w_n,p) \le d^2(x_n,p) + \theta_n d(x_n,x_{n-1})N_2. \tag{3.21}$$

On substituting (3.19) into (3.17), we obtain

$$\begin{aligned}
d(x_{n+1},p) &\le \beta_n\big[kd(x_n,p)+d(f(p),p)\big] + (1-\beta_n)\,[d(x_n,p)+N_1]\\
&= (1-\beta_n(1-k))d(x_n,p) + \beta_n\left[(1-k)\frac{d(f(p),p)+N_1}{1-k}\right]\\
&\ \ \vdots\\
&\le \max\left\{d(x_n,p), \frac{d(f(p),p)+N_1}{1-k}\right\}.
\end{aligned}$$

By induction, we obtain that

$$d(x_{n+1},p) \le \max\left\{d(x_1,p), \frac{d(f(p),p)+N_1}{1-k}\right\}.$$

Hence, the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$ and $\{Tz_n\}$ are bounded. □

**Theorem 3.1.** *Let $f: C\to C$ be a contraction with constant $k\in(0,1)$ and assume conditions $(D1)$-$(D4)$ holds. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges to $p\in\Omega$, where $p = P_\Omega f(p)$ and $P_\Omega$ is the nearest point projection of $C$ onto $\Omega$.*

*Proof.* Let $p\in\Omega$, then using Proposition 2, we obtain

$$\begin{aligned}
d^2(u_n,p) &= d(\gamma_n^2(1-\delta_n),p)\\
&\le (1-\delta_n)d^2(\gamma_n^2(0),p) + \delta_n d^2(\gamma_n^2(1),p)\\
&- \delta_n(1-\delta_n)d^2(\gamma_n^2(0),\gamma_n^2(1))\\
&\le (1-\delta_n)d^2(z_n,p) + \delta_n d^2(T(z_n),Tp)\\
&- \delta_n(1-\delta_n)d^2(z_n,Tz_n)\\
&\le (1-\delta_n)d^2(z_n,p) + \delta_n d^2(z_n,p)\\
&- \delta_n(1-\delta_n)d^2(z_n,Tz_n)\\
&= d^2(z_n,p) - \delta_n(1-\delta_n)d^2(z_n,Tz_n).
\end{aligned} \tag{3.22}$$

By substituting (3.14) and (3.21) into (3.22), we have

$$
\begin{aligned}
d^2(u_n,p) &\le d^2(w_n,p) - (1-\alpha_n^2\frac{\mu^2}{\alpha_{n+1}^2})d^2(y_n,w_n) - \delta_n(1-\delta_n)d^2(z_n,Tz_n)\\
&\le d^2(x_n,p) + \theta_n d(x_n,x_{n-1})N_2 - (1-\alpha_n^2\frac{\mu^2}{\alpha_{n+1}^2})d^2(y_n,w_n)\\
&- \delta_n(1-\delta_n)d^2(z_n,Tz_n). \qquad (3.23)
\end{aligned}
$$

Fix $n \ge 1$ and let $v = f(x_n), u = u_n$ and $w = f(p)$. Consider the following geodesic triangles with their respective comparison triangles $\Delta(v,u,w)$ and $\Delta(v',u',w'), \Delta(w,u,v)$ and $\Delta(w',u',v'), \Delta(w,u,p)$ and $\Delta(w',u',p')$. Applying Lemma 2.3, we get $d(v,u) = \|v'-u'\|, d(v,w) = \|v'-w'\|, d(v,p) = \|v'-p'\|, d(u,w) = \|u'-w'\|$ and $d(w,p) = \|w'-p'\|$. From Algorithm 3.1, we have $x_{n+1} = \exp_v(1-\beta_n)\exp_v^{-1}u$. The comparison point of $x_{n+1} \in \mathbb{R}^2$ is $x'_{n+1} = \beta_n v' + (1-\beta_n)u'$. Let $\phi$ and $\phi'$ denote the angle and comparison angle at $p$ and $p'$ in the triangles $\Delta(w,x_{n+1},p)$ and $\Delta(y',x'_{n+1},p')$ respectively. Therefore, $\phi \le \phi'$ and $\cos\phi' \le \cos\phi$.

By applying Lemma 2.4 and the property of $f$, we obtain

$$
\begin{aligned}
d^2(x_{n+1},p) &\le \|x'_{n+1}-p'\|^2\\
&= \|\beta_n(v'-p') + (1-\beta_n)(u'-p')\|^2\\
&\le \|\beta_n(v'-w') + (1-\beta_n)(u'-p')\|^2 + 2\beta_n\langle x'_{n+1}-p', w'-p'\rangle\\
&\le (1-\beta_n)\|u'-p'\|^2 + \beta_n\|v'-w'\|^2 + 2\beta_n\|x'_{n+1}-p'\|\|w'-p'\|\cos\phi'\\
&\le (1-\beta_n)d^2(u,p) + \beta_n d^2(v,w) + 2\beta_n d(x_{n+1},p)d(w,p)\cos\phi\\
&= (1-\beta_n)d^2(u_n,p) + \beta_n d^2(f(x_n),f(p)) + 2\beta_n d(x_{n+1},p)d(w,p)\cos\phi. \qquad (3.24)
\end{aligned}
$$

It is obvious that $d(x_{n+1},p)d(f(p),p)\cos\phi = \langle \exp_p^{-1} f(p), \exp_p^{-1} x_{n+1}\rangle$, then by substituting (3.23) into (3.24), we obtain

$$
\begin{aligned}
d^2(x_{n+1},p) &\le (1-\beta_n)d^2(u_n,p) + \beta_n d^2(f(x_n),f(p)) + 2\beta_n\langle \exp_p^{-1} f(p), \exp_p^{-1} x_{n+1}\rangle\\
&\le (1-\beta_n)d^2(x_n,p) + (1-\beta_n)\theta_n d(x_n,x_{n-1})N_2\\
&- (1-\beta_n)(1-\alpha_n^2\frac{\mu^2}{\alpha_{n+1}^2})d^2(y_n,w_n)\\
&+ \beta_n d^2(f(x_n),f(p)) + 2\beta_n\langle \exp_p^{-1} f(p), \exp_p^{-1} x_{n+1}\rangle\\
&- (1-\beta_n)\delta_n(1-\delta_n)d^2(z_n,Tz_n)\\
&= \big(1-\beta_n(1-k)\big)d^2(x_n,p)\\
&+ \beta_n(1-k)\Bigg[\frac{\frac{\theta_n}{\beta_n}d(x_n,x_{n-1})N_2 + 2\langle \exp_p^{-1} f(p), \exp_p^{-1} x_{n+1}\rangle}{(1-k)}\Bigg]\\
&- (1-\beta_n)(1-\alpha_n^2\frac{\mu^2}{\alpha_{n+1}^2})d^2(y_n,w_n) - (1-\beta_n)\delta_n(1-\delta_n)d^2(z_n,Tz_n) \qquad (3.25)\\
&= \big(1-\beta_n(1-k)\big)d^2(x_n,p) + \beta_n(1-k)Z_n, \qquad (3.26)
\end{aligned}
$$

where

$$Z_n = \left[\frac{\frac{\theta_n}{\beta_n} d(x_n, x_{n-1}) N_2 + 2\langle \exp_p^{-1} f(p), \exp_p^{-1} x_{n+1}\rangle}{(1-k)}\right.$$
$$- (1-\beta_n)(1-\alpha_n^2 \frac{\mu^2}{\alpha_{n+1}^2}) d^2(y_n, w_n)$$
$$- (1-\beta_n)\delta_n(1-\delta_n) d^2(z_n, Tz_n).$$

From (3.25), we obtain

$$\begin{aligned}
&(1-\beta_n)(1-\alpha_n^2 \frac{\mu^2}{\alpha_{n+1}^2}) d^2(y_n, w_n) \\
&- (1-\beta_n)\delta_n(1-\delta_n) d^2(z_n, Tz_n) \\
&\leq d^2(x_n, p) - d^2(x_{n+1}, p) \\
&+ \beta_n(1-k) N_3, \qquad (3.27)
\end{aligned}$$

where $N_3 := \sup\limits_{n\in\mathbb{N}} Z_n$.

To show that $d(x_n, p) \to 0$ as $n \to \infty$. Let $a_n = d(x_n, p)$ and $d_n = \beta_n(1-k)$. It is very easy to see that the inequality (3.26) satisfies

$$a_{n+1} \leq (1-d_n)a_n + d_n b_n.$$

In view of Lemma 2.7, we claim that $\limsup\limits_{k\to\infty} Z_{n_k} \leq 0$ for a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{n\to\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0.$$

Now, from (3.27), we get

$$\begin{aligned}
&\limsup_{k\to\infty} \left[(1-\beta_{n_k})\left(1 - \alpha_{n_k}\frac{\mu^2}{\alpha_{n_k+1}}\right) d^2(y_{n_k}, w_{n_k})\right. \\
&\left. + (1-\beta_{n_k})\delta_{n_k} d^2(z_{n_k}, T(z_{n_k}))\right] \\
&\leq \limsup_{k\to\infty}\left[d^2(x_{n_k}, p) - d^2(x_{n_k+1}, p) + \beta_{n_k}(1-k)N_3\right] \\
&= -\liminf_{k\to\infty}(d^2(x_{n_{k+1}}, p) - d^2(x_{n_k}, p)) \\
&\leq 0. \qquad (3.28)
\end{aligned}$$

By applying the condition on $\beta_{n_k}, \delta_{n_k}$ and the fact that

$$\lim_{k\to\infty}\left(1 - \alpha_{n_k}\frac{\mu^2}{\alpha_{n_k+1}^2}\right) = 1 - \mu^2 > 0,$$

thus, we obtain that

$$\lim_{k\to\infty} d(y_{n_k}, w_{n_k}) = 0 = \lim_{k\to\infty} d(z_{n_k}, T(z_{n_k})). \qquad (3.29)$$

From Algorithm 3.1 and replacing $x_{n_k}$ with $p$ in (3.18), it is clear that

$$\begin{aligned}\lim_{k\to\infty} d(w_{n_k}, x_{n_k}) &\leq \lim_{k\to\infty} \beta_{n_k} \cdot \frac{\theta_{n_k}}{\beta_{n_k}} \|x'_{n_k} - x'_{n_k-1}\| \\ &\leq \lim_{k\to\infty} \beta_{n_k} \cdot \frac{\theta_{n_k}}{\beta_{n_k}} d(x_{n_k}, x_{n_k-1}) \\ &= 0. \end{aligned} \tag{3.30}$$

The following are easy to establish from Algorithm 3.1, (3.29) and (3.30):

$$\begin{cases} \lim\limits_{k\to\infty} d(u_{n_k}, z_{n_k}) = 0, \\ \lim\limits_{k\to\infty} d(x_{n_k+1}, u_{n_k}) = 0, \\ \lim\limits_{k\to\infty} d(z_{n_k}, y_{n_k}) = 0, \\ \lim\limits_{k\to\infty} d(x_{n_k+1}, z_{n_k}) = 0, \\ \lim\limits_{k\to\infty} d(x_{n_k+1}, y_{n_k}) = 0, \\ \lim\limits_{k\to\infty} d(x_{n_k+1}, w_{n_k}) = 0, \\ \lim\limits_{k\to\infty} d(y_{n_k}, x_{n_k}) = 0, \\ \lim\limits_{k\to\infty} d(x_{n_k+1}, x_{n_k}) = 0. \end{cases} \tag{3.31}$$

Since $\{x_{n_k}\}$ and $\{y_{n_k}\}$ are bounded, there exist subsequences $\{x_{n_{k_l}}\}$ and $\{y_{n_{k_l}}\}$ which converge to $x^*$. Using the fact that $y_{n_{k_l}} = P_C(\exp_{w_{n_{k_l}}}(-\alpha_{n_{k_l}} F(w_{n_{k_l}}))$ and by Lemma 2.1, we get

$$\begin{aligned} &\langle \exp^{-1}_{y_{n_{k-l}}} \exp_{w_{n_{k_l}}}(-\alpha_{n_{k_l}} F(w_{n_{k_l}})), \exp^{-1}_{y_{n_{k_l}}} x\rangle \\ &= \langle \exp^{-1}_{y_{n_{k_l}}} w_{n_{k_l}} - \alpha_{n_{k_l}} P_{y_{n_{k_l}}}, w_{n_{k_l}} F(w_{n_{k_l}}), \\ &\exp^{-1}_{y_{n_{k_l}}} x\rangle \leq 0. \end{aligned} \tag{3.32}$$

Using Remark 2.2 and (2.1), the inequality (3.32) becomes

$$\begin{aligned} 0 &\geq \langle \exp^{-1}_{y_{n_{k_l}}} w_{n_{k_l}} - \alpha_{n_{k_l}} P_{y_{n_{k_l}}}, w_{n_{k_l}} F(w_{n_{k_l}}), \exp^{-1}_{y_{n_{k_l}}} x\rangle \\ &= \langle \exp^{-1}_{y_{n_{k_l}}} w_{n_{k_l}}, \exp^{-1}_{y_{n_{k_l}}} x\rangle \\ &- \alpha_{n_{k_l}} \langle P_{y_{n_{k_l}}}, w_{n_{k_l}} F(w_{n_{k_l}}), \exp^{-1}_{y_{n_{k_l}}} x\rangle \\ &\geq \langle \exp^{-1}_{y_{n_{k_l}}} w_{n_{k_l}}, \exp^{-1}_{y_{n_{k_l}}} x\rangle \\ &- \alpha_{n_{k_l}} \langle P_{y_{n_{k_l}}}, w_{n_{k_l}} F(w_{n_{k_l}}), \exp^{-1}_{y_{n_{k_l}}} w_{n_{k_l}}\rangle \\ &- \alpha_{n_{k_l}} \langle P_{y_{n_{k_l}}}, w_{n_{k_l}} F(w_{n_{k_l}}), P_{y_{n_{k_l}}}, w_{n_{k_l}} \exp^{-1}_{w_{n_{k_l}}} x\rangle \\ &= \langle \exp^{-1}_{y_{n_{k_l}}} w_{n_{k_l}}, \exp^{-1}_{y_{n_{k_l}}} x\rangle \\ &- \alpha_{n_{k_l}} \langle P_{y_{n_{k_l}}}, w_{n_{k_l}} F(w_{n_{k_l}}), \exp^{-1}_{y_{n_{k_l}}} w_{n_{k_l}}\rangle \\ &- \alpha_{n_{k_l}} \langle F(w_{n_{k_l}}), \exp^{-1}_{w_{n_{k_l}}} x\rangle. \end{aligned}$$

In view of Lemma 2.2, the Lipschitz continuity of $F$ and $\lim_{l\to\infty} \alpha_{n_{k_l}} = \alpha > 0$, it follows that

$$\langle F(x^*), \exp_{x^*}^{-1} x\rangle \geq 0, \ \forall \ x \in C.$$

Hence, $x^* \in VIP(C, F)$. Also, by applying (3.29), we have that $x^* \in F(T)$. Therefore, we conclude that $x^* \in \Omega$.
Next, we claim that $\limsup_{k\to\infty} Z_{n_k} \leq 0$. To establish this, we need to show that

$$\limsup_{k\to\infty} \langle \exp_p^{-1} f(p), \exp_p^{-1} x_{n_k+1}\rangle \leq 0.$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ which converges to $x^* \in \mathbb{M}$ such that

$$\begin{aligned}\lim_{l\to\infty} \langle \exp_p^{-1} f(p), \exp_p^{-1} x_{n_{k_l}}\rangle &= \limsup_{k\to\infty} \langle \exp_p^{-1} f(p), \exp_p^{-1} x_{n_k}\rangle \\ &= \langle \exp_p^{-1} f(p), \exp_p^{-1} x^*\rangle \\ &\leq 0. \end{aligned} \tag{3.33}$$

By substituting (3.33) into (3.26) and applying Lemma 2.7, we conclude that $\{x_n\}$ converges to $p \in \Omega$. □

## 4. Numerical example

In this section, we present two numerical examples in the framework of Hadamard manifolds to illustrate the performance of our iterative method. Let $\mathbb{M} := \mathbb{R}^{++} = \{u \in \mathbb{R} : u > 0\}$ and $(\mathbb{R}^{++}, \langle ., .\rangle)$ be the Riemannian manifold with the Riemannian metric $\langle ., .\rangle$ defined by

$$\langle x, y\rangle := \frac{1}{u^2} xy, \tag{4.1}$$

for all vectors $x, y \in T_u\mathbb{M}$, where $T_u\mathbb{M}$ is the tangent space at $x \in \mathbb{M}$. For $u \in \mathbb{M}$, the tangent space $T_u\mathbb{M}$ at $u$ equals $\mathbb{R}$. Also, the parallel transport is known as the identity mapping. The Riemannian distance (see 35) $d : \mathbb{M} \times \mathbb{M} \to \mathbb{R}^+$ is defined by

$$d(x, y) := |\ln \frac{x}{y}|, \forall \ x, y \in \mathbb{M}. \tag{4.2}$$

Then $(\mathbb{R}^{++}, \langle ., .\rangle)$ is an Hadamard manifold, and the unique geodesic $\omega : \mathbb{R} \to \mathbb{M}$ with initial value $\omega(0) = x$ with $v = \omega'(0) \in T_x\mathbb{M}$ is defined by $\omega(t) := xe^{(\frac{vt}{x})}$. In addition, the inverse exponential map is defined by

$$\exp_x^{-1} y = \omega'(0) = x\frac{\ln y}{x}. \tag{4.3}$$

**Example 4.1.** Let $C = [1, 2]$ be a geodesic convex subset of $\mathbb{R}^+$ and $F : C \to \mathbb{R}$ be a single-valued vector field defined by

$$Fx := -x \ln \frac{2}{x}, \in C \tag{4.4}$$

Now, let $x, y \in C$ and $\langle Fx, \exp_x^{-1} y\rangle \geq 0$. Then we get

$$\begin{aligned}
\langle Fy, \exp_y^{-1} x\rangle &\leq \langle Fy, \exp_y^{-1} x\rangle + \langle Fx, \exp_x^{-1} y\rangle \\
&= \frac{1}{y^2}\Big(-y\ln\frac{2}{y}\Big)\Big(y\ln\frac{x}{y}\Big) + \frac{1}{x^2}\Big(-x\ln\frac{2}{x}\Big)\Big(x\ln\frac{y}{x}\Big) \\
&= \Big(-\ln\frac{2}{y}\Big)\Big(\ln\frac{x}{y}\Big) + \Big(-\ln\frac{2}{x}\Big)\Big(\ln\frac{y}{x}\Big) \\
&= -\ln 2\ln\frac{x}{y} + \ln y\ln\frac{x}{y} - \ln 2\ln\frac{y}{x} + \ln x\ln\frac{y}{x} \\
&= \ln y\ln\frac{x}{y} + \ln x\ln\frac{y}{x} \\
&= (\ln x\ln y - \ln^2 y + \ln x\ln y - \ln^2 x) \\
&= -\ln^2\frac{x}{y} \\
&\leq 0. \qquad (4.5)
\end{aligned}$$

Hence, we conclude that $F$ is pseudomonotone and 1-Lipschitz continuous. Therefore, the variational inequality problem has a unique solution, i.e

$$\begin{aligned}
\langle Fp, \exp_p^{-1} y\rangle =& \frac{1}{p^2}\Big(-p\ln\frac{2}{p}\Big)\Big(p\ln\frac{y}{p}\Big) \\
&= -\ln\frac{2}{p}\ln\frac{y}{p} \geq 0, \forall y \in C \\
&\Leftrightarrow p = 2. \qquad (4.6)
\end{aligned}$$

We deduce that $VIP(F, C) = \{2\}$, therefore $\Omega \neq \emptyset$. Let $f$ be a continuous mapping and $T$ be a nonexpansive mapping defined by $f(x) = \frac{1}{2}x$ and $T(x) = x$ for all $x \in C$. Choose $\eta_n = \frac{100}{(n+1)^{(1.1)}}, \beta_n = \frac{0.1}{n+1}, \delta_n = \frac{n}{3n+7}$, $\mu = 0.5$, $\epsilon_n = (\frac{1}{2})^n$ and $\alpha_n = \frac{1}{2} - \frac{1}{n+3}$ for Khammahawong et al. [16, Algorithm 1]. The termination criterion is $d(x_n, x_{n+1}) \leq \epsilon$. For this numerical experiment we take $x_0 = 1$, $x_1 = 1.1$ and compare our algorithm with [16, Algorithm 1] with $\epsilon = 10^{-3}$ and $\epsilon = 10^{-4}$.

**Example 4.2.** Let $C = [1, 10]$ be a geodesic convex subset of $\mathbb{R}^+$ and $F : C \to \mathbb{R}$ be a single-valued vector field defined by

$$Fx = x\ln x, \forall x \in C \qquad (4.7)$$

Now let $x, y \in C$ and $\langle Fx, \exp_x^{-1} y\rangle \geq 0$. Then we have

$$\begin{aligned}
\langle Fy, \exp_y^{-1} x\rangle &\leq \langle Fy, \exp_y^{-1} x\rangle + \langle Fx, \exp_x^{-1} y\rangle \\
&= \frac{1}{y^2}\cdot y\ln y\cdot y\ln\frac{x}{y} + \frac{1}{x^2}\cdot x\ln x\cdot x\ln\frac{y}{x} \\
&= -(\ln y - \ln x)^2 \\
&= -\ln^2\frac{y}{x} \\
&\leq 0.
\end{aligned}$$

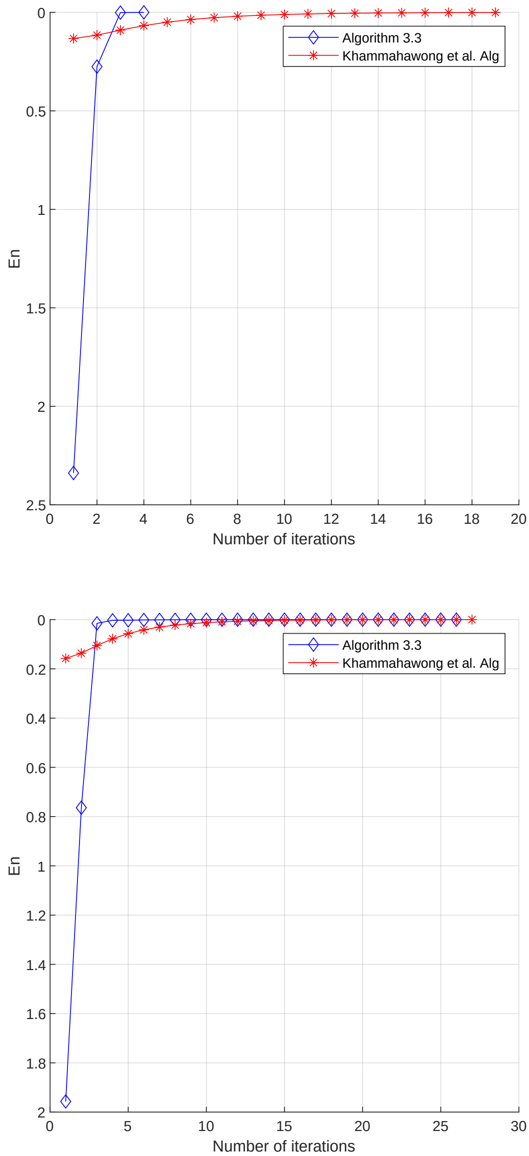
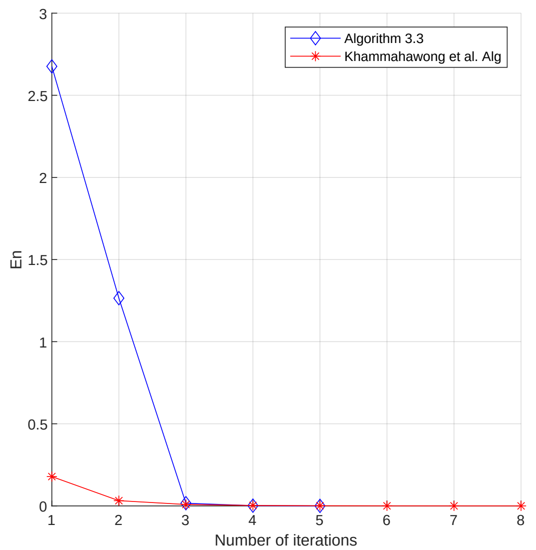



FIGURE 1. Numerical report for Example 4.1.

Hence, $F$ is pseudomonotone. Also, the variational inequality problem has a unique solution i.e

$$\begin{aligned}\langle Fp, \exp_p^{-1} q\rangle =& \frac{1}{p^2}(p\ln p)\cdot p\cdot \ln\frac{q}{p} \\ &= \ln p \ln\frac{q}{p} \geq 0,\ \forall\, q\in C \\ &\Longleftrightarrow p = 1. \end{aligned} \tag{4.8}$$

Thus, $\Omega \neq \emptyset$. For this example, choose $\eta_n = \frac{100}{(n+1)^{(1.1)}}, \beta_n = \frac{0.1}{100n+1}, \delta_n = \frac{n}{3n+7}$, $\mu = 0.5$, $\epsilon_n = (\frac{1}{2})^n$ and $\alpha_n = \frac{1}{2} - \frac{1}{n+3}$ for Khammahawong et al. [16, Algorithm 1]. The termination criterion is $d(x_n, x_{n+1}) \leq \epsilon$. For this numerical experiment we take $\epsilon = 10^{-4}$ and compare our algorithm with Khammahawong et al. [16, Algorithm 1] with varying initial points $x_0$ and $x_1$. It can be seen from figures that our iterative method converges faster that of Khammahawong et al. [16].



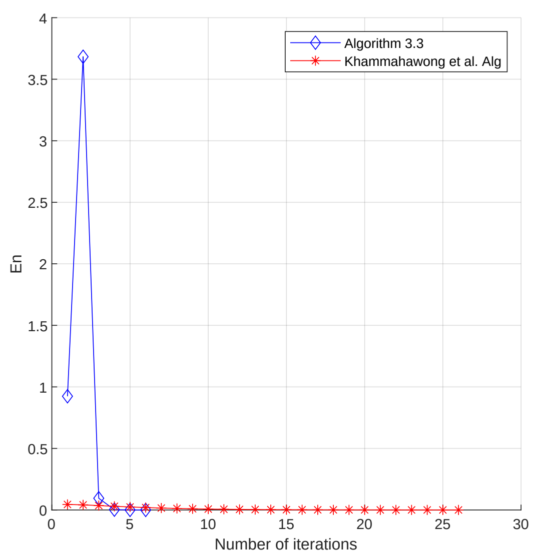




FIGURE 2. Numerical report for Example 4.2. Left: $x_0 = 0.8$ and $x_1 = 0.5$; Right: $x_0 = 1.2$ and $x_1 = 0.9$.

## 5. Conclusion

In this article, we considered an inertial extrapolation method which is known to speed up the rate of convergence of iterative method together with a Tseng's method to solve variational inequality problem involving pseudomonotone function and fixed point of a nonexpansive mapping in the settings of a Hadamard manifold. We employed a self-adaptive procedure which generates dynamic stepsizes converging to a positive constant. Several examples were illustrated and compared with the result of [16].

To generalize problem (1.1) in our future research, we will consider VIP involving quasi-monotone function together with a projection and contraction method in the setting of Hadamard manifolds.

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