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ON THE CLASS OF n-QUASI-m-SYMMETRIC OPERATORS

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Abstract. The aim of this paper is to extend some properties of m-symmetric operators to the class of $n-$quasi$-m-$symmetry. It is shown that if T is an n_1-quasi$-m-$symmetry and S is an n_2-quasi-l-symmetry such that T and S are double commuting, then TS is an $n-$quasi$-(m+l-1)-$symmetry where $n = \max\{n_1, n_2\}$. Also we study some spectral properties and $\mathcal{C}_0-$semigroup of this class.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ denote to the algebra of all bounded linear operators on $\mathcal{H}$. For every $T \in \mathcal{B}(\mathcal{H})$, we denote T^*, $N(T)$ and $R(T)$ the adjoint, the null space and the range of T, respectively. As usual $\overline{M}$ denotes the closure of $M \subset \mathcal{H}$, while $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ stand for the spectrum, the point spectrum and the approximate point spectrum of T, respectively.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be m-isometry for some positive integer m if

$$\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}T^{*k}T^{k} = 0,$$

where $\begin{pmatrix} m \\ k \end{pmatrix}$ is the binomial coefficient. This class has been generalized to the class of n-quasi-m-isometry i.e., T is an n-quasi-m-isometry if

$$T^{*n}\left(\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}T^{*k}T^{k}\right)T^{n} = 0,$$

for some positive integers m and n. This class has been studied in [2, 4, 8, 10, 13].

Let m be a positive integer. $T \in \mathcal{B}(\mathcal{H})$ is said to be m-symmetry if it satisfies

$$\alpha_m(T) = \sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}T^{*(m-k)}T^{k} = 0.$$

For $m = 1$, we obtain that T is selfadjoint. Moreover, if T is selfadjoint, then T is m-symmetric for every positive integer m; hence the class of m-symmetry is a generalization of selfadjoint operators. This class has been studied by many authors see [1, 3, 6, 7, 11, 12, 14].

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In [16] the authors generalize the class of $m-$symmetry to the class of n-quasi-m-symmetry where m and n are positive integers, i.e., T is an n-quasi-m-symmetry if

$$T^{*n}\left(\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}T^{*(m-k)}T^{k}\right)T^{n}=0.$$

Recall that T is an n-quasi strict m-symmetry if T is an n-quasi-m-symmetry, and T is not an n-quasi-$(m-1)$-symmetry. It is shown that if T is an n-quasi-m-symmetry and $R\left(T^{n}\right)$ is dense, then T is an m-symmetric operator [16]. It has been proved in [16] that T is an n-quasi-m-symmetric operator if and only if

$$T=\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $\mathcal{H}=\overline{R\left(T^{n}\right)}\oplus N\left(T^{*n}\right)$, where T_1 is an m-symmetric operator and $T_3^n=0$. A well-known property of n-quasi-m-symmetries is the power of any n-quasi-m-symmetry is also an n-quasi-m-symmetry. Moreover, if T is an n-quasi-m-symmetry, then T is an k-quasi$-l-$symmetry for $k\geq n$ and $l\geq m$.

Our purpose in this paper is to extend the properties of m-symmetry to the class of n-quasi-m-symmetry and we discuss which properties remain valid and which properties are not valid for the class of n-quasi-m-symmetry. The paper is organized as follows. Section 2 begins with some lemmas that are needed throughout this work. Then we study some properties of m-symmetric operators. Next we devote our interest to the study of properties of n-quasi-m-symmetries. Section 3 is dedicated to discuss $\mathcal{C}_0-$semigroup of n-quasi-m-symmetry. More precisely, we prove that if $\{T(t)\}_{t\geq 0}\subset\mathcal{B}(\mathcal{H})$ is a $\mathcal{C}_0-$semigroup on $\mathcal{H}$, then $T(t)$ is an n-quasi-m-symmetry for all $t\geq 0$ if and only if $T(t)$ is an m-symmetry for all $t\geq 0$.

2. Properties of n-quasi-m-symmetry

First, we present some lemmas which are needed throughout this work.
Let $T,S\in\mathcal{B}(\mathcal{H})$. T and S are said to be double commuting if T commutes with S and S^*.

Lemma 2.1. [5] *Let $T,S,Q\in\mathcal{B}(\mathcal{H})$. Then*

(i) *If T and Q commutes, then*

$$\alpha_m(T+Q)=\sum_{k=0}^{m}\sum_{j=0}^{m-k}(-1)^k\begin{pmatrix} m \\ k \end{pmatrix}\begin{pmatrix} m-k \\ j \end{pmatrix}Q^{*j}\alpha_{m-k-j}(T)Q^k.$$

(ii) *If T and S are double commuting, then*

$$\alpha_m(TS)=\sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix}T^{*k}\alpha_k(S)\alpha_{m-k}(T)S^{m-k}.$$

Lemma 2.2. [5] *Let $T,S\in\mathcal{B}(\mathcal{H})$. Assume that T and S are double commuting. If T is $m-$symmetric and S is ℓ-symmetric, then operator TS is $(m+\ell-1)$-symmetric.*

Lemma 2.3. [9] *Let $T \in \mathcal{B}(\mathcal{H})$ be an $2m-$symmetry. Then T is an $(2m-1)-$symmetry.*

Proposition 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. If T^r and T^{r+1} are selfadjoint for some $r \in \mathbb{N}$, then T is selfadjoint.*

Proof. Since T is invertible with T^r and T^{r+1} are selfadjoint, then

$$\begin{aligned} T^{r+1} &= (T^{r+1})^* = (T^r)^*T^* = T^rT^* \\ &\Longrightarrow T^{r+1} = T^rT^* \\ &\Longrightarrow T^* = T. \end{aligned}$$

Therefore, T is selfadjoint. □

The following example shows that Proposition 2.1 is not necessarily true if T is not invertible.

Example 2.1. Let $\mathcal{H} = \mathbb{C}^3$, and $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, it is clear that T is not selfadjoint even though T^2 and T^3 are selfadjoint, since $T^2 = T^3 = 0$.

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be an invertible operator. If T^r is an $m-$symmetry and T^s is an $l-$symmetry with $T^{*r}T^s = T^sT^{*r}$ for some positive integers r and s, then T^h is an $(m+l-1)-$symmetry where $h = \max\{r, s\} - min\{r, s\}$.*

Proof. Assume that $r \leq s$, since $T^rT^s = T^sT^r = T^{r+s}$, then $T^{-r}T^s = T^sT^{-r}$. Similarly, since $T^{*r}T^s = T^sT^{*r}$, we deduce that $(T^*)^{-r}T^s = T^s(T^*)^{-r}$.
That means T^{-r} and T^s are double commuting, and since T^r is an $m-$symmetry, thus so is T^{-r}. Applying Lemma 2.2 on T^s and T^{-r}, we get $T^h = T^sT^{-r}$ is an $(m+l-1)-$symmetry where $h = s - r$. □

Corollary 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be an invertible operator. Then we have*

(1) *If T^r is an $m-$symmetry and T^{r+1} is an $l-$symmetry for some $r \in \mathbb{N}$, such that $T^{*r}T^{r+1} = T^{r+1}T^{*r}$, then T is an $(m+l-1)-$symmetry.*
(2) *If T^r is selfadjoint and T^{r+1} is an $m-$symmetry, then T is an $m-$symmetry.*

Theorem 2.2. *Let $T, S \in \mathcal{B}(\mathcal{H})$. If T is an n_1-quasi$-m-$symmetry and S is an n_2-quasi-l-symmetry such that T and S are double commuting, then TS is an $n-$quasi$-(m+l-1)-$symmetry, where $n = \max\{n_1, n_2\}$. Moreover, TS is an $n-$quasi strict $(m+l-1)-$symmetry if and only if $T^{*(n+l-1)}\alpha_{m-1}(T)T^nS^{*n}\alpha_{l-1}(S)$ S^{n+l-1} is not the zero operator.*

Proof. Since $TS = ST$ and $T^*S = ST^*$, and by Lemma 2.1, we obtain

$$(T^*S^*)^n\alpha_{m+l-1}(TS)(TS)^n$$

$$= T^{*n}S^{*n}\sum_{k=0}^{m+l-1}\binom{m+l-1}{k}T^{*k}\alpha_k(S)\alpha_{m+l-1-k}(T)S^{m+l-1-k}T^nS^n.$$

For $k \leq l-1$, then $m+l-1-k \geq m$, thus for all $k \leq l-1$, we have

$$T^{*n}\alpha_{m+l-1-k}(T)T^n = 0.$$

For $k \geq l$, then $S^{*n}\alpha_k(S)S^n = 0$ for all $k \geq 0$, hence

$$(T^*S^*)^n \alpha_{m+l-1}(TS)(TS)^n = 0.$$

Similarly, we get

$$(T^*S^*)^n \alpha_{m+l-2}(TS)(TS)^n = T^{*(n+l-1)}\alpha_{m-1}(T)T^n S^{*n}\alpha_{l-1}(S)S^{n+l-1}.$$

Therefore, TS is an $n-$quasi strict $(m+l-1)-$symmetry if and only if $T^{*(n+l-1)}$ $\alpha_{m-1}(T)T^n S^{*n}\alpha_{l-1}(S)S^{n+l-1}$ is not the zero operator. □

The following example shows that Theorem 2.2 is not necessarily true if T and S are not double commuting.

Example 2.2. Let $\mathcal{H} = \mathbb{C}^3$, and $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, by a simple calculation we obtain that T is an $1-$quasi$-3-$symmetry, and S is clearly selfadjoint. We have that T and S are not commuting, then by straightforward calculation we get that TS is not a $1-$quasi$-3-$symmetry.

Theorem 2.3. *Let $T, Q \in \mathcal{B}(\mathcal{H})$. If T is an $n-$quasi$-m-$symmetry and Q is a nilpotent operator of order p such that $TQ = QT$, then $T + Q$ is an $(n + p - 1)-$quasi$-(m + 2p - 2)-$symmetry. Furthermore, $T + Q$ is an $(n + p - 1)-$quasi strict $-(m + 2p - 2)-$symmetry if and only if $(T^* + Q^*)^{n+p-1}Q^{*(p-1)}\alpha_{m-1}(T)$ $Q^{p-1}(T + Q)^{n+p-1}$ is not the zero operator.*

Proof. Set $r = n + p - 1$, $l = m + 2p - 2$ and $R = T + Q$, since $TQ = QT$ and applying Lemma 2.1, we obtain

$$R^{*r}\alpha_l(R)R^r$$

$$=(T^* + Q^*)^r \sum_{k=0}^{l}\sum_{j=0}^{l-k}(-1)^k \binom{l}{k}\binom{l-k}{j} Q^{*j}\alpha_{l-k-j}(T)Q^k(T+Q)^r$$

$$=\sum_{i=0}^{r} a_i T^{*r-i}Q^{*i} \sum_{k=0}^{l}\sum_{j=0}^{l-k}(-1)^k b_k c_{k,j} Q^{*j}\alpha_{l-k-j}(T)Q^k \sum_{i=0}^{r} a_i T^{r-i}Q^i,$$

where $a_i = \binom{r}{i}$, $b_k = \binom{l}{k}$ and $c_{k,j} = \binom{l-k}{j}$.

Note that if $k \geq p$ or $j \geq p$, then $Q^k = 0$ or $Q^{*j} = 0$, thus

$$Q^{*j}\alpha_{l-k-j}(T)Q^k = 0.$$

If $k \leq p - 1$ and $j \leq p - 1$, we obtain

$$l - k - j = m + 2p - 2 - k - j \geq m + 2p - 2 - (p - 1) - (p - 1) = m.$$

Since T is an $n-$quasi$-m-$symmetry and Q is a nilpotent operator of order p, we get

$$T^{*n+p-1-i}\alpha_{l-k-j}(T)T^{n+p-1-i} = 0 \qquad \text{for } i \leq p - 1.$$

For $p \leq i \leq n + p - 1$, we obtain

$$T^{*n+p-1-i}Q^{*i}\alpha_{l-k-j}(T)T^{n+p-1-i}Q^i = 0.$$

Then, $R^{*r}\alpha_l(R)R^r = 0$ as desired.
Similarly, we obtain

$$R^{*r}\alpha_{m+2p-3}(R)R^r$$
$$= (-1)^{m+p-2}\binom{m+2p-3}{p-1}\binom{m+p-2}{p-1}R^{*r}Q^{*(p-1)}\alpha_{m-1}(T)Q^{p-1}R^r.$$

Hence $T+Q$ is an $(n+p-1)-$quasi strict $(m+2p-2)-$symmetry if and only if $(T^*+Q^*)^{n+p-1}Q^{*(p-1)}\alpha_{m-1}(T)Q^{p-1}(T+Q)^{n+p-1}$ is not the zero operator. □

Let $\mathcal{H}\otimes\mathcal{H}$ denote the completion, endowed with a reasonable uniform cross-norm of the algebraic tensor product $\mathcal{H}\otimes\mathcal{H}$ of $\mathcal{H}$ and $\mathcal{H}$. For $T, S \in \mathcal{B}(\mathcal{H})$, $T\otimes S \in \mathcal{B}(\mathcal{H}\otimes\mathcal{H})$ denotes to the tensor product operator defined by T and S.

Lemma 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is an $n-$quasi$-m-$symmetry if and only if $T\otimes I$ and $I\otimes T$ are $n-$quasi$-m-$symmetry.*

Proof. We have

$$\alpha_m(T\otimes I) = \alpha_m(T)\otimes I.$$

Then,

$$(T\otimes I)^{*n}\alpha_m(T\otimes I)(T\otimes I)^n = (T^{*n}\otimes I)(\alpha_m(T)\otimes I)(T^n\otimes I) = T^{*n}\alpha_m(T)T^n\otimes I.$$

Hence T is an $n-$quasi$-m-$symmetry if and only if $T\otimes I$, and the same with $I\otimes T$. □

Proposition 2.2. *Let $T, S \in \mathcal{B}(\mathcal{H})$. If T is an n_1-quasi$-m-$symmetry and S is an n_2-quasi$-l-$symmetry, then $T\otimes S$ is an $n-$quasi$-(m+l-1)-$symmetry where $n = \max\{n_1, n_2\}$.*

Proof. Since T is an n_1-quasi$-m-$symmetry, and S is an n_2-quasi-l-symmet-ry, by Lemma 2.4 we obtain that $T\otimes I$ is an n_1-quasi$-m-$symmetry, and $I\otimes S$ is an n_2-quasi$-l-$symmetry. We observe that

$$T\otimes S = (T\otimes I)(I\otimes S) = (I\otimes S)(T\otimes I).$$

Similarly, we obtain

$$T\otimes S^* = (T\otimes I)(I\otimes S^*) = (T\otimes I)(I\otimes S)^* = (I\otimes S)^*(T\otimes I).$$

Hence $T\otimes I$ and $I\otimes S$ are double commuting, by applying Theorem 2.2 on $T\otimes I$ and $I\otimes S$, we obtain that $T\otimes S$ is an $n-$quasi$-(m+l-1)-$symmetry, where $n = \max\{n_1, n_2\}$. □

Proposition 2.3. *Let $T, Q \in \mathcal{B}(\mathcal{H})$. If T is an $n-$quasi$-m-$symmetry and Q is a nilpotent of order p, then $T\otimes I + I\otimes Q$ is an $(n+p-1)-$quasi$-(m+2p-2)-$symmetry.*

Proof. Since T is an $n-$quasi$-m-$symmetry, then so is $T\otimes I$, as Q is a nilpotent of order p, thus $I\otimes Q$ is a nilpotent of order p as well, we have

$$T\otimes Q = (T\otimes I)(I\otimes Q) = (I\otimes Q)(T\otimes I).$$

Applying Theorem 2.3 on $T\otimes I$ and $I\otimes Q$, we obtain that $T\otimes I + I\otimes Q$ is is an $(n+p-1)-$quasi$-(m+2p-2)-$symmetry. □

The following example shows that if T is an n_1-quasi$-m-$symmetry and S is an n_2-quasi$-l-$symmetry with T and S are double commuting, then $T+S$ is not necessarily $n-$quasi$-(m+l-1)-$symmetry.

Example 2.3. Let $\mathcal{H}=\mathbb{C}^3$, $T=\begin{pmatrix}0&0&0\\1&0&0\\0&2&0\end{pmatrix}$ and $S=I$, T is an $1-$quasi-3-symmetry and S is selfadjoint with T and S are double commuting. By a simple calculation we obtain that $T+I$ is not $n-$quasi-3-symmetry for any $n\in\mathbb{N}$.

Theorem 2.4. *Let $T,S\in\mathcal{B}(\mathcal{H})$. If T is an n_1-quasi$-m-$symmetry and S is an n_2-quasi$-l-$symmetry such that $TS=ST=T^*S=ST^*=0$, then $T+S$ is an $n-$quasi$-q-$symmetry, where $n=\max\{n_1,n_2\}$ and $q=\max\{m,l\}$.*

Proof. Since $TS=ST=0$, then for any strictly positive integer r, we have

$$(T+S)^r=\sum_{i=0}^{r}\binom{r}{i}T^{r-i}S^i=T^r+S^r,$$

$$(T^*+S^*)^r=\sum_{i=0}^{r}\binom{r}{i}T^{*(r-i)}S^{*i}=T^{*r}+S^{*r}.$$

Therefore,

$$\begin{aligned}&\alpha_q(T+S)\\ =&\sum_{k=0}^{q}(-1)^{q-k}\binom{q}{k}(T^{*(q-k)}+S^{*(q-k)})(T^k+S^k)\\ =&(-1)^qT^{*q}+(-1)^qS^{*q}+\sum_{k=1}^{q-1}(-1)^{q-k}\binom{q}{k}(T^{*(q-k)}+S^{*(q-k)})(T^k+S^k)\\ &+T^q+S^q.\end{aligned}$$

Since $T^*S=ST^*=0$, we obtain

$$\begin{aligned}\alpha_q(T+S)=&(-1)^qT^{*q}+(-1)^qS^{*q}+\sum_{k=1}^{q-1}(-1)^{q-k}\binom{q}{k}(T^{*(q-k)}T^k+S^{*(q-k)}S^k)\\ &+T^q+S^q\\ =&\alpha_q(T)+\alpha_q(S).\end{aligned}$$

We observe that

$$(T+S)^{*n}\alpha_q(T+S)(T+S)^n=(T^{*n}+S^{*n})(\alpha_q(T)+\alpha_q(S))(T^n+S^n).$$

We have $TS=ST=T^*S=ST^*=0$, then

$$(T+S)^{*n}\alpha_q(T+S)(T+S)^n=T^{*n}\alpha_q(T)T^n+S^{*n}\alpha_q(S)S^n.$$

Since $n=\max\{n_1,n_2\}$ and $q=\max\{m,l\}$, using the assumption that T is an n_1-quasi$-m-$symmetry and S be an n_2-quasi$-l-$symmetry, we get

$$T^{*n}\alpha_q(T)T^n=S^{*n}\alpha_q(S)S^n=0.$$

Hence $(T+S)^{*n}\alpha_q(T+S)(T+S)^n=0$ as required. $\square$

Proposition 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be an $n-$quasi$-2m-$symmetry. Then T is an $n-$quasi$-(2m-1)-$symmetry.*

Proof. • If $R(T^n)$ is dense, then T is an $2m-$symmetry, by Lemma 2.3, we get that T is an $(2m-1)-$symmetry.
• If $R(T^n)$ is not dense, then $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{R(T^n)} \oplus N(T^{*n})$, where T_1 is an $2m$-symmetric operator and $T_3^n = 0$.
By Lemma 2.3, we obtain that T_1 is an $(2m-1)-$symmetry. Hence we get that T is an $n-$quasi$-(2m-1)-$symmetry. □

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be an $n-$quasi$-m-$symmetry. The following statements hold:*

1. *$\sigma(T) \subset \mathbb{R}$.*
2. *If $\lambda \in \sigma_p(T) \setminus \{0\}$ i.e., there exists $x \in \mathcal{H}$ such that $Tx = \lambda x$, if $x \notin N(T^{*n})$, then $\lambda \in \sigma_p(T^*)$.*
3. *If $\lambda \in \sigma_{ap}(T) \setminus \{0\}$ i.e., there exists a sequence $(x_j) \subset \mathcal{H}$ of unit vectors such that $\lim\limits_{j\to\infty}(T-\lambda)x_j = 0$, if $\lim\limits_{j\to\infty} T^{*n}x_j \neq 0$, then $\lambda \in \sigma_{ap}(T^*)$.*

Proof. (1) Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If λ is an approximate point spectrum of T, then there exists a sequence of unit vectors $(x_j) \subset \mathcal{H}$ such that

$$\lim_{j\to\infty}(T-\lambda)x_j = 0.$$

Then we can easily prove that for any strictly positive integer r

$$\lim_{j\to\infty}(T^r-\lambda^r)x_j = 0.$$

Now, we prove that for any strictly positive integers r and s

$$\lim_{j\to\infty}\langle T^r x_j, T^s x_j\rangle = \lambda^r\overline{\lambda}^s.$$

Observe that

$$\begin{aligned}
|\langle T^r x_j, T^s x_j\rangle - \lambda^r\overline{\lambda}^s| =& |\langle T^r x_j, T^s x_j\rangle - \langle \lambda^r x_j, \lambda^s x_j\rangle| \\
=& |\langle T^r x_j, T^s x_j\rangle - \langle \lambda^r x_j, T^s x_j\rangle + \langle \lambda^r x_j, T^s x_j\rangle \\
& - \langle \lambda^r x_j, \lambda^s x_j\rangle| \\
=& |\langle (T^r-\lambda^r)x_j, T^s x_j\rangle + \langle \lambda^r x_j, (T^s-\lambda^s)x_j\rangle| \\
\leq& |\langle (T^r-\lambda^r)x_j, T^s x_j\rangle| + |\langle \lambda^r x_j, (T^s-\lambda^s)x_j\rangle| \\
\leq& \|(T^r-\lambda^r)x_j\|\|T^s x_j\| + \|\lambda^r x_j\|\|(T^s-\lambda^s)x_j\| \\
\leq& \|T^s\|\|(T^r-\lambda^r)x_j\| + |\lambda|^r\|(T^s-\lambda^s)x_j\|.
\end{aligned}$$

We have

$$\lim_{j\to\infty}(T^r-\lambda^r)x_j = \lim_{j\to\infty}(T^s-\lambda^s)x_j = 0,$$

we deduce that

$$\lim_{j\to\infty}|\langle T^r x_j, T^s x_j\rangle - \lambda^r\overline{\lambda}^s| = 0.$$

Since T is an $n-$quasi$-m-$symmetry, we obtain

$$\begin{aligned}0 &= \sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\langle T^{*(n+m-k)}T^{k+n}x_j, x_j\rangle \\ &= \sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\langle T^{k+n}x_j, T^{n+m-k}x_j\rangle.\end{aligned}$$

Since

$$\lim_{j\to\infty}\langle T^r x_j, T^s x_j\rangle = \lambda^r\overline{\lambda}^s,$$

we obtain

$$\begin{aligned}0 =& \sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\lambda^{n+k}\overline{\lambda}^{n+m-k} \\ =&|\lambda|^{2n}\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\lambda^{k}\overline{\lambda}^{m-k} \\ =&|\lambda|^{2n}(\lambda-\overline{\lambda})^m.\end{aligned}$$

Thus,

$$2^m|\lambda|^{2n}Im(\lambda)^m = 0.$$

We have $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then $\lambda \neq 0$, this implies that $Im(\lambda) = 0$. This contradicts the hypothesis of $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Hence $\lambda \notin \sigma_{ap}(T)$, then $\sigma_{ap}(T) \subset \mathbb{R}$. It follows that $\partial\sigma(T) \subset \sigma_{ap}(T) \subset \mathbb{R}$. Therefore, $\sigma(T) \subset \mathbb{R}$.

(2) Let $\lambda \in \sigma_p(T)\setminus\{0\}$ i.e. there exists $x \in \mathcal{H}$ such that $Tx = \lambda x$, and assume that $x \notin N(T^{*n})$. Since T is an $n-$quasi$-m-$symmetry, we get

$$\begin{aligned}0 =& \sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}T^{*(n+m-k)}T^{k+n}x \\ =& \sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}T^{*(n+m-k)}\lambda^{k+n}x \\ =&\lambda^n T^{*n}\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\lambda^k T^{*(m-k)}x \\ =&(-1)^m\lambda^n T^{*n}(T^*-\lambda)^m x.\end{aligned}$$

Since $\lambda \neq 0$, we obtain

$$T^{*n}(T^*-\lambda)^m x = 0.$$

Let $S \in \mathcal{B}(\mathcal{H})$ such that $N(S) = \{0\}$. Then $N(S^r) = \{0\}$ for any positive integer r. By induction, for $r = 1$, we have $N(S) = \{0\}$.
Assume that $N(S^r) = \{0\}$, we prove that $N(S^{r+1}) = \{0\}$.
Suppose that there exists a nonzero $x \in N(S^{r+1})$. Using $N(S) = N(S^r) = \{0\}$, we get

$$S^{r+1}x = 0 \Longrightarrow S(S^r x) = 0 \Longrightarrow S^r x = 0 \Longrightarrow x = 0.$$

Therefore, $N(S^{r+1}) = \{0\}$ as desired.
If $\lambda \notin \sigma_p(T^*)$, then $N(T^* - \lambda) = \{0\}$, thus $N((T^* - \lambda)^m) = \{0\}$.
Observe that

$$T^{*n}(T^* - \lambda)^m x = (T^* - \lambda)^m T^{*n} x = 0 \Longrightarrow T^{*n} x = 0.$$

This contradicts the assumption of $x \notin N(T^{*n})$. Hence $\lambda \in \sigma_p(T^*)$.

(3) Let $\lambda \in \sigma_{ap}(T) \setminus \{0\}$ i.e., there exists a sequence $(x_j) \subset \mathcal{H}$ of unit vectors such that $\lim\limits_{j\to\infty} (T - \lambda)x_j = 0$. Assume that $\lim\limits_{j\to\infty} T^{*n}x_j \neq 0$.
Since T is an $n-$quasi$-m-$symmetry,

$$\|\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix} T^{*(n+m-k)}T^{n+k}x_j\| = 0.$$

Hence

$$\begin{aligned}
&\|\lambda^n T^{*n}\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\lambda^k T^{*(m-k)}x_j\| \\
&=\|\lambda^n T^{*n}\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\lambda^k T^{*(m-k)}x_j\| \\
&-\|\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix} T^{*(n+m-k)}T^{n+k}x_j\| \\
&\leq\|T^{*n}\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix} T^{*(m-k)}(T^{k+n} - \lambda^{k+n})x_j\| \\
&\leq\sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix}\|T^{*(n+m-k)}\|\|(T^{k+n} - \lambda^{k+n})x_j\|.
\end{aligned}$$

Note that for every $0 \leq k \leq m$,

$$\lim_{j\to\infty}\|(T^{k+n} - \lambda^{k+n})x_j\| = 0.$$

Therefore,

$$\lim_{j\to\infty}\|\lambda^n T^{*n}\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\lambda^k T^{*(m-k)}x_j\| = 0.$$

We have

$$\lambda^n T^{*n}\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\lambda^k T^{*(m-k)}x_j = (-1)^m\lambda^n T^{*n}(T^* - \lambda)^m x_j.$$

Since $\lambda \neq 0$, we get

$$\lim_{j\to\infty}\|T^{*n}(T^* - \lambda)^m x_j\| = 0.$$

If $\lambda \notin \sigma_{ap}(T^*)$, then $T^* - \lambda$ is bounded from below

$$\exists\, c > 0,\ \ \forall x \in \mathcal{H}:\ \ \|(T^* - \lambda)x\| \geq c\|x\|.$$

We observe that for all $x \in \mathcal{H}$

$$\|(T^* - \lambda)^m x\| \geq c^m\|x\|.$$

Thus,

$$\|T^{*n}(T^* - \lambda)^m x_j\| = \|(T^* - \lambda)^m T^{*n} x_j\| \geq c^m \|T^{*n} x_j\|.$$

Since $\lim_{j\to\infty} \|T^{*n}(T^* - \lambda)^m x_j\| = 0$, we deduce that

$$\lim_{j\to\infty} \|T^{*n} x_j\| = 0.$$

This contradicts the assumption of $\lim\limits_{j\to\infty} \|T^{*n} x_j\| \neq 0$.

Hence $\lambda \in \sigma_{ap}(T^*)$.

□

Proposition 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a hyponormal operator (i.e., $T^*T \geq TT^*$). If T is an $n-$quasi$-m-$symmetry, then T is selfadjoint.*

Proof. By Theorem 2.5, we have $\sigma(T) \subset \mathbb{R}$. From [15, Corollary 3], it follows that T is selfadjoint. □

Corollary 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. If T is a strict $n-$quasi$-m-$symmetry, then T is not hyponormal.*

Proposition 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be an $n-$quasi$-m-$symmetry. If there exists a strictly positive integer $r \leq n-1$ such that $N(T^{*r}) = N(T^{*(r+1)})$, then T is an $r-$quasi$-m-$symmetry.*

Proof. Since $N(T^{*r}) = N(T^{*(r+1)})$, we can prove that $N(T^{*r}) = N(T^{*n})$.
We have T is an $n-$quasi$-m-$symmetry, thus

$$T^{*n} \sum_{k=0}^{m} (-1)^{m-k} \begin{pmatrix} m \\ k \end{pmatrix} T^{*(m-k)} T^k T^n = 0$$

$$\Longrightarrow T^{*r} \sum_{k=0}^{m} (-1)^{m-k} \begin{pmatrix} m \\ k \end{pmatrix} T^{*(m-k)} T^k T^n = 0$$

$$\Longrightarrow (T^{*r} \sum_{k=0}^{m} (-1)^{m-k} \begin{pmatrix} m \\ k \end{pmatrix} T^{*(m-k)} T^k T^n)^* = 0$$

$$\Longrightarrow T^{*n} \sum_{k=0}^{m} (-1)^{m-k} \begin{pmatrix} m \\ k \end{pmatrix} T^{*k} T^{m-k} T^r = 0$$

$$\Longrightarrow T^{*r} \sum_{k=0}^{m} (-1)^{m-k} \begin{pmatrix} m \\ k \end{pmatrix} T^{*k} T^{m-k} T^r = 0.$$

Hence T is an $r-$quasi$-m-$symmetry. □

3. $\mathcal{C}_0-$semigroup of $n-$quasi$-m-$symmetry

Let $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(\mathcal{H})$. $\{T(t)\}_{t\geq 0}$ is said to be $\mathcal{C}_0-$semigroup if it satisfies the following:
(1) $T(0) = I$,
(2) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$,
(3) $\lim_{t\to 0^+} T(t)x = x$, for all $x \in \mathcal{H}$ in the strong operator topology.

Lemma 3.1. *Let $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(\mathcal{H})$ be a $\mathcal{C}_0-$semigroup on $\mathcal{H}$. The following statements are equivalent:*
(i) *$T(t)$ is an $n-$quasi$-m-$symmetry for all $t \geq 0$.*
(ii) *$T(t)$ is an $n-$quasi$-m-$symmetry for all $t \in]0, t^{'}]$ with $t^{'} > 0$.*

Proof. $(i) \Longrightarrow (ii)$ is obvious.
$(ii) \Longrightarrow (i)$ For any $t \geq t^{'}$, there exists $r = [\frac{t}{t'}] + 1$ and $s \in]0, t^{'}]$ such that $t = rs$, where $[\frac{t}{t'}]$ denotes the greatest integer $\leq \frac{t}{t'}$. Since $s \in]0, t^{'}]$, $T(s)$ is an $n-$quasi$-m-$symmetry, thus $T^r(s)$ is an $n-$quasi$-m-$symmetry as well. Observe that $T^r(s) = T(rs) = T(t)$. Hence $T(t)$ is an $n-$quasi$-m-$symmetry. □

Theorem 3.1. *Let $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(\mathcal{H})$ be a $\mathcal{C}_0-$semigroup on $\mathcal{H}$. Then $T(t)$ is an n-quasi-m-symmetry for all $t \geq 0$ if and only if $T(t)$ is an $m-$symmetry for all $t \geq 0$.*

Proof. • If $T(t)$ is an $m-$symmetry for all $t \geq 0$, then $T(t)$ is clearly an n-quasi-m-symmetry for all $t \geq 0$.
• Suppose that $T(t)$ is an $n-$quasi$-m-$symmetry for all $t \geq 0$, then by Lemma 3.1 for all $t^{'} > 0$, we have $T(t)$ is an $n-$quasi$-m-$symmetry for all $t \in]0, t^{'}]$. Observe that for all $x \in \mathcal{H}$

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t) T^k(t) T^n(t)x, T^n(t)x \rangle = 0.$$

Hence for all $x \in \overline{R(T^n(t))}$, we have

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t) T^k(t)x, x \rangle = 0.$$

Therefore, for all $x \in \bigcap_{t\in]0,t']} \overline{R(T^n(t))}$, we have

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t) T^k(t)x, x \rangle = 0.$$

However, we can easily prove that $\bigcap_{t\in]0,t']} \overline{R(T^n(t))} = \overline{R(T^n(t^{'}))}$. Then for all $t \in]0, t^{'}]$ and for all $x \in \overline{R(T^n(t^{'}))}$, we have

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t) T^k(t)x, x \rangle = 0.$$

That means for all $t \in]0, t^{'}]$, $T(t)$ is an $m-$symmetry on $\overline{R(T^n(t^{'}))}$. Using Lemma 3.1 again, we obtain that $T(t)$ is an $m-$symmetry on $\overline{R(T^n(t^{'}))}$ for all $t > 0$ i.e.

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t) T^k(t)x, x \rangle = 0, \ \forall x \in \overline{R(T^n(t^{'}))}, \ \forall t > 0.$$

We observe for all $t > 0$ and $x \in \mathcal{H}$

$$\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\langle T^{*(n-k)}(t)T^k(t)T^n(t^{'})x, T^n(t^{'})x\rangle = 0.$$

Now, for all $x \in \mathcal{H}$ we have

$$\begin{aligned}
&|\langle T^{*(n-k)}(t)T^k(t)T^n(t^{'})x, T^n(t^{'})x\rangle - \langle T^{*(n-k)}(t)T^k(t)x, x\rangle| \\
&=|\langle T^{*(n-k)}(t)T^k(t)T^n(t^{'})x, T^n(t^{'})x\rangle - \langle T^{*(n-k)}(t)T^k(t)x, T^n(t^{'})x\rangle \\
&+\langle T^{*(n-k)}(t)T^k(t)x, T^n(t^{'})x\rangle - \langle T^{*(n-k)}(t)T^k(t)x, x\rangle| \\
&=|\langle T^{*(n-k)}(t)T^k(t)T^n(t^{'})x - T^{*(n-k)}(t)T^k(t)x, T^n(t^{'})x\rangle \\
&+\langle T^{*(n-k)}(t)T^k(t)x, T^n(t^{'})x - x\rangle| \\
&\leq\|T^{*(n-k)}(t)T^k(t)T^n(t^{'})x - T^{*(n-k)}(t)T^k(t)x\|\|T^n(t^{'})x\| \\
&+\|T^{*(n-k)}(t)T^k(t)x\|\|T^n(t^{'})x - x\| \\
&\leq(\|T^{*(n-k)}(t)T^k\|\|T^n(t^{'})x\| + \|T^{*(n-k)}(t)T^k(t)x\|)\|T^n(t^{'})x - x\|.
\end{aligned}$$

Since $\lim_{s\to 0^+} T(s)x = x$, we have

$$\lim_{t^{'}\to 0^+}\|T^n(t^{'})x - x\| = \lim_{t^{'}\to 0^+}\|T(nt^{'})x - x\| = 0.$$

Therefore,

$$\lim_{t^{'}\to 0^+}|\langle T^{*(n-k)}(t)T^k(t)T^n(t^{'})x, T^n(t^{'})x\rangle - \langle T^{*(n-k)}(t)T^k(t)x, x\rangle| = 0.$$

Thus, for all $t > 0$ and $x \in \mathcal{H}$

$$\begin{aligned}
&\lim_{t^{'}\to 0^+}\sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\langle T^{*(n-k)}(t)T^k(t)T^n(t^{'})x, T^n(t^{'})x\rangle = 0, \\
&\Longrightarrow \sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}\langle T^{*(n-k)}(t)T^k(t)x, x\rangle = 0, \\
&\Longrightarrow \sum_{k=0}^{m}(-1)^{m-k}\begin{pmatrix} m \\ k \end{pmatrix}T^{*(n-k)}(t)T^k(t)x = 0.
\end{aligned}$$

Hence $T(t)$ is an $m-$symmetry for all $t \geq 0$. □

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