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BOUNDEDNESS OF DISCRETE HILBERT TRANSFORM ON ORLICZ SEQUENCE SPACES

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Abstract. The Hilbert transform has been well studied on classical function spaces, such as Lebesgue, Morrey, Orlicz spaces. But its discrete version, which also has numerous applications, has not been fully studied in discrete analogues of these spaces. In this paper we study the discrete Hilbert transform on Orlicz sequence spaces. In particular, we obtain its boundedness on the Orlicz sequence spaces.

1. Introduction

The Hilbert transform plays an important role in the theory and practice of signal processing operations in continuous system theory because of its relevance to problems such as envelope detection and demodulation, as well as its use in relating the real and imaginary components, and the magnitude and phase components of spectra. The Hilbert transform was the motivation for the development of modern harmonic analysis. Its discrete version is also widely used in many areas of science and technology and plays an important role in digital signal processing. The essential motivation behind studying discrete transforms is that experimental data are most frequently not taken in a continuous manner but sampled at discrete time values. Since much of the data collected in both physical sciences and engineering are discrete, the discrete Hilbert transform is a rather useful tool in these areas for the general analysis of this type of data.

Denote by l_p, $p \geq 1$, the class of sequences of complex numbers $b = \{b_n\}_{n\in\mathbb{Z}}$ satisfying the condition

$$\|b\|_{l_p} = \left(\sum_{n\in\mathbb{Z}} |b_n|^p\right)^{1/p} < \infty,$$

where $\mathbb{Z}$ is the set of integers.

Let $b = \{b_n\}_{n\in\mathbb{Z}} \in l_p$, $p \geq 1$. The sequence $\tilde{b} = \{\tilde{b}_n\}_{n\in\mathbb{Z}}$ is called discrete Hilbert transform of the sequence $b = \{b_n\}_{n\in\mathbb{Z}}$, where

$$\tilde{b}_n = \sum_{m\neq n} \frac{b_m}{n-m}, \quad n \in \mathbb{Z}.$$

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M. Riesz proved (see [30]) that if $b \in l_p$, $p > 1$, then $\tilde{b} \in l_p$ and the inequality

$$\|\tilde{b}\|_{l_p} \leq c_p \|b\|_{l_p} \tag{1.1}$$

holds, where c_p is a constant, depending only p. Weighted analogues of (1.1) are investigated in the works [5, 7, 8, 13, 17, 26, 28, 32].

If $b \in l_1$ then the sequence $\tilde{b}$ belongs to the class $\bigcap_{p>1} l_p$, but generally it does not belong to the class l_1 (see [3]). In this case, R.Hunt, B.Muckenhoupt and R.Wheeden (see [17]) proved that the distribution function $\tilde{b}(\lambda) := \sum_{\{n\in\mathbb{Z}:|\tilde{b}_n|>\lambda\}} 1$ of $\tilde{b}$ satisfies the weak condition

$$\forall\lambda > 0 \qquad |\tilde{b}(\lambda)| \leq \frac{c_0}{\lambda}\|b\|_{l_1},$$

where c_0 is an absolute constant. In [3], it was proved that, if the sequence $b \in l_1$ satisfies the conditions $\sum_{n\in\mathbb{Z}} b_n = 0$ (this condition is necessary for the summability of the discrete Hilbert transform) and $\sum_{n\in\mathbb{Z}} |b_n| \ln(e + |n|) < \infty$, then $\tilde{b} \in l_1$ and the following inequality holds:

$$\|\tilde{b}\|_{l_1} \leq 6 \sum_{n\in\mathbb{Z}} |b_n| \ln(e + |n|).$$

In [2] there was introduced the concept of Q-summability of series and using this notion it was proved that the Hilbert transform of a sequence $b \in l_1$ is Q-summable and its Q-sum is equal to zero. In [1, 4, 15, 16, 18] discrete analogues of harmonic analysis operators on discrete Morrey spaces were studied.

In this paper we study the discrete Hilbert transform on Orlicz sequence spaces. In particular, we obtain its boundedness on the Orlicz sequence spaces using the boundedness of the Hilbert transform on Orlicz spaces.

2. Orlicz sequence spaces

Definition 2.1. A function $\Phi : [0, +\infty) \to [0, +\infty]$ is called a Young function, if Φ is convex, left-continuous, $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$, and $\lim_{r\to+\infty} \Phi(r) = +\infty$.

It follows from the definition that any Young function is increasing and satisfies the properties

$$\Phi\left(\sum_{k\in\mathbb{Z}} \alpha_k t_k\right) \leq \sum_{k\in\mathbb{Z}} \alpha_k \Phi(t_k) \text{ for } \sum_{k\in\mathbb{Z}} \alpha_k = 1,\ \alpha_k \geq 0,\quad t_k \geq 0,\ k \in \mathbb{Z}. \tag{2.1}$$

Denote by $\mathcal{Y}$ the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \text{ for } 0 < r < +\infty.$$

Every function $\Phi \in \mathcal{Y}$ is absolutely continuous on every closed interval in $[0, +\infty)$, and bijective from $[0, +\infty)$ to itself.

Definition 2.2. For a Young function Φ, the set

$$L_\Phi(\mathbb{R}) = \left\{ f \in L_1^{loc}(\mathbb{R}) : \int_{\mathbb{R}} \Phi(k|f(x)|)dx < +\infty \text{ for some } \quad k > 0 \right\}$$

is called Orlicz space.

Note that if $\Phi(r) = r^p$, $1 \le p < \infty$, then $L_\Phi(\mathbb{R}) = L_p(\mathbb{R})$; if $\Phi(r) = 0$ $(0 \le r \le 1)$ and $\Phi(r) = \infty$ $(r > 1)$, then $L_\Phi(\mathbb{R}) = L_\infty(\mathbb{R})$. We refer to [21, 22, 29] for the theory of Orlicz spaces.

$L_\Phi(\mathbb{R})$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

It follows from the Fatou theorem that

$$\int_{\mathbb{R}} \Phi\left(\frac{|f(x)|}{\|f\|_{L_\Phi}}\right) dx \le 1.$$

For a measurable function f, and $t > 0$, let

$$m(f,t) = |\{x \in \mathbb{R} : \ |f(x)| > t\}|.$$

Definition 2.3. For a Young function Φ, the weak Orlicz space

$$WL_\Phi(\mathbb{R}) = \left\{f \in L_1^{loc}(\mathbb{R}) : \|f\|_{WL_\Phi} < +\infty\right\}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf\left\{\lambda > 0 : \sup_{t>0} \Phi(t) m\left(\frac{f}{\lambda}, t\right) \le 1\right\}.$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r) \quad \text{for} \quad r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr) \quad \text{for} \quad r > 0$$

for some $k > 1$. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both conditions Δ_2 and ∇_2.

For a Young function Φ, the complementary function $\tilde{\Phi}(r)$ is defined by

$$\tilde{\Phi}(r) = \sup\{rs - \Phi(s) : s \in [0,\infty)\}, \quad r \in [0,\infty).$$

The complementary function $\tilde{\Phi}$ is also a Young function and $\tilde{\tilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\tilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\tilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\tilde{\Phi}(r) = r^{p'}/p'$. Note that $\Phi \in \nabla_2$ if and only if $\tilde{\Phi} \in \Delta_2$. It is well known that

$$r \le \Phi^{-1}(r)\tilde{\Phi}^{-1}(r) \le 2r \quad \text{for} \quad r \ge 0.$$

The following analogue of the Hölder inequality is well known.

Theorem 2.1. *For a Young function Φ and its complementary function $\tilde{\Phi}$, the following inequality holds:*

$$\|fg\|_{L_1(\mathbb{R})} \le 2\|f\|_{L_\Phi}\|g\|_{L_{\tilde{\Phi}}}$$

Definition 2.4. For a Young function Φ, the set of all sequences of scalars $b = \{b_n\}_{n\in\mathbb{Z}}$ such that

$$\sum_{n\in\mathbb{Z}} \Phi(k|b_n|) < +\infty \quad \text{for some} \quad k > 0$$

is called Orlicz sequence space and denoted by l_Φ.

The space l_Φ with the norm

$$\|b\|_{l_\Phi} = \inf\left\{\lambda > 0 : \sum_{n\in\mathbb{Z}} \Phi\left(\frac{|b_n|}{\lambda}\right) dx \le 1\right\}.$$

becomes a Banach space.

Definition 2.5. For a Young function Φ, weak Orlicz sequence space

$$Wl_\Phi = \{b = \{b_n\}_{n\in\mathbb{Z}} : \|b\|_{Wl_\Phi} < +\infty\}$$

is defined by the norm

$$\|b\|_{Wl_\Phi} = \inf\left\{\lambda > 0 : \sup_{t>0} \Phi(t) b(\lambda t) \le 1\right\},$$

where $b(\lambda) := \sum_{\{n\in\mathbb{Z}:\ |b_n|>\lambda\}} 1$ is the distribution function of the sequence $b = \{b_n\}_{n\in\mathbb{Z}}$.

The properties of Orlicz sequence spaces are investigated in the works [6, 10, 11, 12, 19, 20, 23, 24, 25, 27, 31].

3. Boundedness of the discrete Hilbert transform on Orlicz sequence spaces

Necessary and sufficient conditions for the boundedness of singular integral operators in Orlicz spaces were obtained in [9]. To formulate the results from [9], we recall that, for functions Φ and Ψ from $[0,\infty)$ into $[0,\infty]$, the function Ψ is said to dominate Φ globally if there exists a positive constant C such that $\Phi(s) \le \Psi(Cs)$ for all $s > 0$.

Theorem 3.1. [9]. *Let T be any singular integral operator having the form*

$$(Tf)(x) = \lim_{\varepsilon\to 0+} \int_{(|y|\ge\varepsilon)} \frac{g(y)}{|y|^n} \cdot f(x-y)dy, \quad x \in \mathbb{R}^n, \tag{3.1}$$

where g is a non-identically zero odd function on $\mathbb{R}^n$, homogeneous of degree 0, satisfying the "Dini-type" condition $\int_0 \frac{\omega(\delta)}{\delta} d\delta < \infty$ [1]

on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$, $\omega(\delta) = \sup\{|g(x)-g(y)| : x, y \in \mathbb{S}^{n-1}, |x-y| \le \delta\}$. Let Φ and Ψ be Young functions. Then

(i) T is of weak type (Φ,Ψ) if and only if $\int_0 \tilde{\Phi}(t)/t^2 dt < \infty$ and $\tilde{\Psi}(s)$ dominates the Young function $s\int_0^s \tilde{\Phi}(t)/t^2 dt$ globally;

(ii) T is of strong type (Φ,Ψ) if and only if $\int_0 \Psi(t)/t^2 dt < \infty$, $\int_0 \tilde{\Phi}(t)/t^2 dt < \infty$, $\Phi(s)$ dominates the Young function $s\int_0^s \Psi(t)/t^2 dt$ globally and $\tilde{\Psi}(s)$ dominates the Young function $s\int_0^s \tilde{\Phi}(t)/t^2 dt$ globally.

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[1]Here and below, $\int_0 f(t)dt < \infty$ means the existence of $\eta > 0$ such that $\int_0^\eta f(t)dt$ converges.

Observe that the Hilbert transform

$$(Hf)(t) = \frac{1}{\pi} v.p. \int_{\mathbb{R}} \frac{f(t)}{x-t} dt := \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{\{\tau \in \mathbb{R}:\ |\tau - t| > \varepsilon\}} \frac{f(\tau)}{t-\tau} d\tau, \quad t \in R$$

is of type (3.1).

Theorem 3.2. *Let Φ and Ψ be Young functions.*

(i) If $\int_0 \frac{\tilde{\Phi}(t)}{t^2} dt < \infty$ and $\tilde{\Psi}(s)$ dominates the Young function $s \int_0^s \frac{\tilde{\Phi}(t)}{t^2} dt$ globally, then the discrete Hilbert transform is bounded from l_Φ to Wl_Ψ, that is for any $b \in l_\Phi$ we have $\tilde{b} \in Wl_\Psi$, and there exists a positive constant C_1 such that

$$\|\tilde{b}\|_{Wl_\Psi} \leq C_1 \cdot \|b\|_{l_\Phi}$$

for all $b \in l_\Phi$.

(ii) If $\int_0 \frac{\Psi(t)}{t^2} dt < \infty$, $\int_0 \frac{\tilde{\Phi}(t)}{t^2} dt < \infty$, $\Phi(s)$ dominates the Young function $s \int_0^s \frac{\Psi(t)}{t^2} dt$ globally and $\tilde{\Psi}(s)$ dominates the Young function $s \int_0^s \frac{\tilde{\Phi}(t)}{t^2} dt$ globally, then the discrete Hilbert transform is bounded from l_Φ to l_Ψ, that is for any $b \in l_\Phi$ we have $\tilde{b} \in l_\Psi$, and there exists a positive constant C_2 such that

$$\|\tilde{b}\|_{l_\Psi} \leq C_2 \cdot \|b\|_{l_\Phi}$$

for all $b \in l_\Phi$.

Proof. (ii). At first we note that if $\tilde{\Psi}(s)$ dominates the Young function $s \int_0^s \tilde{\Phi}(t)/t^2 dt$ globally, then it follows from the inequality

$$s \int_0^s \frac{\tilde{\Phi}(t)}{t^2} dt \geq s \int_{s/2}^s \frac{\tilde{\Phi}(t)}{t^2} dt \geq s \tilde{\Phi}\left(\frac{s}{2}\right) \int_{s/2}^s \frac{dt}{t^2} = \tilde{\Phi}\left(\frac{s}{2}\right)$$

that $\tilde{\Psi}$ dominates the Young function $\tilde{\Phi}$ globally and, therefore, Φ dominates the Yung function Ψ globally. Hence, for any $b \in l_\Phi$ we have $b \in l_\Psi$, and there exists a positive constant C_3 such that

$$\|b\|_{l_\Psi} \leq C_3 \cdot \|b\|_{l_\Phi}$$

holds for all $b \in l_\Phi$.

Let $b \in l_\Phi$. We define the function $f(x)$ to be $\pi[(n+1-x)b_n + (x-n)b_{n+1}]$ for $x \in [n, n+1)$, $n \in \mathbb{Z}$. We first show that $f \in L_\Phi$. Indeed, for any $k > 0$ it follows from the inequality

$$\int_{\mathbb{R}} \Phi(k|f(x)|) dx = \sum_{n \in \mathbb{Z}} \int_n^{n+1} \Phi(\pi k |(n+1-x)b_n + (x-n)b_{n+1}|) dx$$

$$\leq \sum_{n \in \mathbb{Z}} \int_n^{n+1} \Phi((n+1-x)\pi k |b_n| + (x-n)\pi k |b_{n+1}|) dx$$

$$\leq \sum_{n \in \mathbb{Z}} \int_n^{n+1} ((n+1-x)\Phi(\pi k |b_n|) + (x-n)\Phi(\pi k |b_{n+1}|)) dx = \sum_{n \in \mathbb{Z}} \Phi(\pi k |b_n|)$$

that $f \in L_\Phi$ and

$$\|f\|_{L_\Phi} \leq C_4 \cdot \|b\|_{l_\Phi},$$

where $C_4 > 0$ is a constant depending only on Φ.

Then it follows from Theorem 3.1 that $Hf \in L_\Psi$ and there exists $C_5 > 0$ such that

$$\|Hf\|_{L_\Psi} \le C_5 \|b\|_{l_\Phi}. \tag{3.2}$$

We define the function $F(x)$ to be $\tilde{b}_n$ for $x \in [n, n+1)$, $n \in \mathbb{Z}$ and

$$G(x) = (Hf)(x) - F(x). \tag{3.3}$$

We prove that $G(x) \in L_\Psi$. For every $x \in (n, n+1)$, $n \in \mathbb{Z}$ we have

$$\begin{aligned}
G(x) &= \frac{1}{\pi} v.p. \int_{\mathbb{R}} \frac{f(t)}{x-t} dt - \tilde{b}_n \\
&= \sum_{m\in\mathbb{Z}} \int_m^{m+1} \frac{(m+1-t)b_m + (t-m)b_{m+1}}{x-t} dt - \sum_{m\neq n} \frac{b_m}{n-m} \\
&= \sum_{m\in\mathbb{Z}/\{n-1,n,n+1,n+2\}} b_m \left[\int_{m-1}^{m} \frac{t-(m-1)}{x-t} dt + \int_m^{m+1} \frac{m+1-t}{x-t} dt - \frac{1}{n-m} \right] \\
&\quad + b_{n-1} \left[\int_{n-2}^{n-1} \frac{t-(n-2)}{x-t} dt + \int_{n-1}^{n} \frac{n-t}{x-t} dt - 1 \right] \\
&\quad + b_n \left[\int_{n-1}^{n} \frac{t-(n-1)}{x-t} dt + v.p. \int_n^{n+1} \frac{n+1-t}{x-t} dt \right] \\
&\quad + b_{n+1} \left[v.p. \int_n^{n+1} \frac{t-n}{x-t} dt + \int_{n+1}^{n+2} \frac{n+2-t}{x-t} dt + 1 \right] \\
&\quad + b_{n+2} \left[\int_{n+1}^{n+2} \frac{t-(n+1)}{x-t} dt + \int_{n+2}^{n+3} \frac{n+3-t}{x-t} dt + \frac{1}{2} \right] \\
&= G_1(x) + G_2(x) + G_3(x) + G_4(x) + G_5(x).
\end{aligned} \tag{3.4}$$

For any $m \in \mathbb{Z}/\{n-1, n, n+1, n+2\}$ it follows from

$$\begin{aligned}
&\int_{m-1}^{m} \frac{t-(m-1)}{x-t} dt + \int_m^{m+1} \frac{m+1-t}{x-t} dt \\
&\le \int_{m-1}^{m} \frac{t-(m-1)}{n-m} dt + \int_m^{m+1} \frac{m+1-t}{n-(m+1)} dt = \frac{1}{n-m} + \frac{1}{2} \frac{1}{(n-m)(n-m-1)},
\end{aligned}$$

$$\begin{aligned}
&\int_{m-1}^{m} \frac{t-(m-1)}{x-t} dt + \int_m^{m+1} \frac{m+1-t}{x-t} dt \\
&\ge \int_{m-1}^{m} \frac{t-(m-1)}{n-m+2} dt + \int_m^{m+1} \frac{m+1-t}{n-m+1} dt \\
&= \frac{1}{n-m} - \frac{1}{2} \frac{1}{(n-m)(n-m+1)} - \frac{1}{(n-m)(n-m+2)}
\end{aligned}$$

that

$$\left| \int_{m-1}^{m} \frac{t-(m-1)}{x-t} dt + \int_m^{m+1} \frac{m+1-t}{x-t} dt - \frac{1}{n-m} \right| \le \frac{6}{|n-m|^2}.$$

Therefore, for any $x \in (n, n+1)$, $n \in \mathbb{Z}$ we have

$$|G_1(x)| \le \sum_{m\neq n} \frac{6|b_m|}{|n-m|^2}. \tag{3.5}$$

For any $k > 0$ it follows from (3.5) and (2.1) that

$$\int_{\mathbb{R}} \Psi\left(\frac{k}{2\pi^2}|G_1(x)|\right) dx \le \sum_{n\in\mathbb{Z}} \int_n^{n+1} \Psi\left(\sum_{m\neq n} \frac{3}{\pi^2|n-m|^2} \cdot k|b_m|\right) dx$$
$$= \sum_{n\in\mathbb{Z}} \Psi\left(\sum_{m\neq n} \frac{3}{\pi^2|n-m|^2} \cdot k|b_m|\right) \le \sum_{n\in\mathbb{Z}} \sum_{m\neq n} \frac{3}{\pi^2|n-m|^2} \cdot \Psi(k|b_m|)$$
$$= \sum_{m\in\mathbb{Z}} \sum_{n\neq m} \frac{3}{\pi^2|n-m|^2} \cdot \Psi(k|b_m|) = \sum_{m\in\mathbb{Z}} \Psi(k|b_m|). \tag{3.6}$$

Inequality (3.6) shows that $G_1 \in L_\Psi$ and there exists $C_6 > 0$ such that

$$\|G_1\|_{L_\Psi} \le C_6 \|b\|_{l_\Psi} \le C_7 \|b\|_{l_\Phi}, \tag{3.7}$$

where $C_7 = C_3 \cdot C_6$.

Let us show that $G_i \in L_\Psi$ for $i = 2, 3, 4, 5$. For any $x \in (n, n+1)$, $n \in \mathbb{Z}$ we have

$$|G_2(x)| = |b_{n-1}| \cdot \left| \int_{n-2}^{n-1} \frac{t-(n-2)}{x-t} dt + \int_{n-1}^{n} \frac{n-x}{x-t} dt \right|$$
$$\le |b_{n-1}| \cdot \left[\left| \int_{n-2}^{n-1} (t-(n-2)) dt \right| + (x-n) \left| \ln \frac{x-n}{x-(n-1)} \right| \right]$$
$$= |b_{n-1}| \cdot \left[\frac{1}{2} + (x-n) \ln\left(1 + \frac{1}{x-n}\right) \right] \le \frac{3}{2} |b_{n-1}|; \tag{3.8}$$

$$|G_3(x)| = |b_n| \cdot \left| \int_{n-1}^{n} \frac{t-(n-1)}{x-t} dt + v.p. \int_n^{n+1} \frac{n+1-t}{x-t} dt \right|$$
$$= |b_n| \cdot \left| \int_{n-1}^{n} \frac{x-(n-1)}{x-t} dt + v.p. \int_n^{n+1} \frac{n+1-x}{x-t} dt \right|$$
$$= |b_n| \cdot |(x-(n-1)) \ln(x-(n-1)) - 2(x-n) \ln(x-n) - (n+1-x) \ln(n+1-x)|$$
$$\le 5|b_n|; \tag{3.9}$$

$$|G_4(x)| = |b_{n+1}| \cdot \left| v.p. \int_n^{n+1} \frac{t-n}{x-t} dt + \int_{n+1}^{n+2} \frac{n+2-t}{x-t} dt + 1 \right|$$
$$= |b_{n+1}| \cdot \left| v.p. \int_n^{n+1} \frac{x-n}{x-t} dt + \int_{n+1}^{n+2} \frac{n+2-x}{x-t} dt + 1 \right|$$
$$= |b_{n+1}| \cdot |(x-n) \ln(x-n) + 2(n+1-x) \ln(n+1-x) - (n+2-x) \ln(n+2-x) + 1|$$
$$\le 6|b_{n+1}|; \tag{3.10}$$

$$|G_5(x)| = |b_{n+2}| \cdot \left| \int_{n+1}^{n+2} \frac{t-(n+1)}{x-t} dt + \int_{n+2}^{n+3} \frac{n+3-t}{x-t} dt + \frac{1}{2} \right|$$
$$\le |b_{n+2}| \cdot \left[\left| \int_{n+1}^{n+2} \frac{x-(n+1)}{x-t} dt \right| + \frac{3}{2} + \left| \int_{n+2}^{n+3} (n+3-t) dt \right| \right]$$
$$= |b_{n+2}| \cdot \left[2 + (n+1-x) \ln\left(1 + \frac{1}{n+1-x}\right) \right] \le 3|b_{n+2}|. \tag{3.11}$$

It follows from (3.8), (3.9), (3.10) and (3.11) that $G_i \in L_\Psi$, $i = 2, 3, 4, 5$ and there exists $C_8 > 0$ such that

$$\|G_2\|_{L_\Psi} + \|G_3\|_{L_\Psi} + \|G_4\|_{L_\Psi} + \|G_5\|_{L_\Psi} \le C_8 \|b\|_{l_\Phi}. \tag{3.12}$$

Hence, owing to (3.4), (3.7) and (3.12), we conclude that $G \in L_\Psi$:

$$\|G\|_{L_\Psi} \le (C_7 + C_8)\|b\|_{l_\Phi}. \tag{3.13}$$

Since $F(x) = (Hf)(x) - G(x)$, by (3.2) and (3.13) we get that $F \in L_\Psi$:

$$\|F\|_{L_\Psi} \le (C_5 + C_7 + C_8)\|b\|_{l_\Phi}.$$

Therefore it follows from

$$\sum_{n\in\mathbb{Z}} \Psi\left(\frac{|\tilde{b}_n|}{\lambda}\right) = \sum_{n\in\mathbb{Z}} \int_n^{n+1} \Psi\left(\frac{|F(x)|}{\lambda}\right) dx = \int_{\mathbb{R}} \Psi\left(\frac{|F(x)|}{\lambda}\right) dx$$

that $\tilde{b} \in l_\Psi$ and

$$\|\tilde{b}\|_{l_\Psi} \le (C_5 + C_7 + C_8)\|b\|_{l_\Phi}.$$

This completes the proof of part (ii). The proof of part (i) is similar to the proof of part (ii).

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