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TRANSFORMATION OPERATORS FOR THE PERTURBED MODIFIED BESSEL EQUATION

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Abstract. The perturbed modified Bessel equation on the entire axis is considered. By means of transformation operator an integral representation of the Jost-type solution is found. An estimate is obtained with respect to the kernel of the transformation operator. A connection is established between the kernel of the integral representation and the perturbation potential.

1. Introduction

Consider the equation

$$x^2u'' + xu' - x^2u + q(x)\,u = \nu^2 u,\ x > 0, \tag{1.1}$$

where the coefficient $q(x)$ satisfies the conditions

$$q(x) \in C^1(0,\infty),\ \int_0^1 x^{-1}\left|q(x)\ln x\right| dx + \int_1^\infty x\left|q(x)\right| dx < \infty. \tag{1.2}$$

For $q(x) = 0$ we get the equation

$$x^2u'' + xu' - x^2u = \nu^2 u,\ x > 0. \tag{1.3}$$

which is called the modified Bessel equation. It is well known (see [1], [8]) that this equation has two linearly independent solutions $\varphi(x,\nu) = I_\nu(x)$ and $\psi(x,\nu) = K_\nu(x)$, where $I_\nu(x)$ and $K_\nu(x)$ are modified functions of the first and second kind, respectively(e.g., see [1]). We will be interested in solutions $\Phi(x,\nu)$ and $\Psi(x,\nu)$, of the perturbed equation (1.1) with asymptotics

$$\Psi(x,\nu) = \psi(x,\nu)\left[1 + o(1)\right],\ x \to +\infty, \tag{1.4}$$

$$\Phi(x,\nu) = \varphi(x,\nu)\left[1 + o(1)\right],\ x \to 0. \tag{1.5}$$

In this paper, the existence of solutions $\Phi(x,\nu)$ and $\Psi(x,\nu)$ is proved and by means of transformation operators their triangular representations are found. The obtained results can be useful in studying direct and inverse scattering problems for equation (1.1). Note that for various Sturm-Liouville equations, transformation operators were constructed in the works [6], [7]. Transformation operators for differential equations with Bessel operators are studied in detail in

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the works [8], [11]. Inverse spectral problems for the self-adjoint Bessel operator were studied in papers [2], [5], [9], [10].

2. Special solutions of the perturbed Bessel equation

To study the perturbed modified Bessel equation, we use the established connection in work [3],[4]. Consider the equation (1.1). If substitute variables $x = e^t$, $y(t) = u\left(e^t\right)$, $\nu = i\lambda$, equation (1.1) takes the form

$$-y'' + e^{2t}y + p(t)\,y = \lambda^2 y, \tag{2.1}$$

where

$$p(t) = q\left(e^t\right). \tag{2.2}$$

From relations (1.2) and (2.2) it follows that $p(t)$ is a continuously differentiable function on the whole axis and satisfies the conditions

$$\int_0^\infty e^{2t}\left|p(t)\right| dt < \infty, \tag{2.3}$$

$$\int_{-\infty}^0 \left|tp(t)\right| dt < \infty. \tag{2.4}$$

Let

$$z = \sqrt{2\left(e^{2\xi} - e^{2\xi_0}\right)\left(ch2\eta_0 - ch2\eta\right)},\ \xi_0 < \xi < \infty,\ 0 < \eta < \eta_0. \tag{2.5}$$

$$R(\xi, \eta, \xi_0, \eta_0) = J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}, \tag{2.6}$$

where $J_0(z)$ is a Bessel function of the first kind, It is directly verified that the function $R(\xi, \eta, \xi_0, \eta_0)$ satisfies the equation

$$\frac{\partial^2 R}{\partial\xi\,\partial\eta} - 2e^{2\xi}\,sh2\eta\,R = 0 \tag{2.7}$$

and

$$R(\xi_0, \eta; \xi_0, \eta_0) = R(\xi, \eta_0; \xi_0, \eta_0) = 1. \tag{2.8}$$

In other words, $R(\xi, \eta, \xi_0, \eta_0)$ is the Riemann function of the equation (2.7) and has the symmetric property

$$R(\xi, \eta; \xi_0, \eta_0) = R(\xi_0, \eta_0; \xi, \eta). \tag{2.9}$$

For all $\xi_0 \le \xi < \infty,\ 0 \le \eta \le \eta_0$ the following inequality holds:

$$|R| \le 1. \tag{2.10}$$

In addition, for each $\eta \in (0,\ \eta_0)$, the following relations hold:

$$\begin{gathered}\frac{\partial R}{\partial\xi} = O\left(e^{2\xi}\right),\ \frac{\partial R}{\partial\eta} = O\left(e^{2\xi}\right),\ \frac{\partial^2 R}{\partial\xi\partial\eta} = O\left(e^{2\xi}\right), \xi \to \infty,\\ \frac{\partial^2 R}{\partial\xi^2} = O\left(e^{2\xi}\right),\ \frac{\partial^2 R}{\partial\eta^2} = O\left(e^{2\xi}\right),\ \xi \to \infty.\end{gathered} \tag{2.11}$$

We shall use the following notation

$$\sigma(x) = \int_x^\infty \left|p(t)\right| dt,\ \sigma_1(x) = \int_x^\infty \sigma(t)\,dt.$$

Theorem 2.1. *If $p(t)$ satisfies condition (2.3), then the integral equation*

$$U(\xi_0,\eta_0)=\frac{1}{2}\int_{\xi_0}^{\infty}R(\xi,0,\xi_0,\eta_0)\,p(\xi)\,d\xi-$$

$$-\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}U(\xi,\eta)\,R(\xi,\eta,\xi_0,\eta_0)\,p(\xi-\eta)\,d\eta. \tag{2.12}$$

has one and only one solution $U(\xi_0,\eta_0)$. Furthermore,

$$|U(\xi_0,\eta_0)|\le\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}. \tag{2.13}$$

Proof. We will solve this integral equation by the method of successive approximations. Let us put

$$U_0(\xi_0,\eta_0)=\frac{1}{2}\int_{\xi_0}^{\infty}R(\xi,0;\xi_0,\eta_0)p(\xi)d\xi,$$

$$U_n(\xi_0,\eta_0)=-\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}U_{n-1}(\xi,\eta)p(\xi-\eta)R(\xi,\eta;\xi_0,\eta_0)d\eta.$$

Taking into account the known estimate $|R|\le 1$, we have

$$|U_0(\xi_0,\eta_0)|\le\frac{1}{2}\int_{\xi_0}^{\infty}|\,R(\xi,0;\xi_0,\eta_0)\,|\;|p(\xi)|\,d\xi\le\frac{1}{2}\int_{\xi_0}^{\infty}|p(\xi)|\;d\xi,$$

because $\xi>\xi_0,\ \ \eta<\eta_0$. Then

$$|U_0(\xi_0,\eta_0)|\le\frac{1}{2}\sigma(\xi_0).$$

Next, taking into account (2.10), we find

$$|U_1(\xi_0,\eta_0)|\le\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}|U_0(\xi,\eta)|\cdot|p(\xi-\eta)|\cdot R(\xi,\eta;\,\xi_0,\eta_0)\,d\eta\le$$

$$\le\frac{1}{2}\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}\sigma(\xi)\;|p(\xi-\eta)|\,d\eta\le\frac{1}{2}\int_{\xi_0}^{\infty}\sigma(\xi)\,d\xi\int_0^{\eta_0}|p(\xi-\eta)|\;d\eta\le$$

$$\le\frac{\sigma(\xi_0)}{2}\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}|p(\xi-\eta)|\;d\eta=\frac{\sigma(\xi_0)}{2}\int_{\xi_0}^{\infty}d\xi\int_{\xi-\eta_0}^{\xi}|p(\alpha)|\;d\alpha\le$$

$$\le\frac{\sigma(\xi_0)}{2}\int_{\xi_0}^{\infty}d\xi\int_{\xi-\eta_0}^{\infty}|p(\alpha)|\,d\alpha\le$$

$$\le\frac{\sigma(\xi_0)}{2}\int_{\xi_0}^{\infty}\sigma(\xi-\eta_0)\,d\xi\;=\frac{\sigma(\xi_0)}{2}\sigma_1(\xi_0-\eta_0).$$

Let now

$$|U_{n-1}(\xi_0,\eta_0)|\le\frac{1}{2}\,\sigma(\xi_0)\,\frac{(\sigma_1(\xi_0-\eta_0))^{n-1}}{(n-1)!},$$

then

$$|U_n(\xi_0,\eta_0)|\le\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}|p(\xi-\eta)R(\xi,\eta,\,\xi_0,\eta_0)U_{n-1}(\xi,\eta)|\,d\eta\;\le$$

$$\le\frac{1}{2}\,\sigma(\xi_0)\int_{\xi_0}^{\infty}\frac{(\sigma_1(\xi-\eta_0))^{n-1}}{(n-1)!}\int_{\xi-\eta_0}^{\xi}|p(\alpha)|\,d\alpha d\xi\le$$

$$\leq \frac{1}{2}\,\sigma\left(\xi_0\right)\int_{\xi_0}^{\infty}\frac{\left(\sigma_1\left(\xi-\eta_0\right)\right)^{n-1}}{(n-1)!}\int_{\xi-\eta_0}^{\infty}\left|p\left(\alpha\right)\right|d\alpha d\xi \leq$$

$$=-\frac{1}{2}\,\sigma\left(\xi_0\right)\int_{\xi_0}^{\infty}\frac{\left(\sigma_1\left(\xi-\eta_0\right)\right)^{n-1}}{(n-1)!}d\sigma_1\left(\xi-\eta_0\right)=\frac{1}{2}\,\sigma\left(\xi_0\right)\frac{\left(\sigma_1\left(\xi_0-\eta_0\right)\right)^{n}}{n!}.$$

From this it obviously follows that the series $U\left(\xi_0,\eta_0\right)=\sum\limits_{n=0}^{\infty}U_n\left(\xi_0,\eta_0\right)$ converges absolutely and uniformly, and its sum is a solution to Eq. (2.12) and $U\left(\xi_0,\eta_0\right)$ satisfies the inequality (2.13). The theorem has been proved. □

Now suppose $p\left(y\right)$ is a continuously differentiable function and satisfies the condition

$$\int_{0}^{+\infty}e^{2t}\left[\left|p\left(t\right)\right|+\left|p'\left(t\right)\right|\right]dt<\infty, \tag{2.14}$$

Differentiating equation (2.12) and using (2.9), (2.11) and (2.14), we find that the function $U\left(\xi_0,\eta_0\right)$ is twice continuously differentiable in the region $0\leq\eta_0\leq\xi_0<\infty$ and the following relations hold:

$$\begin{gathered}\frac{\partial U}{\partial\xi_0}=O\left(e^{\xi_0+\eta_0}\right),\ \frac{\partial U}{\partial\eta_0}=O\left(e^{\xi_0+\eta_0}\right),\ \frac{\partial^2 U}{\partial\xi_0\partial\eta_0}=O\left(e^{\xi_0+\eta_0}\right),\xi\to\infty,\\ \frac{\partial^2 U}{\partial\xi_0^2}=O\left(e^{\xi_0+\eta_0}\right),\ \frac{\partial^2 U}{\partial\eta_0^2}=O\left(e^{\xi_0+\eta_0}\right),\ \xi_0+\eta_0\to\infty.\end{gathered} \tag{2.15}$$

Moreover, the solution $U\left(\xi_0,\eta_0\right)$ of the integral equation (2.12) satisfies the following differential equation

$$\frac{\partial^2 U\left(\xi_0,\eta_0\right)}{\partial\xi_0\partial\eta_0}+\left[-2e^{2\xi}\,sh2\eta+p\left(\xi-\eta\right)\right]U\left(\xi_0,\eta_0\right)=0 \tag{2.16}$$

and

$$U\left(\xi_0,0\right)=\frac{1}{2}\int_{\xi_0}^{+\infty}p\left(\xi\right)d\xi. \tag{2.17}$$

From this and from (2.15), it follows that the function $A\left(u,v\right)=U\left(\frac{v+u}{2},\frac{v-u}{2}\right)$ is twice continuously differentiable in the region $0<u\leq v<\infty$ and due to (2.14) for each fixed x

$$\frac{\partial A(u,v)}{\partial u}=O\left(e^{v}\right),\ \frac{\partial A(u,v)}{\partial v}=O\left(e^{v}\right),$$

$$\frac{\partial^2 A(u,v)}{\partial u^2}=O\left(e^{v}\right),\ \frac{\partial^2 A(u,v)}{\partial v^2}=O\left(e^{v}\right),v\to\infty.$$

From this and from (2.16), (2.17) it follows that the function $A\left(u,v\right)=U\left(\frac{v+u}{2},\frac{v-u}{2}\right)$ satisfies the following problem

$$\frac{\partial^2 A\left(u,v\right)}{\partial u^2}-\frac{\partial^2 A\left(u,v\right)}{\partial v^2}-\left(e^{2u}-e^{2v}+p\left(u\right)\right)\ A\left(u,v\right)=0,\ 0<u<v, \tag{2.18}$$

$$A\left(u,u\right)=\frac{1}{2}\int_{u}^{\infty}p\ \left(t\right)\ dt, \tag{2.19}$$

$$\lim_{u+v\to\infty}\ A\left(u,v\right)=0. \tag{2.20}$$

Furthermore, the following estimate holds

$$\left|A\left(u,v\right)\right|\leq\sigma\left(\frac{v+u}{2}\right)e^{\sigma_1(u)}. \tag{2.21}$$

Then, using (2.18)-(2.20) and direct differentiation, we find that

$$f(t,\lambda) = K_{i\lambda}\left(e^t\right) + \int_t^\infty A(t,s)\ K_{i\lambda}(e^s)\,ds, \tag{2.22}$$

satisfies the equation (2.1). We note that the requirement (2.14) for the function $p(t)$ can be omitted, since a function from the class (2.3) can be approximated by functions from the class (2.14) and then extrapolation to the limit carried out (see [7]).

Let

$$\rho(x) = \int_x^\infty |q(t)|\,t^{-1}dt,\ \rho_1(x) = \int_x^\infty \rho(t)\,t^{-1}dt. \tag{2.23}$$

Theorem 2.2. *If $q(x)$ is a continuously differentiable function and satisfies the condition (1.2), then, for all values of ν, Eq.* (1.1) *has solutions $\Psi(x,\nu)$, representable as*

$$\Psi(x,\nu) = \psi(x,\nu) + \int_x^{+\infty} K(x,t)\,\psi(t,\nu)\,t^{-1}dt, \tag{2.24}$$

where the kernel $K(x,t)$ is continuous function and satisfies the following conditions:

$$|K(x,t)| \le \rho\left(\sqrt{xt}\right)e^{\rho_1(x)}, \tag{2.25}$$

$$K(x,x) = \frac{1}{2}\int_x^\infty q\ (t)\,t^{-1}dt. \tag{2.26}$$

Proof. We put $\nu = i\lambda$, $e^t = x$, $K(x,t) = A(\ln x, \ln t)$. Then (2.22) entails the validity of representation (2.24). In addition, from (2.2), (2.21) it follows that

$$|K(x,t)| \le \sigma\left(\frac{\ln x + \ln t}{2}\right)e^{\sigma_1(\ln x)} = \sigma\left(\ln\sqrt{xt}\right)e^{\sigma_1(\ln x)} =$$

$$= e^{\int_{\ln x}^\infty \sigma(t)dt}\int_{\ln\sqrt{xt}}^\infty |p(t)|\,dt = e^{\rho_1(x)}\int_{\sqrt{xt}}^\infty |p(\ln u)|\,u^{-1}du = \rho\left(\sqrt{xt}\right)e^{\rho_1(x)}.$$

On the other hand, from equality (2.2), (2.19) we obtain that

$$K(x,x) = A(\ln x, \ln x) = \frac{1}{2}\int_{\ln x}^\infty p\ (t)\ dt =$$

$$\frac{1}{2}\int_x^\infty p\ (\ln s)\,s^{-1}\,ds = \frac{1}{2}\int_x^\infty q\ (s)\,s^{-1}\,ds.$$

The theorem has been proved. □

Let us now study the solution $\Phi(x,\nu)$ with asymptotic behavior (1.5). We shall use the following notation

$$\theta(x) = \int_{-\infty}^x |p(t)|\,dt,\ \theta_1(x) = \int_{-\infty}^x \theta(t)\,dt.$$

It is known [1] that the function $I_\nu(z)$ has the asymptotic behavior

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \Gamma^{-1}(\nu+1)\,(1+o(1))\,,\ z\to 0.$$

Then, according to the general theory [6], [7], under condition (2.4), equation (2.1) for all λ from the closed upper half-plane has a solution $e(x,\lambda)$, representable in the form

$$e(x,\lambda) = I_{-i\lambda}(e^{x}) + \int_{-\infty}^{x} B(x,t)\, I_{-i\lambda}\left(e^{t}\right) dt,$$

where the kernel $B(x,t)$ is a continuously differentiable function and satisfy the relations

$$|B(t,s)| \leq \theta\left(\frac{s+t}{2}\right) e^{\theta_1(t)},$$

$$B(t,t) = \frac{1}{2}\int_{-\infty}^{t} p(\alpha)\, d\alpha.$$

Using the last three formulas and reasoning as in the proof of Theorem 2.2, we obtain the validity of the following theorem.

Let

$$\xi(x) = \int_0^x |q(t)|\, t^{-1} dt,\ \xi_1(x) = \int_0^x \xi(t)\, t^{-1} dt.$$

Theorem 2.3. *If $q(x)$ is a continuously differentiable function and satisfies the condition* (1.3)*, then, for all values of ν from the closed left half-plane, Eq.* (1.1) *has solutions $\Phi(x,\nu)$, representable as*

$$\Phi(x,\nu) = \varphi(x,\nu) + \int_0^x G(x,t)\, \varphi(t,\nu)\, t^{-1} dt,$$

where the kernel $G(x,t)$ is continuous function and satisfies the following conditions:

$$|G(x,t)| \leq \xi\left(\sqrt{xt}\right) e^{\xi_1(x)},$$

$$G(x,x) = \frac{1}{2}\int_0^x q(t)\ t^{-1} dt.$$

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