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NODAL SOLUTIONS OF SOME NONLINEAR STURM-LIOUVILLE PROBLEM DEPENDING ON A PARAMETER

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Abstract. In this paper, we consider the nonlinear Sturm-Liouville problem depending on a parameter. Using bifurcation technique, we find interval for this parameter, in which this problem has nodal solutions.

1. Introduction

We consider the following nonlinear Sturm-Liouville problem

$$\ell(y) \equiv -\left(p\,(x)y'(x)\right)' + q(x)y(x) = \chi r(x)h(y(x)),\ x \in (0,1), \tag{1.1}$$

$$b_0 y(0) + d_0 p\,(0)y'(0) = 0, \tag{1.2}$$

$$b_1 y(1) + d_1 p\,(1)y'(1) = 0, \tag{1.3}$$

where $p \in C^2([0,1];(0,+\infty))$, $q \in C([0,1];\mathbb{R})$, χ is a real parameter, $r \in C([0,1];(0,+\infty))$, b_0, d_0, b_1 and d_1 are real constants such that $|b_0| + |d_0| > 0$ and $|b_1| + |d_1| > 0$. Here the nonlinear term h is a real-valued continuous function on $\mathbb{R}$ that satisfies the following conditions: there exist $h_0,\, h_\infty \in (0,+\infty)$, $h_0 \neq h_\infty$, such that

$$h_0 = \lim_{|s|\to 0} \frac{h(s)}{s},\ h_\infty = \lim_{|s|\to +\infty} \frac{h(s)}{s}\,. \tag{1.4}$$

The purpose of this paper is to determine the interval for χ in which the problem (1.1)-(1.3) has nodal solutions.

Let E be a Banach space $C^1[0,1] \cap (b.c.)$ with the usual norm $||y||_1 = ||y||_\infty + ||y'||_\infty$, where $(b.c.)$ denotes the set of functions satisfying the boundary conditions (1.2) and (1.3), and $||y||_\infty = \max\limits_{x\in[0,1]} |y\,(x)|$.

From now on ν will denote an element of $\{+,-\}$ that is, either $\nu = +$ or $\nu = -$.

For each $k \in \mathbb{N}$ and each ν by S_k^ν we denote the set of functions $y \in E$ such that (i) y has only simple nodal zeros in $[0,1]$ and exactly $k-1$ such zeros in the

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interval $(0,1)$; (ii) νy is positive in a deleted neighborhood of $x=0$. Moreover, let $S_k = S_k^+ \cup S_k^-$, $k \in \mathbb{N}$ (see [10]).

It is well known (see, e.g., [2]) that the eigenvalues of the linear Sturm-Liouville problem

$$\begin{cases} \ell(y)(x) = \lambda r(x)y(x), \ x \in (0,1), \\ y \in (b.c.), \end{cases} \tag{1.5}$$

are real and simple and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$; the eigenfunction $y_k(x)$, $k \in \mathbb{N}$, corresponding to the eigenvalue λ_k, lies in S_k.

The following theorems are the main results of this paper.

Theorem 1.1. *Let for some $k \in \mathbb{N}$ one of the following conditions holds:*

(i) $\frac{\lambda_k}{h_0} < \chi < \frac{\lambda_k}{h_\infty}$;

(ii) $\frac{\lambda_k}{h_\infty} < \chi < \frac{\lambda_k}{h_0}$.

Then problem (1.1)-(1.3) *has solutions y_k^+ and y_k^- such that $y_k^+ \in \mathcal{S}_k^+$ and $y_k^- \in \mathcal{S}_k^-$, respectively.*

Theorem 1.2. *Let for some $k \in \mathbb{N}$ one of the following conditions holds:*

(i) $\frac{\lambda_{k+1}}{h_0} < \chi < \frac{\lambda_k}{h_\infty}$;

(ii) $\frac{\lambda_{k+1}}{h_\infty} < \chi < \frac{\lambda_k}{h_0}$.

Then problem (1.1)-(1.3) *has solutions y_k^+, y_k^-, y_{k+1}^+ and y_{k+1}^- such that $y_k^+ \in S_k^+$, $y_k^- \in S_k^-$, $y_{k+1}^+ \in S_{k+1}^+$ and $y_{k+1}^- \in S_{k+1}^-$, respectively.*

The problem (1.1)-(1.3) in a particular case was considered in [1, 4-7, 9] and in the general case in [3, 8]. In these papers, it was shown the existence of nodal solutions of considered problems under the condition $sh(s) > 0$, $s \in \mathbb{R}$, as well as under the conditions $q \geq 0$ and b_0, d_0, b_1, $d_1 \in [0, +\infty)$, $b_0 d_1 - b_1 d_0 + b_0 b_1 > 0$. Note that the last three conditions guarantee the positivity of the smallest eigenvalue of problem (1.5). As can be seen from Theorems 1.1 and 1.2, we prove existence of nodal solutions to problem (1.1)-(1.3) without these conditions. It should be noted that only in [1] problem (1.1)-(1.3) is considered under the conditions that the first eigenvalue of problem (1.5) is positive.

The rest of this paper is organized as follows. In Section 2, we consider an auxiliary nonlinear eigenvalue problem and study the global bifurcation of nontrivial solutions from zero and from infinity to this problem. We prove the existence of two families of unbounded continua of nontrivial solutions, branching from the line of trivial solutions and contained in the classes $\mathbb{R} \times S_k^+$ and $\mathbb{R} \times S_k^-$, $k \in \mathbb{N}$. Here we also show the existence of two families of global continua of solutions, branching from the line $\mathbb{R} \times \{\infty\}$ and containing in the classes $\mathbb{R} \times S_k^+$ and $\mathbb{R} \times S_k^-$, $k \in \mathbb{N}$, in some neighborhood of the asymptotic bifurcation points. Moreover, these continua either meet another bifurcation points, or meet the line $\mathbb{R} \times \{0\}$, or have unbounded projections onto $\mathbb{R} \times \{0\}$. In Section 3, we show that the global continua bifurcating from $\mathbb{R} \times \{\infty\}$ are contained in the classes $\mathbb{R} \times S_k^+$ and $\mathbb{R} \times S_k^-$, $k \in \mathbb{N}$ and meet the line $\mathbb{R} \times \{0\}$. Here we also prove that the global continua bifurcating from zero and from infinity, contained in the same oscillating classes, coincide. Next, using this statement, we define an interval for a parameter χ, in which there exist solutions to problem (1.1)-(1.3) with fixed oscillation count.

2. Global bifurcation from zero and infinity in some nonlinear Sturm-Liouville problems

By (1.4) we have

$$h(s) = h_0 s + h^0(s) \text{ and } h(s) = h_\infty s + h^\infty(s),\ s \in \mathbb{R}, \tag{2.1}$$

where

$$h^0(s) = o(|s|) \text{ as } |s| \to 0 \text{ and } h^\infty(s) = o(|s|) \text{ as } |s| \to \infty, \tag{2.2}$$

respectively.

Remark 2.1. Throughout what follows we will assume that χ is contained in some bounded interval not containing 0.

We consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda\chi h_0 r(x)y(x) + \chi r(x)h^0(y(x)),\ x \in (0,1), \\ y \in (b.c.). \end{cases} \tag{2.3}$$

which can be rewritten in the form

$$\begin{cases} \ell(y)(x) = \lambda\chi h_0 r(x)y(x) + g_0(x, y(x), y'(x), \lambda),\ x \in (0,1), \\ y \in (b.c.), \end{cases}$$

where

$$g_0(x,s,t,\lambda) = \chi r(x)h^0(s),\ (x,s,t,\lambda) \in [0,1] \times \mathbb{R}^3.$$

It is obvious that $g_0 \in C([0,1] \times \mathbb{R}^3)$. Moreover, by the first relation of (2.2) and Remark 2.1, from the relation

$$\frac{|h^0(s)|}{|s|+|t|} \le \frac{|h^0(s)|}{|s|}$$

it follows that

$$g_0(x,s,t,\lambda) = o(|s|+|t|) \text{ as } |s|+|t| \to 0, \tag{2.4}$$

uniformly for $(x,\lambda) \in [0,1] \times \mathbb{R}$. Consequently, to problem (2.3) is applicable [10, Theorem 2.3] in view of (2.4). Then, by Theorem 2.3 of [10], we have the following result.

Theorem 2.1. *For each $k \in \mathbb{N}$ and each ν there exists a continuum C_k^ν of the set of nontrivial solutions of problem* (2.3) *which meets $(\widetilde{\lambda}_k, 0)$, is contained in $\mathbb{R} \times S_k^\nu$ and is unbounded in $\mathbb{R} \times E$, where $\widetilde{\lambda}_k$ is the kth eigenvalue of the linear Sturm-Liouville problem*

$$\begin{cases} \ell(y)(x) = \lambda\chi h_0 r(x)y(x),\ x \in (0,1), \\ y \in (b.c.). \end{cases} \tag{2.5}$$

It is seen from (2.5) that

$$\widetilde{\lambda}_k = \frac{\lambda_k}{\chi h_0}. \tag{2.6}$$

By (2.1) we get

$$h^0(s) = h(s) - h_0 s = (h_\infty - h_0)s + h^\infty(s),\ s \in \mathbb{R}. \tag{2.7}$$

Then (2.3) takes the following form

$$\begin{cases} \ell(y) = \lambda\chi h_0 r(x)y + \chi(h_\infty - h_0)r(x)y + \chi r(x)h^\infty(y),\ x \in (0,1), \\ y \in (b.c.). \end{cases} \tag{2.8}$$

Obviously, problem (2.8) can be rewritten as follows:

$$\begin{cases} \ell(y) = \lambda\chi h_0 r(x)y + \chi(h_\infty - h_0)r(x)y + g_\infty(x, y, y', \lambda),\ x \in (0,1), \\ y \in (b.c.), \end{cases}$$

where continuous on $[0,1] \times \mathbb{R}^3$ function $g_\infty(x, s, t, \lambda)$ is defined by

$$g_\infty(x, s, t, \lambda) = \chi r(x)h^\infty(s).$$

By the second relation of (2.2) we get for any sufficiently small $\varepsilon > 0$ there is a sufficiently large $K_\varepsilon > 0$ such that

$$\frac{|h^\infty(s)|}{|s|} < \varepsilon \ \text{ for } \ |s| > K_\varepsilon. \tag{2.9}$$

In view of the first relation from (2.2), there is a sufficiently small $\kappa_\varepsilon > 0$ such that

$$\frac{|h^0(s)|}{|s|} < \varepsilon \ \text{ for } \ 0 < |s| < \kappa_\varepsilon,$$

whence, by (2.7), implies that

$$\frac{|h^\infty(s)|}{|s|} < |h_\infty - h_0| + \varepsilon \ \text{ for } \ 0 < |s| < \kappa_\varepsilon. \tag{2.10}$$

The condition $h \in C(\mathbb{R};\mathbb{R})$ ensures the existence of a positive constants M_ε and N_ε such that

$$|h^\infty(s)| \le M_\varepsilon \ \text{ for } \ |s| \le K_\varepsilon \ \text{ and } \ \frac{|h^\infty(s)|}{|s|} \le N_\varepsilon \ \text{ for } \ \kappa_\varepsilon \le |s| \le K_\varepsilon. \tag{2.11}$$

Let $N_\varepsilon^1 = \max\{N_\varepsilon,\ |h_\infty - h_0| + \varepsilon\}$. Then by (2.10) and the second relation of (2.11), we get

$$\frac{|h^\infty(s)|}{|s|} \le N_\varepsilon^1 \ \text{ for } \ 0 < |s| \le K_\varepsilon. \tag{2.12}$$

We denote by K_ε^1 a positive number satisfying the following conditions:

$$K_\varepsilon^1 > K_\varepsilon,\ M_\varepsilon < \varepsilon K_\varepsilon^1. \tag{2.13}$$

If $(s,t) \in \mathbb{R}^2$ such that $|s| + |t| > K_\varepsilon^1$, then by (2.9), (2.13) and the first relation of (2.11) we obtain

$$\frac{|h^\infty(s)|}{|s| + |t|} \le \frac{|h^\infty(s)|}{|s|} < \varepsilon \ \text{ if } \ |s| > K_\varepsilon,$$

$$\frac{|h^\infty(s)|}{|s| + |t|} < \frac{M_\varepsilon}{K_\varepsilon^1} < \varepsilon \ \text{ if } \ |s| \le K_\varepsilon.$$

By Remark 2.1 from the last relations we get

$$g_\infty(x, s, t, \lambda) = o(|s| + |t|) \ \text{as} \ |s| + |t| \to \infty, \tag{2.14}$$

uniformly in $(x, \lambda) \in [0,1] \times \mathbb{R}$. Therefore, to problem (2.8) are applicable [11, Theorem 2.4] and [12, Theorem 3.1] in view of (2.14). Then, by these theorems, we have the following result.

Theorem 2.2. *For each $k \in \mathbb{N}$ and each ν there exists connected component D_k^ν of the set of nontrivial solutions of problem* (2.8) *which meets* $(\widehat{\lambda}_k, \infty)$ *and either* (i) *the set D_k^ν meets $(\widehat{\lambda}_{k'}, \infty)$ with respect to $\mathbb{R} \times \mathcal{S}_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$, or* (ii) *the set D_k^ν meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$, or* (iii) *the projection of the set D_k^ν onto $\mathbb{R} \times \{0\}$ is unbounded, where $\widehat{\lambda}_k$ is the kth eigenvalue of the linear Sturm-Liouville problem*

$$\begin{cases} \ell(y)(x) = \left(\lambda + \frac{h_\infty}{h_0} - 1\right) \chi h_0 r(x) y(x),\ x \in (0,1), \\ y \in (b.c.). \end{cases} \tag{2.15}$$

It follows from (2.15) that

$$\left(\widehat{\lambda}_k + \frac{h_\infty}{h_0} - 1\right) \chi h_0 = \lambda_k,$$

which implies that

$$\widehat{\lambda}_k = \frac{\lambda_k}{\chi h_0} - \frac{h_\infty}{h_0} + 1. \tag{2.16}$$

3. The connection between the sets C_k^ν and D_k^ν, $k \in \mathbb{N}$, $\nu \in \{+, -\}$

In this section we explore the connection between global continua of nontrivial solutions to problem (2.3) (or (2.8)), bifurcating from the line of trivial solutions and from the line $\mathbb{R} \times \{\infty\}$.

The following result holds.

Theorem 3.1. *For each $k \in \mathbb{N}$ and each ν we have the following relation:*

$$C_k^\nu = D_k^\nu. \tag{3.1}$$

Proof. Let $(\lambda, y) \in \mathbb{R} \times \partial S_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+, -\}$, is a solution of problem (2.8). Then (λ, y) is also a solution to problem (2.3). By [5, Lemma 2.2], we have $y \equiv 0$, and therefore, D_k^ν cannot intersect the boundary of the set $\mathbb{R} \times S_k^\nu$. Hence we get

$$D_k^\nu \subset \mathbb{R} \times S_k^\nu. \tag{3.2}$$

It follows from the last relation that the set D_k^ν cannot meets $I_{k'} \times \{\infty\}$ with respect to $\mathbb{R} \times \mathcal{S}_{k'}^{\nu'}$ for any $(k', \nu') \neq (k, \nu)$, i.e. alternative (i) of Theorem 2.2 cannot hold.

If alternative (iii) of Theorem 2.2 hold then there exists sequence $\{(\eta_n, v_n)\}_{n=1}^\infty \subset \mathbb{R} \times \mathcal{S}_k^\nu$ such that

$$\begin{cases} \ell(v_n) = \eta_n \chi h_0 r(x) v_n + \chi(h_\infty - h_0) r(x) v_n + \chi r(x) h^\infty(v_n),\ x \in (0,1), \\ v_n \in (b.c.), \end{cases} \tag{3.3}$$

and

$$\lim_{n\to\infty} \eta_n = -\infty \ \text{ or } \ \lim_{n\to\infty} \eta_n = +\infty. \tag{3.4}$$

Let

$$\varphi_n^\infty(x) = \begin{cases} \frac{h^\infty(v_n(x))}{v_n(x)} & \text{if } v_n(x) \neq 0, \\ 0 & \text{if } v_n(x) = 0. \end{cases} \tag{3.5}$$

By (2.9) and (2.12) we get

$$\frac{|h^\infty(s)|}{|s|} \le N_1^1 \ \text{ for } \ s \in \mathbb{R}, \ s \ne 0.$$

Then it follows from (3.5) that

$$|\varphi_n^\infty(x)| \le N_1^1, \ x \in [0,1]. \tag{3.6}$$

In view of (3.5) we can rewrite (3.3) in the following form

$$\begin{cases} \ell(v_n) = \eta_n \chi h_0 r(x) v_n + \chi(h_\infty - h_0) r(x) v_n + \chi r(x) \varphi_n^\infty(x) v_n, \ x \in (0,1), \\ v_n \in (b.c.). \end{cases} \tag{3.7}$$

It is clear from (3.7) that (η_n, v_n) for each $n \in \mathbb{N}$ solves the following linear eigenvalue problem

$$\begin{cases} \ell(y) = \lambda \chi h_0 r(x) y + \chi(h_\infty - h_0) r(x) y + \chi r(x) \varphi_n^\infty(x) y, \ x \in (0,1), \\ y \in (b.c.). \end{cases} \tag{3.8}$$

By the max-min property of eigenvalues (see [2, pp. 405-406]) we have

$$\widehat{\lambda}_{k,n} \chi h_0 = \max_{V^{(k-1)}} \min_{y \in B.C.} \left\{ \widehat{R}_n[y] : \int_0^1 r(x)\, y(x)\, \varphi(x) dx = 0, \ \varphi \in V^{(k-1)} \right\}, \tag{3.9}$$

where $\widehat{\lambda}_{k,\,n}$ is the kth eigenvalue of the linear problem (3.8),

$$\widehat{R}_n[y] = \frac{\int_0^1 \left\{ p y'^2 + q y^2 \right\} dx - \chi(h_\infty - h_0) \int_0^1 r y^2 dx - \chi \int_0^1 \varphi_n^\infty r y^2 dx + N[y]}{\int_0^1 r y^2 dx}, \tag{3.10}$$

$$N[y] = \begin{cases} \frac{b_0}{d_0} y^2(0) + \frac{b_1}{d_1} y^2(1) & \text{if } \ b_0 b_1 \ne 0, \\ \frac{b_0}{d_0} y^2(0) & \text{if } \ b_0 \ne 0, \ b_1 = 0, \\ \frac{b_1}{d_1} y^2(1) & \text{if } \ b_0 = 0, \ b_1 \ne 0, \\ 0 & \text{if } \ b_0 = b_1 = 0, \end{cases}$$

and $V^{(k-1)}$ denotes any set of $(k-1)$ linearly independent functions with $\varphi_j(x) \in (b.c.)$, $1 \le j \le k-1$.

It should be noted that

$$\widehat{\lambda}_k = \max_{V^{(k-1)}} \min_{y \in B.C.} \left\{ \widehat{R}[y] : \int_0^1 r(x)\, y(x)\, \varphi(x) dx = 0, \ \varphi \in V^{(k-1)} \right\}, \tag{3.11}$$

where

$$\widehat{R}[y] = \frac{\int_0^1 \left\{ p y'^2 + q y^2 \right\} dx + \chi(h_\infty - h_0) \int_0^1 r y^2 dx + N[y]}{\int_0^1 r y^2 dx}, \tag{3.12}$$

By (3.12) from (3.10) we get

$$\widehat{R}_n\,[y] = \widehat{R}\,[y] - \frac{\chi \int\limits_0^1 \varphi_n^\infty r y^2 dx}{\int\limits_0^1 r y^2 dx},$$

which, by (3.6), implies that

$$\widehat{R}\,[y] - \chi N_1^1 \leq \widehat{R}_n\,[y] \leq \widehat{R}\,[y] + \chi N_1^1. \tag{3.13}$$

In view of (3.9), (3.11), by (3.13) we get

$$\widehat{\lambda}_k + \chi N_1^1 \leq \widehat{\lambda}_{k,n} \leq \widehat{\lambda}_k + \chi N_1^1. \tag{3.14}$$

Since $\eta_n = \widehat{\lambda}_{k,n}$ for each $n \in \mathbb{N}$ it follows from (3.14) that relation (3.4) is impossible in view of Remark 2.1. Therefore, alternative (iii) of Theorem 2.2 cannot hold.

Thus, we have shown that only alternative (ii) of Theorem 2.2 holds, and consequently, D_k^ν meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$. Then by (3.2) it follows from Theorem 2.1 that D_k^ν meets only $(\widetilde{\lambda}_k, 0)$ with respect to the set $\mathbb{R} \times S_k^\nu$.

By following the above arguments we can show that C_k^ν meets only $(\widehat{\lambda}_k, \infty)$ with respect to the set $\mathbb{R} \times S_k^\nu$. Therefore, we come to the conclusion that for each $k \in \mathbb{N}$ and each ν the sets C_k^ν and D_k^ν coincide. The proof of this theorem is complete.

4. Proofs of the main results

In this section we prove the main results of this paper.

Proof of Theorem 1.1. Consider two possible cases.

Case 1. Let $\lambda_k = 0$ for some $k \in \mathbb{N}$. In this case the result is trivial. Indeed, the eigenfunction $y_k \in S_k$ corresponding to λ_k of the linear Sturm-Liouville (1.5) is a solution of the following problem

$$\begin{cases} \ell(y)(x) = 0,\ x \in (0,1), \\ y \in (b.c.). \end{cases}$$

This eigenfunction y_k is made unique by requiring that $y_k \in S_k^+$ and $||y_k||_1 = 1$. Consequently, for $\chi = 0$ problem (1.1)-(1.3) has solutions $y_k^+ = y_k \in S_k^+$ and $y_k^- = -\,y_k \in S_k^-$.

Case 2. Let $\lambda_k \neq 0$ for some $k \in \mathbb{N}$. We suppose that $\lambda_k < 0$ and the condition (i) of Theorem 1.1 holds, i.e., let

$$\frac{\lambda_k}{h_0} < \chi < \frac{\lambda_k}{h_\infty}\,.$$

Then it follows from the left hand side of this relation that

$$\widetilde{\lambda}_k = \frac{\lambda_k}{\chi h_0} > 1.$$

Moreover, from the right hand side of this relation we obtain

$$h_\infty > \frac{\lambda_k}{\chi},$$

whence implies that

$$\widehat{\lambda}_k = \frac{\lambda_k}{\chi h_0} - \frac{h_\infty}{h_0} + 1 < \frac{\lambda_k}{\chi h_0} - \frac{\lambda_k}{\chi h_0} + 1 = 1.$$

Thus we have

$$\widehat{\lambda}_k < 1 < \widetilde{\lambda}_k. \tag{4.1}$$

In view of (3.2) for each ν the set D_k^ν lies in $\mathbb{R} \times S_k^\nu$ and by (3.1) meets $(\widetilde{\lambda}_k, 0)$ and $(\widehat{\lambda}_k, \infty)$ and is connected in $\mathbb{R} \times E$. Then it follows from (4.1) that for each ν there exists a solution $(1, y_k^\nu)$ of problem (2.3) such that $y_k^\nu \in S_k^\nu$. Then y_k^ν for each ν solves problem (1.1)-(1.3).

The remainder cases are considered similarly. The proof of this theorem is complete.

Proof of Theorem 1.2. We suppose that the condition (i) of Theorem 1.2 holds, i.e., let

$$\frac{\lambda_{k+1}}{h_0} < \chi < \frac{\lambda_k}{h_\infty}$$

Then the following cases are possible: (a) $\lambda_k < \lambda_{k+1} < 0$; (b) $0 < \lambda_k < \lambda_{k+1}$.

Consider case (a). In this case it follows from the proof of Theorem 1.1 that

$$\widehat{\lambda}_{k+1} < 1 < \widetilde{\lambda}_k,$$

and consequently,

$$\widehat{\lambda}_k < \widehat{\lambda}_{k+1} < 1 < \widetilde{\lambda}_k < \widetilde{\lambda}_{k+1}. \tag{4.2}$$

By following the arguments at the end of the proof of Theorem 1.1, from relation (4.2) we obtain the statement of Theorem 1.2.

The remaining cases are treated similarly. The proof of this theorem is complete.

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