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PUTNAM-FUGLEDE THEOREMS AND ORTHOGONALITY OF AN ELEMENTARY OPERATOR IN $\mathscr{C}_p$ CLASSES

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Abstract. Given Hilbert space commuting operators $T, S \in \mathcal{L}(H)$, such that T is w-hyponormal with $\ker T \subseteq \ker T^*$ and S is normal. Let $\phi_{T,S} \in \mathcal{L}(\mathcal{L}(H))$ be the elementary operator defined by $\phi_{T,S}(X) = TXS^* - SXT^*$. In this paper, we show firstly that (1) $\ker(\phi_{T,S} \mid \mathscr{C}_p) \subset \ker(\phi_{T^*,S^*} \mid \mathscr{C}_p)$; (2) The range of $\phi_{T,S} \mid \mathscr{C}_p$ is orthogonal to the kernel of $\phi_{T,S} \mid \mathscr{C}_p$ ($\mathcal{R}(\phi_{T,S} \mid \mathscr{C}_p) \perp \ker(\phi_{T,S} \mid \mathscr{C}_p)$) if and only if $\ker T \cap \ker S = \{0\}$. Secondly, we will extend these results to the elementary operator $\Phi \in \mathcal{L}(\mathcal{L}(H))$ defined by $\Phi(X) = AXD - CXB$ where $[A, C] = [B, D] = 0$. Related orthogonality results for the elementary operator Φ are also given.

1. Introduction

Let H be a separable complex Hilbert space and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on H into itself. The familiar Putnam-Fuglede theorem [10, Problem 152], asserts that if $T, S \in \mathcal{L}(H)$ are normal operators and $TX = XS$ for some $X \in L(H)$, then $T^*X = XS^*$. The problem of extending the Putnam-Fuglede theorem has been considered by a large number authors, and numerous generalizations of this theorem have appeared over the recent past. The cited references [3, 12, 14, 15] are among various extensions of this celebrated theorem for non-normal classes of operators.

G. Weiss [16] obtained an interesting generalization of the Putnam-Fuglede theorem involving four normal operators. In a way that, if (A, C) and (B, D) are two pairs of commuting normal operators on H, then $AXD = CXB$ implies $A^*XD^* = C^*XB^*$ for all $X \in \mathcal{L}(H)$. This result was generalized by T. Furuta [7] to hyponormal operators with the Hilbert-Schmidt hypothesis on X. In other words, if $A, B, C, D \in \mathcal{L}(H)$, with A, B^*, C and D^* are hyponormal, $CA^* = A^*C$ and $BD^* = D^*B$, then $AXD = CXB$ implies $A^*XD^* = C^*XB^*$ for every X in the Hilbert-Schmidt class.

Let $T \in \mathcal{L}(H)$ be compact, and let $s_1(T) \geqslant s_2(T) \geqslant \cdots \geqslant 0$ denote the singular values of T arranged in their decreasing order. The operator T is said to belong

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to the Schatten p-class $\mathscr{C}_p$ if

$$\|T\|_p = \begin{cases} \left(\sum_{j=1}^{\infty} s_j(T)^p\right)^{1/p} = (\mathrm{tr}(T)^p)^{1/p} < \infty, & 1 \leqslant p < \infty, \\ s_1(T), & p = \infty, \end{cases}$$

where 'tr' denotes the trace functional. Given subspaces M and N of a Banach space V with norm $\|.\|$, then M is said to be orthogonal to N in the sense of Birkhof-James, denoted $M \perp N$, if $\|m + n\| \geqslant \|n\|$ for all $m \in M$ and $n \in N$. The range-kernel orthogonality of elementary operators has been studied by a number of authors over recent decades. For $A, B \in \mathcal{L}(H)$, the generalized derivation $\delta_{A,B} \in \mathcal{L}(\mathcal{L}(H))$ is defined by $\delta_{A,B}(X) = AX - XB$. In [2] Bouali and Cherki proved that if A and B are normal, then the range $\mathcal{R}(\delta_{A,B} \mid \mathscr{C}_p)$ of $\delta_{A,B}$ is orthogonal to its kernel $\ker(\delta_{A,B} \mid \mathscr{C}_p)$. These results have been extended to a diversity of elementary operators $\Phi \in \mathcal{L}(\mathcal{L}(H))$, where $\Phi(X) = AXD - CXB$, for a variety of choices of tuples of commuting operators (A, C) and (B, D) (see [5, 9, 11, 13] for further references). In particular, A. Turnsěk [13] proved that if A, C respectively B, D are nonzero normal commuting operators, then $\mathcal{R}(\Phi \mid \mathscr{C}_p) \perp \ker(\Phi \mid \mathscr{C}_p)$ if and only if

$$\ker A \cap \ker C = \ker B^* \cap \ker D^* = \{0\}.$$

An operator T is called w-hyponormal if $|\tilde{T}| \geq |T| \geq \left|(\tilde{T})^*\right|$ where $|T| = (T^*T)^{\frac{1}{2}}$ and $\tilde{T} = |T|^{1/2}U|T|^{1/2}$, is the Aluthge transform of T. We say that T is w_*-hyponormal and we note $T \in w_* - H$, if T is w-hyponormal and $\ker(T) \subseteq \ker(T^*)$. In this paper we consider the elementary operator $\phi_{T,S}(X) = TXS^* - SXT^*$ where $T \in w_* - H$ and S is normal operator with $TS = ST$. It will be shown the inclusion $\ker(\phi_{T,S} \mid \mathscr{C}_p) \subset \ker(\phi_{T^*,S^*} \mid \mathscr{C}_p)$. This implies that if $\Phi \in \mathcal{L}(\mathcal{L}(H))$ is the elementary operator $\Phi(X) = AXD - CXB$, where $A, B^* \in w_* - H$, C and D are normal such that $AC = CA$ and $BD = DB$, then $\ker(\Phi \mid \mathscr{C}_p) \subset \ker(\Phi_* \mid \mathscr{C}_p)$, where $\Phi_* : X \in \mathcal{L}(H) \mapsto A^*XD^* - C^*XB^*$. Which gives both an extension of the Putnam-Fuglede property and the Weiss's theorem. Another purpose of this paper is to investigate the range-kernel orthogonality of the elementary operators $\phi_{T,S}$ and Φ in $\mathscr{C}_p$ classes. We conclude this section with some notations.

For X a linear operator acting on Banach space E, we denote by X^*, $\ker(X)$, $\ker X^{\perp}$, $R(X)$ and $X|_M$ respectively the adjoint, the kernel, the orthogonal complement of the kernel, the range of X and the restriction of X to an invariant subspace M. For g and ω two vectors in H, we define $g \otimes \omega \in \mathcal{L}(H)$ as follows:

$$g \otimes \omega(x) = \langle x, \omega \rangle g \text{ for all } x \in H.$$

Recall that an operator $T \in \mathcal{L}(H)$ is said to be hyponormal if $T^*T \geq TT^*$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators [8]. An operator T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ for $p \in]0, 1]$ and an operator T is said to be log-hyponormal if T is invertible and $\log |T| \geq \log |T^*|$. p-hyponormal and log-hyponormal operators are defined as extension of hyponormal operator. The classes of log- and w-hyponormal operators were introduced and their properties were studied in [1]. In particular, it was shown in [1] that the class of w-hyponormal operators contains both p- and log-hyponormal operators. Moreover, it is easy to see that w-hyponormal operators T are paranormal (i.e., $\|Tx\|^2 \leqslant \|T^2x\|$ for all unit vectors $x \in H$). The w-hyponormal operators have some interesting properties, amongst them that the restriction of a w-hyponormal operator to an invariant subspace is again a w-hyponormal operator, the inverse of an invertible w-hyponormal operator is again w-hyponormal.

2. Main results

Lemma 2.1. [3] *If $A, B \in w_* - H$ are such that $[A, B] = [A^*, B] = 0$ and B is invertible, then $AB^{-1} \in w_* - H$.*

Theorem 2.1. *Let $T \in w_* - H$ and $S \in \mathcal{L}(H)$ is normal such that $TS = ST$. If T or S is injective, then for $1 \leq p < \infty$, the following assertions holds:*

(i) $\ker(\phi_{T,S} \mid \mathscr{C}_p) \subset \ker(\phi_{T^*,S^*} \mid \mathscr{C}_p)$.

(ii) *For all $X, Y \in \mathcal{L}(H)$ such that $Y \in \ker(\phi_{T,S} \mid \mathscr{C}_p)$ we have*

$$\|\phi_{T,S}(X) + Y\|_p \geq \|Y\|_p .$$

Proof. Firstly, assume that S is injective and for a natural number n, let $\Delta_n = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1/n\}$ and let $E_S(\Delta_n)$ denote the corresponding spectral projection. Set $I - E_S(\Delta_n) = P_n$; then $P_n \to I$ in the strong topology. Since $TS = ST$, the Fuglede's Theorem implies $TS^* = S^*T$ and so $R(P_n)$ reduces both T and S. Hence

$$T = T_{1,n} \oplus T_{2,n} \text{ and } S = S_{1,n} \oplus S_{2,n} \text{ on } H_n = H = \ker(P_n) \oplus R(P_n),$$

where $T_{i,n}$ are w_*-hyponormal $(i = 1, 2)$, $S_{1,n}$ is normal and $S_{2,n}$ is invertible normal. Now and for $Y \in \ker(\phi_{T,S} \mid \mathscr{C}_p)$, let $Y_n = P_nYP_n$, hence $Y_n \longrightarrow Y$ weakly (even, strongly). Also, if we set $R_n = T_{2,n}S_{2,n}^{-1}$, then we have

$$\begin{aligned} P_n\phi_{T,S}(Y)P_n &= P_n(TYS^* - SYT^*)P_n \\ &= T_{2,n}(P_nYP_n)S_{2,n}^* - S_{2,n}(P_nYP_n)T_{2,n}^* \\ &= T_{2,n}Y_nS_{2,n}^* - S_{2,n}Y_nT_{2,n}^* \\ &= S_{2,n}(R_nY_n - Y_nR_n^*)S_{2,n}^*, \end{aligned}$$

which means that $Y_n \in \ker(\delta_{R_n,R_n^*})$. Since R_n is w_*-hyponormal by Lemma 2.1, then [3, Lemma 2.4] implies that $Y_n \in \ker(\delta_{R_n^*,R_n})$. Hence

$$\begin{aligned} P_n\phi_{T^*,S^*}(Y)P_n &= P_n(T^*YS - S^*YT)P_n \\ &= S_{2,n}^*(R_n^*Y_n - Y_nR_n)S_{2,n} \\ &= 0, \end{aligned}$$

and as result $Y \in \ker(\phi_{T^*,S^*} \mid \mathscr{C}_p)$. Secondly, it results from [2], Theorem 2.2, that

$$\left\|\delta_{R_n,R_n^*}(Z_n) + Y_n\right\|_p \geq \|Y_n\|_p \text{ for all } Z_n \in \mathcal{L}(R(P_n)).$$

Thus for $Z_n = S_{2,n}X_nS_{2,n}^*$, we would have

$$\left\|T_{2,n}X_nS_{2,n}^* - S_{2,n}X_nT_{2,n}^* + Y_n\right\|_p \geq \|Y_n\|_p ,$$

for all $X_n = P_nXP_n \in \mathcal{L}(R(P_n))$. It follows that

$$\|\phi_{T,S}(X)+Y\|_p \geq \|P_n(\phi_{T,S}(X)+Y)P_n\|_p = \left\|T_{2,n}X_nS^*_{2,n} - S_{2,n}X_nT^*_{2,n} + Y_n\right\|_p \geq \|Y_n\|_p .$$

Therefore, since $\|Y_n\|_p \longrightarrow \|Y\|_p$,

$$\|\phi_{T,S}(X)+Y\|_p \geq \|Y\|_p \text{ for all } X \in \mathcal{L}(H) \text{ and } Y \in \ker(\phi_{T,S} \mid \mathscr{C}_p).$$

Now, suppose that T is injective and let $Y \in \ker(\phi_{T,S} \mid \mathscr{C}_p)$. So, we can write T, S and Y on $H_0 = H = (\ker(S))^\perp \oplus \ker(S)$ as

$$T = N \oplus R, S = S_1 \oplus 0, \text{ and } Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix},$$

with $NY_1S_1^* = S_1Y_1N^*$, i.e $Y_1 \in \ker(\phi_{N,S_1} \mid \mathscr{C}_p)$ and $S_1Y_2R^* = RY_3S_1^* = 0$, hence $Y_2 = Y_3 = 0$. As result, since $N \in w_* - H$ and S_1 is injective, the first case implies that $Y_1 \in \ker(\phi_{N^*,S_1^*} \mid \mathscr{C}_p)$, which equivalent to $Y \in \ker(\phi_{T^*,S^*} \mid \mathscr{C}_p)$. More things, let $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in \mathcal{L}(H_0)$, then we have:

$$\|\phi_{T,S}(X)+Y\|_p = \left\|\begin{pmatrix} \phi_{N,S_1}(X_1)+Y_1 & * \\ * & Y_4 \end{pmatrix}\right\|_p \geq \left\|\begin{pmatrix} \phi_{N,S_1}(X_1)+Y_1 & 0 \\ 0 & Y_4 \end{pmatrix}\right\|_p$$

since the norm of an operator matrix always dominates the norm of its diagonal part. Also, we can deduce from the first case that:

$$\|\phi_{T,S}(X)+Y\|_p \geq \left(\|\phi_{N,S_1}(X_1)+Y_1\|_p^p + \|Y_4\|_p^p\right)^{\frac{1}{p}} \geq \left(\|Y_1\|_p^p + \|Y_4\|_p^p\right)^{\frac{1}{p}} = \|Y\|_p.$$

□

Remark 2.1. It's easy to check that Theorem 2.1 remains valid for the following assumptions: T is normal and S is w_*-hyponormal, by using the fact that $\phi_{T,S} = -\phi_{S,T}$.

We consider the elementary operator $\Phi \in \mathcal{L}(\mathcal{L}(H))$ defined by $\Phi(X) = AXD - CXB$, with $\Phi_*(X) = A^*XD^* - C^*XB^*$. We therefore deduce that:

Corollary 2.1. *Let $A, B, C, D \in \mathcal{L}(H)$ such that $AC = CA$ and $BD = DB$. If one of the following conditions hold:*

(i) $A, B^* \in w_* - H$ are injective, C and D are normal operators,
(ii) $A, B^* \in w_* - H$, C and D are normal injective operators,
(iii) $C, D^* \in w_* - H$ are injective, A and B are normal operators,
(iv) $C, D^* \in w_* - H$, A and B are normal injective operators.

Then for $1 \leq p < \infty$, $\ker(\Phi \mid \mathscr{C}_p) \subset \ker(\Phi_* \mid \mathscr{C}_p)$ and we have

$$\|\Phi(X)+T\|_p \geq \|T\|_p ,$$

for all $X, T \in \mathcal{L}(H)$ such that $T \in \ker(\Phi \mid \mathscr{C}_p)$.

Proof. Let $T \in \ker(\Phi \mid \mathscr{C}_p)$ and put $\hat{T} = \begin{pmatrix} B^* & 0 \\ 0 & A \end{pmatrix}$, $S = \begin{pmatrix} D^* & 0 \\ 0 & C \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$ be defined on $H \oplus H$. This gives us $\hat{T}S = S\hat{T}$ and $Y \in \ker(\phi_{\hat{T},S} \mid \mathscr{C}_p)$. By our assumptions and Theorem 2.1, we get $Y \in \ker(\phi_{\hat{T}^*,S^*} \mid \mathscr{C}_p)$ which equivalent to $T \in \ker(\Phi_* \mid \mathscr{C}_p)$, furthermore, for all $\hat{X} = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \in \mathcal{L}(H \oplus H)$, we have

$$\left\| \phi_{\hat{T},S}(\hat{X}) + Y \right\|_p = \left\| \begin{pmatrix} 0 & 0 \\ AXD - CXB + T & 0 \end{pmatrix} \right\|_p \geq \|Y\|_p .$$

Therefore, we obtain the desired result. □

Remark 2.2. If we take $C = D = I$, we can see that the preceding corollary generalizes Lemma 2.4. of [3] for $d_{A,B} = \delta_{A,B} \mid \mathscr{C}_p$. In addition, it extends Bouali and Cherki's inequality [2] for w_*-hyponormal operators.

Theorem 2.2. *Let $T \in w_* - H$ and $S \in \mathcal{L}(H)$ be normal operator such that $TS = ST$. If $TXS^* = SXT^*$ for some operator $X \in \mathscr{C}_p$, then we have $T^*XS = S^*XT$.*

Proof. Since $TS = ST$ and S is normal then by Fuglede's Theorem $T^*S = ST^*$. It follows that $(\ker S)^\perp$ reduce T. Therefore

$$T = N \oplus R \text{ on } H_0 = H = (\ker(S))^\perp \oplus \ker(S).$$

We can write S and X on H_0 as:

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

Since $TS = ST$ we have $NS_1 = S_1N$. Thus if $TXS^* = SXT^*$ then $NX_1S_1^* = S_1X_1N^*$, with $N \in w_* - H$ and S_1 is normal injective. So by Theorem 2.1, we get $N^*X_1S_1 = S_1^*X_1N$. By the same way, we obtain that $S_1X_2R^* = 0$ and $RX_3S_1^* = 0$. Thus, by the Putnam-Fuglede theorem, we have

$$S_1(X_2R^*) = (X_2R^*) \cdot 0 \text{ and } 0 \cdot (RX_3) = (RX_3)S_1^* \implies$$
$$S_1^*(X_2R^*) = (X_2R^*) \cdot 0^* \text{ and } 0^* \cdot (RX_3) = (RX_3)S_1.$$

We have also $R \in w_* - H$, then [3, Lemma 2.4] ensures that:

$$0 \cdot (S_1^*X_2) = (S_1^*X_2)R^* \text{ and } R(X_3S_1) = (X_3S_1) \cdot 0 \implies$$
$$0^* \cdot (S_1^*X_2) = (S_1^*X_2)R \text{ and } R^*(X_3S_1) = (X_3S_1) \cdot 0^*.$$

Hence $S_1^*X_2R = R^*X_3S_1 = 0$ and as a result $T^*XS = S^*XT$. □

Corollary 2.2. *Let $A, B, C, D \in \mathcal{L}(H)$ such that $AC = CA$ and $BD = DB$. Then, $AXD = CXB$ and $X \in \mathscr{C}_p$ implies $A^*XD^* = C^*XB^*$ in each of the following cases:*

(i) $A, B^* \in w_* - H$, C and D are normal,
(ii) $C, D^* \in w_* - H$ and A and B are normal.

Remark 2.3. Our results generalize the Weiss's Theorem [16], as well as the Putnam-Fuglede property $(F,P)_{\mathscr{C}_p}$ for w_*-hyponormal operators by taking $C = D = I$. Also, as a consequence of Corollary 2.2 and Duggal's result [6], We'll get:

Corollary 2.3. *Let $A, B, C, D \in \mathcal{L}(H)$ such that $AC = CA$ and $BD = DB$. Then,*

$$\|\Phi(X) + T\|_2^2 = \|\Phi(X)\|_2^2 + \|T\|_2^2 \text{ for all } X \in \mathcal{L}(H) \text{ and } T \in \ker(\Phi \mid \mathscr{C}_2).$$

in each of the following cases:

(i) $A, B^* \in w_* - H$, *C and D are normal,*
(ii) $C, D^* \in w_* - H$ *and A and B are normal.*

Proposition 2.1. *Let* $A, B, C, D \in \mathcal{L}(H)$ *such that* $AC = CA$ *and* $BD = DB$. *If one of the following assertions is verified*

(i) $A, B^*, D^* \in w_* - H$, *C is normal, D is invertible and* $BD^* = D^*B$,
(ii) $A, B^*, C \in w_* - H$, *D is normal, C is invertible and* $AC^* = C^*A$,
(iii) $C, B^*, D^* \in w_* - H$, *A is normal, B is invertible and* $BD^* = D^*B$,
(iv) $A, B^*, D^* \in w_* - H$, *B is normal, A is invertible and* $AC^* = C^*A$. *Then we have the implication*

$$AXD = CXB \implies A^*XD^* = C^*XB^* \text{ for all } X \in \mathscr{C}_p.$$

Proof. It is an immediate consequence of Corollary 2.2 and Lemma 2.1. □

Proposition 2.2. *Let* $T, S \in \mathcal{L}(H)$. *If* $T, S \in w_* - H$ *are doubly commuting operators such that* $|T|S = S|T^*|$. *Then,* $TXS^* = SXT^*$ *implies* $T^*XS = S^*XT$ *for all* $X \in \mathscr{C}_p$.

Proof. By hypothesis, we have $TS - ST = T^*S - ST^* = 0$, It follows that $(\ker S)^\perp$ reduce T and $T|_{(\ker S)^\perp}$ is normal. This means

$$T = N \oplus R \text{ on } H_0 = H = (\ker(S))^\perp \oplus \ker(S).$$

Since $\ker S \subseteq \ker S^*$, $\ker S$ reduces S. Hence, we can write S and X on H_0 as follows:

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

From $TS = ST$ we have $NS_1 = S_1N$. Thus if $TXS^* = SXT^*$ then $NX_1S_1^* = S_1X_1N^*$ with $S_1 \in w_* - H$ and N is normal. So by Theorem 2.2 we get $N^*X_1S_1 = S_1^*X_1N$. Also, we find that $S_1X_2R^* = 0$ and $RX_3S_1^* = 0$ with $R \in w_* - H$. So by using [3], Lemma 2.4, the rest of the proof is similar to that of theorem 2.2. □

Example 2.1. Let $T \in \mathcal{L}(H)$ be p-hyponormal such that $T = T_1 \oplus T_2$ on the space $H = H_1 \oplus H_2$, where T_1 is the normal part of T and T_2 is the pure part of T; i.e., T_2 is p-hyponormal and has no invariant subspace $\mathcal{M}$ such that $T_2|_{\mathcal{M}}$ is normal. If we define $S = N \oplus 0$ on H, where N is a w_*-hyponormal operator on H_1 which commute with T_1 (as example $N = I$). A simple calculation shows that $TS = ST$ and $T^*S = ST^*$ using Fuglede's theorem. Also, $|T|S = S|T^*|$ since $|T^*| = |T_1| \oplus |T_2^*|$. The operators T and S satisfy the hypothesis of Proposition 2.2, hence:

$$TXS^* = SXT^* \implies T^*XS = S^*XT \text{ for all } X \in \mathscr{C}_p.$$

Based on Theorem 2.1, we then give next a generalization of this Theorem, replacing the condition " T or S is injective " by " $\ker T \cap \ker S = \{0\}$ ".

Theorem 2.3. *Let* $T \in w_* - H$ *and* $S \in \mathcal{L}(H)$ *is normal such that* $TS = ST$ *and suppose that* $\ker T \cap \ker S = \{0\}$. *Then for* $1 \le p < \infty$, $\ker(\phi_{T,S} \mid \mathscr{C}_p) \subset \ker(\phi_{T^*,S^*} \mid \mathscr{C}_p)$, *and we have*

$$\|\phi_{T,S}(X) + Y\|_p \ge \|Y\|_p\,,$$

for every $Y \in \ker(\phi_{T,S} \mid \mathscr{C}_p)$ *and for all* $X \in \mathcal{L}(H)$.

Proof. If T or S is injective, we obtain the desired result by using Theorem 2.1. So let us assume that neither T nor S is injective. With respect to the decomposition $H_0 = H = \ker S^{\perp} \oplus \ker S$, we get

$$T = \begin{pmatrix} N & 0 \\ 0 & R \end{pmatrix}, \text{ and } S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $R = T|_{\ker S}$ is injective by hypothesis. Now, we have two cases:

$N = 0$: In this case and if $Y \in \ker(\phi_{T,S} \mid \mathscr{C}_p)$ has the form $Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$ on H_0, we obtain $S_1 Y_2 R^* = R Y_3 S_1^* = 0$, which means that $Y_2 = Y_3 = 0$. Thus, $\phi_{T,S}(Y) = 0 = \phi_{T^*,S^*}(Y)$ and for $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in \mathcal{L}(H_0)$,

$$\|\phi_{T,S}(X) + Y\|_p = \left\| \begin{pmatrix} Y_1 & -S_1 X_1 R^* \\ R X_3 S_1^* & Y_4 \end{pmatrix} \right\|_p \geq \left\| \begin{pmatrix} Y_1 & 0 \\ 0 & Y_4 \end{pmatrix} \right\|_p = \|Y\|_p$$

since the norm of an operator matrix always dominates the norm of its diagonal part.

$N \neq 0$: Since $T|_{\ker S}$ is injective and $N = T|_{\ker S^{\perp}}$ is not injective. Also, of the fact that T is paranormal, $\ker S^{\perp} \ominus \ker N$ is invariant subspace of T. With respect to the decomposition $H_1 = H = \big(\ker S^{\perp} \ominus \ker N\big) \oplus \ker N \oplus \ker S$, we find that

$$T = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_2 \end{pmatrix} \text{ and } S = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where operators T_i and S_i, $1 \leq i \leq 2$, are injective with $T_1 \in w_* - H$. From the hypothesis $TYS^* = SYT^*$, we obtain

$$Y = \begin{pmatrix} Y_{11} & 0 & 0 \\ 0 & Y_{22} & 0 \\ 0 & 0 & Y_{33} \end{pmatrix}, \text{ and } T_1 Y_{11} S_1^* = S_1 Y_{11} T_1^*, \text{ i.e } Y_{11} \in \ker(\phi_{T_1,S_1} \mid \mathscr{C}_p).$$

Then Theorem 2.1 implies that $Y_{11} \in \ker(\phi_{T_1^*,S_1^*} \mid \mathscr{C}_p)$ which equivalent to $Y \in \ker(\phi_{T^*,S^*} \mid \mathscr{C}_p)$. Furthermore and for $X = [X_{ij}]_{i,j=1}^3 \in \mathcal{L}(H_1)$,

$$\begin{aligned} \|\phi_{T,S}(X) + Y\|_p &= \left\| \begin{pmatrix} \phi_{T_1,S_1}(X_{11}) + Y_{11} & * & * \\ * & Y_{22} & * \\ * & * & Y_{33} \end{pmatrix} \right\|_p \\ &\geq \left\| (\phi_{T_1,S_1}(X_{11}) + Y_{11}) \oplus Y_{22} \oplus Y_{33} \right\|_p \end{aligned}$$

since the norm of an operator matrix always dominates the norm of its diagonal part. From Theorem 2.1, we can infer that

$$\begin{aligned} \|\phi_{T,S}(X) + Y\|_p &\geq \left(\|\phi_{T_1,S_1}(X_{11}) + Y_{11}\|_p^p + \|Y_{22}\|_p^p + \|Y_{33}\|_p^p \right)^{\frac{1}{p}} \\ &\geq \left(\|Y_{11}\|_p^p + \|Y_{22}\|_p^p + \|Y_{33}\|_p^p \right)^{\frac{1}{p}} \\ &= \|Y\|_p. \end{aligned}$$

□

Corollary 2.4. *Let $A, B^* \in w_* - H$ and let $C, D \in \mathcal{L}(H)$ be two normal operators, such that $AC = CA$ and $BD = DB$. Suppose that $\ker A \cap \ker C = \ker B^* \cap \ker D = \{0\}$. Then for $1 \le p < \infty$, $\ker(\Phi \mid \mathscr{C}_p) \subset \ker(\Phi_* \mid \mathscr{C}_p)$, and we have*

$$\|\Phi(X) + T\|_p \ge \|T\|_p,$$

for all $T \in \ker(\Phi \mid \mathscr{C}_p)$ and for all $X \in \mathcal{L}(H)$.

Theorem 2.4. *Let $T \in w_* - H$ and S be a normal operator such that $TS = ST$ and $\phi_{T,S} \neq 0$. Then for $1 \le p < \infty$ with $p \neq 2$*

$$\|\phi_{T,S}(X) + Y\|_p \ge \|Y\|_p$$

holds for every $Y \in \ker(\phi_{T,S} \mid \mathscr{C}_p)$ and for all $X \in \mathcal{L}(H)$ if and only if $\ker T \cap \ker S = \{0\}$.

Proof. By Theorem 2.3, it suffices to show that, $\ker T \cap \ker S = \{0\}$ is a necessary condition for $\mathcal{R}(\phi_{T,S} \mid \mathscr{C}_p) \perp \ker(\phi_{T,S} \mid \mathscr{C}_p)$, when T and S are not injective. Suppose that $\ker T \cap \ker S \neq \{0\}$ and decompose $H_0 = H = \ker S^\perp \oplus \ker S$, then

$$T = \begin{pmatrix} N & 0 \\ 0 & R \end{pmatrix} \text{ and } S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ on } H_0.$$

First, assume that $\ker T \neq \ker S$. Without loss of generality, we can also assume that $\ker S \nsubseteq \ker T$. Then $R = T|_{\ker S}$ is a nonzero operator with nontrivial kernel. With respect to the decomposition

$$H_1 = H = \ker S^\perp \oplus (\ker S \ominus \ker R) \oplus \ker R$$

one obtains $T = T_1 \oplus T_2 \oplus 0$ and $S = S_1 \oplus 0 \oplus 0$, where T_2 and S_1 are injective. From the hypothesis $TYS^* = SYT^*$ we get

$$Y = \begin{pmatrix} Y_{11} & 0 & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} \text{ and } T_1Y_{11}S_1^* = S_1Y_{11}T_1^*,$$

so that the other Y entries are arbitrary. Let e be a nonzero vector of H. Choose

$$X = \begin{pmatrix} 0 & e \otimes ie & 0 \\ e \otimes ie & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Y_{23} \\ 0 & Y_{32} & Y_{33} \end{pmatrix}.$$

Then

$$\phi_{T,S}(X) + Y = \begin{pmatrix} 0 & -S_1e \otimes T_2ie & 0 \\ T_2e \otimes S_1ie & 0 & Y_{23} \\ 0 & Y_{32} & Y_{33} \end{pmatrix}.$$

Since $C = \begin{pmatrix} 0 & -S_1e \otimes T_2ie \\ T_2e \otimes S_1ie & 0 \end{pmatrix}$ is a nonzero (S_1 and T_2 are injective) self-adjoint operator of finite rank, we can use [13], Lemma 2.4, for $p \neq 1$ to find operators Y_{23}, Y_{32} and Y_{33} such that

$$\|\phi_{T,S}(X) + Y\|_p < \left\| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Y_{23} \\ 0 & Y_{32} & Y_{33} \end{pmatrix} \right\|_p = \|Y\|_p.$$

For $p = 1$, [13, lemma 2.4] can be used again, because if the operator C is of rank two and has eigenvalues λ_1, λ_2 with $|\lambda_1| = |\lambda_2|$, then a simple calculation shows that $|\lambda_1| = |\lambda_2|$ if and only if $\|T_2e\|\|S_1e\| = 1$ and $\langle T_2e, S_1e\rangle = 0$ for all $e \in H$. Hence $S_1^*T_2 = 0$ which means that $S_1 = 0$ since T_2 is injcetive. We are therefore faced with a contradiction with our hypothesis that $\phi_{T,S} \neq 0$.

In the case $\ker T = \ker S$, it's clear that $R = T|_{\ker S} = 0$ and $N = T|_{\ker S^\perp}$ is injective. From $TYS^* = SYT^*$, it follows that

$$Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \text{ on } H_0 \text{ with } NY_1S_1^* = S_1Y_1N^*,$$

such that the other entries of Y are arbitrary. Let us choose $Y_1 = 0$ and $X = \begin{pmatrix} e\otimes ie & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$\phi_{T,S}(X) + Y = \begin{pmatrix} Ne\otimes S_1ie - S_1e\otimes Nie & Y_2 \\ Y_3 & Y_4 \end{pmatrix}.$$

If $D = Ne\otimes S_1ie - S_1e\otimes Nie = 0$ for all $e \in H$, then, since N and S_1 are injective, we would have $S_1 = cN$ with $c \in \mathbb{R}$. This would imply contrary to our assumption, that $\phi_{T,S} = 0$. Thus D becomes a nonzero self-adjoint operator of finite rank. We can also end the proof in a similar way to the first case. □

Remark 2.4. (1) The condition $\phi_{T,S} \neq 0$ in Theorem 2.4 is essential. It's enough to take $T = 0$ and S is a non-injective operator, to see that $\mathcal{R}(\phi_{T,S} \mid \mathscr{C}_p) \perp \ker(\phi_{T,S} \mid \mathscr{C}_p)$ cannot imply that $\ker T \cap \ker S = \{0\}$ when $\phi_{T,S} = 0$.

(2) Bouali and Cherki prove in [2] that $\ker(\delta_{A,B} \mid \mathscr{C}_p) \subset \ker(\delta_{A^*,B^*} \mid \mathscr{C}_p)$ implies $\mathcal{R}(\delta_{A,B} \mid \mathscr{C}_p) \perp \ker(\delta_{A,B} \mid \mathscr{C}_p)$. But the following example proves that $\ker(\phi_{T,S} \mid \mathscr{C}_p) \subset \ker(\phi_{T^*,S^*} \mid \mathscr{C}_p)$ cannot imply that $\mathcal{R}(\phi_{T,S} \mid \mathscr{C}_p) \perp \ker(\phi_{T,S} \mid \mathscr{C}_p)$ for $1 \leq p < \infty$ with $p \neq 2$.

Example 2.2. Let $H_0 = H \oplus H$ and define the operators:

$$T = \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} \text{ and } S = \begin{pmatrix} iI & 0 \\ 0 & 0 \end{pmatrix},$$

where R is w_*-hyponormal non injective operator on H (as example $R = 0$). Then we have $T \in w_* - H$, S is normal and $TS = ST$. Hence Theorem 2.2 ensures that $\ker(\phi_{T,S} \mid \mathscr{C}_p) \subset \ker(\phi_{T^*,S^*} \mid \mathscr{C}_p)$. But the assumption $\ker T \cap \ker S = \{0\}$ implies that R is injective, which is not the case. Since $\phi_{T,S} \neq 0$ ($\phi_{T,S}(I \oplus 0) \neq 0$), it follows from Theorem 2.4 that $\mathcal{R}(\phi_{T,S} \mid \mathscr{C}_p)$ is not orthogonal to $\ker(\phi_{T,S} \mid \mathscr{C}_p)$.

Corollary 2.5. *Let $A, B^* \in w_* - H$ and let $C, D \in \mathcal{L}(H)$ be two normal operators such that $AC = CA$ and $BD = DB$, and assume that $\Phi \neq 0$ such that A, B, C and D are nonzero. Then for $1 \leq p < \infty$ with $p \neq 2$*

$$\|\Phi(X) + T\|_p \geq \|T\|_p,$$

for all $T \in \ker(\Phi \mid \mathscr{C}_p)$ and for all $X \in \mathcal{L}(H)$ if and only if $\ker A \cap \ker C = \ker B^* \cap \ker D = \{0\}$.

Remark 2.5. The condition $\ker A \cap \ker C = \ker B^* \cap \ker D = \{0\}$ in Corollary 2.5 is not necessary for $\mathcal{R}(\Phi \mid \mathscr{C}_p) \perp \ker(\Phi \mid \mathscr{C}_p)$ if one of the operators A, B, C or D is zero. It's enough to take $A \in w_* - H$ a nonzero operator with nontrivial kernel, $C = 0$ and $D = I$ to find that $\Phi = \delta_{A,0} \neq 0$. It follows from [3], Lemma 2.4, that $(A, 0)$ has $(F, P)_{\mathscr{C}_p}$, which implies that $\mathcal{R}(\Phi \mid \mathscr{C}_p) \perp \ker(\Phi \mid \mathscr{C}_p)$ by [4, Lemma 4].

Proposition 2.3. *Let $A, B, C, D \in \mathcal{L}(H)$ be nonzero operators such that $AC = CA$, $BD = DB$ and $\Phi \neq 0$.*

(1) If $BD^* = D^*B$ and under any one of the following conditions:
(i) $A, B^*, D^* \in w_* - H$, C is normal and D is invertible,
(ii) $C, B^*, D^* \in w_* - H$, A is normal and B is invertible.
For $1 \leq p < \infty$ with $p \neq 2$, there holds

$$\|\Phi(X) + T\|_p \geq \|T\|_p,$$

for all $T \in \ker(\Phi \mid \mathscr{C}_p)$ and for all $X \in \mathcal{L}(H)$ if and only if $\ker A \cap \ker C = \{0\}$.

(2) If $AC^* = C^*A$ and if one of the following conditions hold:
(i) $A, B^*, C \in w_* - H$, D is normal and C is invertible,
(ii) $A, B^*, D^* \in w_* - H$, B is normal and A is invertible.
Then, for $1 \leq p < \infty$ with $p \neq 2$, we have

$$\|\Phi(X) + T\|_p \geq \|T\|_p,$$

for all $T \in \ker(\Phi \mid \mathscr{C}_p)$ and for all $X \in \mathcal{L}(H)$ if and only if $\ker B^* \cap \ker D = \{0\}$.

Proof. By hypothesis of the case (i), we have $T \in \ker(\Phi \mid \mathscr{C}_p)$ if and only if $T \in \ker(\Phi_0 \mid \mathscr{C}_p)$, where $\Phi_0 : X \in \mathcal{L}(H) \mapsto AXI - CXBD^{-1}$. So by Lemma 2.1 and Corollary 2.5, we have $\ker A \cap \ker C = \{0\}$ if and only if

$\|\Phi_0(Y)+T\|_p \geq \|T\|_p$ for all $T \in \ker(\Phi_0 \mid \mathscr{C}_p) = \ker(\Phi \mid \mathscr{C}_p)$ and for all $Y \in \mathcal{L}(H)$,

which equivalent to (by taking $Y = XD$):

$$\|\Phi(X) + T\|_p \geq \|T\|_p \text{ for all } T \in \ker(\Phi \mid \mathscr{C}_p) \text{ and for all } X \in \mathcal{L}(H).$$

We can prove the other cases in the same way. □

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