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LIE SYMMETRY METHOD FOR OBTAINING EXACT SOLUTIONS OF NONLINEAR $K(m, n)$ TYPE EQUATIONS

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Abstract. In this article, we study a type of the $K(m, n)$ equations using Lie symmetry analysis method(LSAM). Calculating the symmetry of the partial differential equations has many applications in various fields of sciences. The Lie point symmetries and its optimal system are given. Then, classical Lie point symmetry operators, infinitesimal generators and invariant solutions are obtained. Finally, exact solutions of a special type of $K(m, n)$ equations are studied using an algebraic method.

1. Introduction

The method of group analysis of differential equations was introduced by Sophus Lie more than one hundred years ago. For analysis of partial differential equations (PDEs), we apply group theory because it is a powerful method to obtain the exact solutions of nonlinear PDEs and problems of mathematical physics. Nonlinear phenomena appear in various scientific fields, such as fluid mechanics, biology, optimal fiber, plasma physics and so on. Exact solutions of nonlinear PDEs play a critical role in better realizing qualitative features and physical interpretations of many occurrences. Many complicated events can be described by these solutions. For this purpose, some techniques have been suggested, such as the Kudryashov method [13], the sub-equation method [14], the $\exp(-\varphi(\xi))$-expansion method [28], the first integral method [11], the sine-Gordon method [6], Lie group method [16], the $\frac{G'}{G^2}$-expansion method [15], the sine-cosine method [10, 26, 27] and so on.
Rosenau and Hyman introduced and studied a type of Korteweg–de Vries (KdV) equations wtih nonlinear dispersion

$$u_t + u^m u_x + (u^n)_{xxx} = 0, \quad m, n > 1, \tag{1.1}$$

which is called $K(m, n)$ by them [5]. Wazwaz solved the following type of Eq.(1.1) when $n = m$ [24, 25, 26, 27]

$$u_t + au^n u_x + (u^n)_{xxx} = 0, \quad n > 1, \tag{1.2}$$

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After that, Biswas introduced a type of $K(m,n)$ equation with generalized evolution term as follows [2]:

$$(u^l)_t + ru^m u_x + s(u^n)_{xxx} = 0, \tag{1.3}$$

where, the first term is the generalized evolution term, while the second term represents the nonlinear term and the third term is the dispersion term. Also, r and s are constants, while l, m and $n \in \mathbb{Z}^+$.

In this paper, we study a special type of $K(m,n)$ equation with $l = 1$, $m = 2$ and $n = 1$ as follows

$$u_t + ru^2 u_x + su_{xxx} = 0. \tag{1.4}$$

Also, it is called modified KdV equation [27]. Much effort has been made on the construction of exact solutions of $K(m,n)$ equation [29, 25, 4, 24]. In this manuscript, we study the Lie symmetry group for $K(m,n)$ type equation and we obtain exact solutions of this equation using an algebraic method.

The paper is organized as follows. In Section 2 we discuss the methodology of Lie symmetry analysis of nonlinear PDEs. The classical symmetries of nonlinear $K(m,n)$ equation and the Lie point symmetries of this equation are calculated in section 3. In Section 4, we explain the group invariant solutions. In Section 5, the optimal system is obtained for one-dimensional subalgebras of nonlinear $K(m,n)$ equation. Finally, we obtain exact solutions for the nonlinear $K(m,n)$ equation in section 6. Conclusions are summarized in section 7.

2. Symmetry group analysis

Many nonlinear equations appear in various fields such as mathematical and physical sciences. Although it is very difficult to solve nonlinear PDEs, but much effort has been made for studying them. Symmetry is one of the most important tools in the area of PDEs. The concepts of Lie theory are based on the invariance of the equation under transformation groups of independent and dependent variables, so called Lie groups. The applications of Lie groups for solving differential equations were introduced in the nineteenth century by Sophus Lie, when he studied the continuous groups of transformations leaving differential equations invariant.

In the last century, the application of the Lie group method has been developed by a number of mathematicians. Baumann [1], Bluman [3], Ibragimov [9], Olver [18] and Ovsiannikov [19] are some of the scientists who have enormous amount of investigations in this field. Now, we consider the following system of PDEs with q dependent and p independent variables [17]:

$$\triangle_\nu(\mathbf{x}, \varphi^{(n)}) = 0, \quad \nu = 1, 2, \ldots, \ell, \tag{2.1}$$

where $\mathbf{x} = (x^1, x^2, \ldots, x^p)$, $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^q)$ and $\varphi^{(n)}$ denotes all the derivatives of φ of all orders from 0 to n. A one-parameter Lie group of infinitesimal transformations of the system (2.1) is

$$\begin{aligned} \bar{x}^i &= x^i + \varepsilon\xi^i(x,\varphi) + O(\varepsilon^2); \quad i = 1, 2, \ldots, p, \\ \bar{\varphi}^\alpha &= \varphi^\alpha + \varepsilon\eta^\alpha(x,\varphi) + O(\varepsilon^2); \quad \alpha = 1, 2, \ldots, q, \end{aligned} \tag{2.2}$$

where ξ^i and η^α are the infinitesimals of the transformations for the independent and dependent variables, and ε is the transformation parameter. We consider the general vector field X as the infinitesimal generator associated with the above group of transformations:

$$X=\sum_{i=1}^{p}\xi^i(x,\varphi)\frac{\partial}{\partial x^i}+\sum_{\alpha=1}^{q}\eta^\alpha(x,\varphi)\frac{\partial}{\partial\varphi^\alpha}. \tag{2.3}$$

The n-th order prolongation of X given by:

$$X^{(n)}=X+\sum_{\alpha=1}^{q}\sum_{J}\eta_J^\alpha(x,\varphi^{(n)})\frac{\partial}{\partial\varphi_J^\alpha}, \tag{2.4}$$

where $J=(i_1,\ldots,i_k), 1\le i_k\le p, 1\le k\le n$, and the sum is over all $J's$ of order $0<\#J\le n$. If $\#J=k$, the coefficient η_J^α of $\frac{\partial}{\partial\varphi_J^\alpha}$ will depend only on k-th and lower order derivatives of φ, and

$$\eta_\alpha^J(x,\varphi^{(n)})=D_J(\eta_\alpha-\sum_{i=1}^{p}\xi^i\varphi_i^\alpha)+\sum_{i=1}^{p}\xi^i\varphi_{J,i}^\alpha, \tag{2.5}$$

where $\varphi_i^\alpha=\frac{\partial\varphi^\alpha}{\partial x^i}$ and $\varphi_{J,i}^\alpha=\frac{\partial\varphi_J^\alpha}{\partial x^i}$.
A vector field X is an infinitesimal symmetry of the system of differential equations (2.1) if and only if it satisfies the infinitesimal invariance condition

$$X^{(n)}(\triangle_\nu(x,\varphi^{(n)})=0 \quad for\ all \quad \nu=1,2,\ldots,\ell. \tag{2.6}$$

3. Classical Symmetries of $K(m,n)$ equation

Let us consider a one-parameter Lie group of infinitesimal transformation:

$$\begin{aligned}\bar{x}&=x+\varepsilon\xi^1(x,t,u)+O(\varepsilon^2),\\ \bar{t}&=t+\varepsilon\xi^2(x,t,u)+O(\varepsilon^2),\\ \bar{u}&=u+\varepsilon\eta(x,t,u)+O(\varepsilon^2),\end{aligned}$$

with a small parameter ε. The symmetry group of (1.4) will be generated by the vector field of the form:

$$X=\xi^1(x,t,u)\frac{\partial}{\partial x}+\xi^2(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u}. \tag{3.1}$$

The third prolongation of X is the vector field

$$\begin{aligned}X^{(3)}=X \quad &+ \quad \eta^x\frac{\partial}{\partial u_x}+\eta^t\frac{\partial}{\partial u_t}+\eta^{2x}\frac{\partial}{\partial u_{2x}}+\eta^{xt}\frac{\partial}{\partial u_{xt}}+\eta^{2t}\frac{\partial}{\partial u_{2t}}\\ &+ \quad \eta^{3x}\frac{\partial}{\partial u_{3x}}+\ldots+\eta^{3t}\frac{\partial}{\partial u_{3t}},\end{aligned} \tag{3.2}$$

with coefficients

$$\eta^J=D_J(\eta-\sum_{i=1}^{2}\xi^i u_i^\alpha)+\sum_{i=1}^{2}\xi^i u_{J,i}^\alpha, \tag{3.3}$$

where $J=(i_1,\ldots,i_k)$, $1\le i_k\le 2, 1\le k\le 3$, and the sum is over all $J's$ of order $0<\#J\le 3$. Applying the third prolongation $(X^{(3)})$ to Eq.(1.4), we can obtain ξ^1,ξ^2 and η.

Theorem 3.1. *The Lie group of point symmetries of Eq.* (1.4) *has a Lie algebra generated by the vector field*

$$X = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}, \tag{3.4}$$

where

$$\begin{aligned} \xi^1 &= \frac{1}{3} c_1\, x + c_3, \\ \xi^2 &= c_1\, t + c_2, \\ \eta &= -\frac{1}{3} c_1\, u. \end{aligned}$$

Here c_1, c_2, c_3 *are arbitrary constants.*

Proof. Applying the third prolongation of the vector field (3.1),$X^{(3)}$, to (1.4) we have

$$X^{(3)}(u_t + ru^2u_x + su_{xxx})|_{(1)=0} = 0. \tag{3.5}$$

Expanding the above equation and solving the obtained system, we obtain

$$\begin{aligned} \xi^1 &= \frac{1}{3} c_1\, x + c_3, \\ \xi^2 &= c_1\, t + c_2, \\ \eta &= -\frac{1}{3} c_1\, u. \end{aligned}$$

The proof of the theorem is completed.

Corollary 3.1. *The Lie algebra of infinitesimal generators of every one-parameter Lie group of the* $K(m,n)$ *equation is spanned by the three vector fields*

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \end{aligned}$$

The table 1 shows the commutation relations between these vector fields. Where the entry in the i-th row and j-th column is defined as $[Xi, Xj] = X_iX_j - X_jX_i,\ i,j = 1,2,3.$

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$\frac{1}{3}X_1$
X_2	0	0	X_2
X_3	$-\frac{1}{3}X_1$	$-X_2$	0

Table 1: Commutation relations satisfied by infinitesimal generators.

4. Group Invariant Solutions

For obtaining the group transformation which is generated by the infinitesimal generators X_i for i = 1, 2, 3, we need to solve the three systems of first order ordinary differential equations

$$\frac{d\bar{x}}{d\varepsilon} = \xi_i^1(\bar{x}(\varepsilon), \bar{t}(\varepsilon), \bar{u}(\varepsilon)), \bar{x}(0) = x, \tag{4.1}$$

$$\frac{d\bar{t}}{d\varepsilon} = \xi_i^2(\bar{x}(\varepsilon), \bar{t}(\varepsilon), \bar{u}(\varepsilon)), \bar{t}(0) = t, \quad i = 1, 2, 3, \tag{4.2}$$

$$\frac{d\bar{u}}{d\varepsilon} = \eta_i(\bar{x}(\varepsilon), \bar{t}(\varepsilon), \bar{u}(\varepsilon)), \bar{u}(0) = u. \tag{4.3}$$

Exponentiating the infinitesimal symmetries of equation (1.4), we get the one-parameter groups $H_i(\varepsilon)$ generated by X_i for i = 1,2,3.

$$H_1(\varepsilon) : (x, t, u) \to (x + \varepsilon, t, u), \tag{4.4}$$

$$H_2(\varepsilon) : (x, t, u) \to (x, t + \varepsilon, u), \tag{4.5}$$

$$H_3(\varepsilon) : (x, t, u) \to (xe^{\frac{\varepsilon}{3}}, te^{\varepsilon}, ue^{-\frac{\varepsilon}{3}}). \tag{4.6}$$

Recall that in general to each one parameter subgroups of the symmetry group of a system there will correspond a family of solutions called invariant solutions[12]. Consequently, we can state the following theorem:

Theorem 4.1. *If $u = f(x, t)$ is a solution of $K(m, n)$ equation, so are the functions*

$$H_1(\varepsilon)f(x, t) = f(x - \varepsilon, t),$$
$$H_2(\varepsilon)f(x, t) = f(x, t - \varepsilon),$$
$$H_3(\varepsilon)f(x, t) = f(xe^{-\frac{\varepsilon}{3}}, te^{-\varepsilon})e^{-\frac{\varepsilon}{3}}.$$

5. Optimal system of one-dimensional subalgebras of $K(m, n)$ equation

In this section, we want to divide the set of all invariant solutions of a given differential equation into equivalence classes. If one solution can be mapped to the another solution by a point symmetry of the PDE, then these solutions are equivalent. Classification simplifies the problem of determining all invariant solutions. we need only to find one invariant solution from each class, then the whole class can be constructed by applying the symmetries. This strategy minimizes the effort needed to obtain invariant solutions [17].

Definition 5.1. The solutions $u = f(x)$ and $u = \bar{f}(x)$ are equivalent if a symmetry maps one to the other. Similarly, the symmetry maps X to $\bar{X}$, so these generators are regarded as equivalent. It is important to classify invariant solutions by classifying the associated symmetry generators. Having done this, one generator from each class is used to obtain the desired set of invariant solutions. A set consisting of exactly one generator from each class is called an optimal system of generators [8].

Theorem 5.1. *Let F and $\bar{F}$ be connected ε-dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras $\mathfrak{f}$ and $\bar{\mathfrak{f}}$ of the Lie algebra $\mathfrak{g}$ of G. Then $\bar{F} = gFg^{-1}$ are conjugate subgroups if and only if $\bar{\mathfrak{f}} = Ad(g(\mathfrak{f}))$ are conjugate subalgebras* [18].

By theorem 5.1, the problem of finding an optimal system of subgroups is equivalent to find an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by nonzero vector in $\mathfrak{g}$. Each $X_i, i = 1, 2, 3$, of the basis symmetries generates an adjoint representation $Ad(exp(\varepsilon X_i))$ defined by the Lie series

$$Ad(exp(\varepsilon.X_i).X_j) = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2}[X_i, [X_i, X_j]] - ..., \tag{5.1}$$

where $[X_i, X_j]$ is the commutator for the Lie algebra, ε is a parameter, and i, j = 1, 2 , 3 [18]. All the adjoint representations of the $K(m, n)$ Lie group, with the (i, j) the entry indicating $Ad(exp(\varepsilon X_i))X_j$ are given In table 2.

$Ad(exp(\varepsilon.X_i).X_j)$	X_1	X_2	X_3
X_1	X_1	X_2	$-\frac{1}{3}X_1 + X_3$
X_2	X_1	X_2	$-\varepsilon X_2 + X_3$
X_3	$\frac{1}{3}X_1$	$(1+\varepsilon)X_2$	X_3

Table 2: Adjoint representation generated by the basis symmetries of the $K(m, n)$ Lie algebra.

Theorem 5.2. *An optimal system of one-dimensional Lie algebras of $K(m, n)$ equation is provided by*

$$(i) : X_2, \quad (ii) : X_3, \quad (iii) : X_1 + X_2, \quad (iv) : X_2 - X_1 \tag{5.2}$$

Proof. Consider the symmetry algebra $\mathfrak{g}$ of the Eq.(1.4) whose adjoint representation was determined in table 2 and let $H_i^s : \mathfrak{g} \to \mathfrak{g}$ defined by $X \to Ad(exp(\varepsilon X_i)X)$ is a linear map, for i = 1, 2, 3. The matrices $M_i^\varepsilon, i = 1, 2, 3$, with respect to basis $\{X_1, X_2, X_3\}$ are

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & \frac{1}{3}\varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \varepsilon \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3^\varepsilon = \begin{pmatrix} e^{-\varepsilon} & 0 & 0 \\ 0 & e^{-\varepsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.3}$$

Let

$$X = a_1X_1 + a_2X_2 + a_3X_3, \tag{5.4}$$

is a nonzero vector. We will simplify as many of the coefficients a_1, a_2 and a_3 as possible through proper adjoint applications on X.
Assume first that $a_3 \neq 0$. We can suppose $a_3 = 1$. According to Table 2, if we accomplish on such a X by $Ad(exp(a_2X_2)X$, we can make the coefficient of X_2 vanish:

$$X'^a = Ad(exp(a_2X_2))X = a_1'X_1 + X_3.$$

Then we accomplish on X'^a using $Ad(exp(3\varepsilon X_1))X'^a$ to vanish the coefficient X_1, so that X is equivalent to $X''^a = X_3$ under the adjoint representation. The remaining one-dimensional subalgebras are spanned by vectors of the above form with $a_3 = 0$. If $a_2 \neq 0$, we put $a_2 = 1$, and then we have

$$X'^b = a_1X_1 + X_2.$$

We can further act on X'^b using the group generated by X_3, this has the net effect of scaling the coefficient of X_1 and X_2:

$$X''^b = Ad(exp(-\varepsilon X_3))X'^b = \frac{1}{3}a_1X_1 + (1+\varepsilon)X_2.$$

This is a scalar multiple of $X'''^b = \frac{1}{3(1+\varepsilon)}a_1X_1 + X_2$, so, depending on the sign of a_1, we can make the coefficient of X_1 either +1, -1 or 0. Thus any one-dimensional subalgebra spanned by X with $a_3 = 0$ and $a_2 \neq= 0$ is equivalent to one spanned by either X_2, $X_1 + X_2$ and $X_2 - X_1$. The further simplifications are not possible. Then an optimal system of the $K(m,n)$ equation is given by

$$(i) : X_2, \quad (ii) : X_3, \quad (iii) : X_1 + X_2, \quad (iv) : X_2 - X_1 \tag{5.5}$$

6. The methodology of $\frac{G'}{G'+G+A}$-expansion method

In this section, we briefly explain the $\frac{G'}{G'+G+A}$-expansion method to obtain exact solutions of the following type of PDEs [7]:

$$R(\varphi, \varphi_x, \varphi_t, \varphi_{xx}, \varphi_{xt}, \varphi_{tt}, \ldots) = 0, \tag{6.1}$$

where $\varphi = \varphi(x,t)$ is an unknown function. After that we use this method to solve a type of $K(m,n)$ equations.
The equation (6.1) can be solved using the following steps:

- Substituting the following transformation

$$\xi = x - kt, \quad k \neq 0, \tag{6.2}$$

  into (6.1), it can be transformed to the following ODE

$$\tilde{R}(\Phi, \Phi', \Phi'', \Phi''', ...) = 0. \tag{6.3}$$

  Here prime denotes the derivative with respect to ξ. On rare occasions, integrating equation (6.3) can be used to adapt the NODE to the homogeneous balancing principle.
- Consider the solution of Eq. (6.3) be:

$$\Phi(\xi) = \sum_{i=0}^{m} a_i(\frac{G'}{G'+G+A})^i, \tag{6.4}$$

  where $a_i\,(a_m \neq 0)$ are constants and $G(\xi)$ is the general solution of the following ODE:

$$G''(\xi) + B\,G'(\xi) + C\,G(\xi) + A\,C = 0, \tag{6.5}$$

  where A, B and C are constants.
- The positive integer m used in (6.4) can be derived by homogeneously balancing the biggest nonlinear component and the highest-order derivative in equation (6.3).
- The terms of $G(\xi)$ are then brought together into similar orders when (6.4) is inserted into (6.4) or the equation that emerges from the integration of (6.4). Applying this process yields an expression in terms of $\frac{G'}{G'+G+A}$. Putting the coefficients in this expression to zero, we then get an algebraic system of equations representing the variables a_i and other related parameters.

- Using Mathematica, the system of nonlinear algebraic equations is analytically solved.
- Analytical soliton solutions for (6.1) are then derived by computing and putting the unknown values into equation (6.4) along with the $G(\xi)$ (the equation (6.5) solutions).

We know that the Eq. (6.5) has the following special solutions:
Case1. If $D = B^2 - 4C > 0$:

$$\frac{G'}{G'+G+A} = 1 + \frac{2c_1 + 2c_2 e^{\sqrt{D}\xi}}{c_1(-2+B+\sqrt{D}) + c_2(-2+B-\sqrt{D})e^{\sqrt{D}\xi}}. \tag{6.6}$$

Case2. If $D = B^2 - 4C < 0$:

$$\frac{G'}{G'+G+A} = \frac{(Bc_1 - \sqrt{-D}c_2)\cos(\frac{\sqrt{-D}\xi}{2}) + (Bc_2 + \sqrt{-D}c_1)\sin(\frac{\sqrt{-D}\xi}{2})}{((B-2)c_1 - \sqrt{-D}c_2)\cos(\frac{\sqrt{-D}\xi}{2}) + ((B-2)c_2 - \sqrt{-D}c_1)\sin(\frac{\sqrt{-D}\xi}{2})}. \tag{6.7}$$

Now we apply this method for the following type of $K(m,n)$ equation

$$u_t + ru^2u_x + su_{xxx} = 0,. \tag{6.8}$$

Using the transformation (6.2), we convert the above equation into the following ODE

$$-k\Phi'(\xi) + r\Phi^2(\xi)\Phi'(\xi) + s\Phi'''(\xi) = 0. \tag{6.9}$$

Balancing Φ''' with $\Phi^2\Phi'$ in (6.9) gives $m = 1$. Therefore, the exact solution of Eq. (6.9) can be written in the form:

$$\Phi(\xi) = a_0 + a_1(\frac{G'}{G'+G+A}), \quad a_1 \neq 0. \tag{6.10}$$

Based on the description of this method, we have

$$\bullet Set\,1: \quad a_0 = \frac{\sqrt{k+4Cs}}{r},\ a_1 = -\frac{3Cs}{\sqrt{r(k+4Cs)}},$$
$$B = 0,\ C = \frac{\sqrt{2s(8s-3k)} - 4s}{3s}. \tag{6.11}$$

Using (6.11), (6.10) and cases (6.6)-(6.7) respectively, we get

$$\varphi_1(x,t) = \frac{\sqrt{k+4Cs}}{r} - \frac{3Cs}{\sqrt{r(k+4Cs)}} \left(1 + \frac{2c_1 + 2c_2 e^{\sqrt{D}\xi}}{c_1(-2+B+\sqrt{D}) + c_2(-2+B-\sqrt{D})e^{\sqrt{D}\xi}}\right),$$

$$\varphi_2(x,t) = \frac{\sqrt{k+4Cs}}{r} - \left(\frac{(Bc_1 - \sqrt{-D}c_2)\cos(\frac{\sqrt{-D}\xi}{2}) + (Bc_2 + \sqrt{-D}c_1)\sin(\frac{\sqrt{-D}\xi}{2})}{((B-2)c_1 - \sqrt{-D}c_2)\cos(\frac{\sqrt{-D}\xi}{2}) + ((B-2)c_2 - \sqrt{-D}c_1)\sin(\frac{\sqrt{-D}\xi}{2})}\right) \times \frac{3Cs}{\sqrt{r(k+4Cs)}}.$$

$$\bullet Set\,2: \qquad A_0=\frac{\sqrt{k}}{\sqrt{r}},\ A_1=\frac{3s(3C-B)}{\sqrt{rk}},$$
$$B=\frac{27C^2s+12Cs+8Ck+2k}{8k+18Cs},\ C=C. \qquad (6.12)$$

Using (6.12), (6.10) and cases (6.6)-(6.7) respectively, we get

$$\varphi_3(x,t)=\frac{\sqrt{k}}{\sqrt{r}}+\frac{3s(3C-B)}{\sqrt{rk}}\left(1+\frac{2c_1+2c_2e^{\sqrt{D}\xi}}{c_1(-2+B+\sqrt{D})+c_2(-2+B-\sqrt{D})e^{\sqrt{D}\xi}}\right),$$

$$\varphi_4(x,t)=\frac{\sqrt{k}}{\sqrt{r}}+\left(\frac{(Bc_1-\sqrt{-D}c_2)\cos(\frac{\sqrt{-D}\xi}{2})}{((B-2)c_1-\sqrt{-D}c_2)\cos(\frac{\sqrt{-D}\xi}{2})}\frac{+(Bc_2+\sqrt{-D}c_1)\sin(\frac{\sqrt{-D}\xi}{2})}{+((B-2)c_2-\sqrt{-D}c_1)\sin(\frac{\sqrt{-D}\xi}{2})}\right)\times\frac{3s(3C-B)}{\sqrt{rk}}.$$

7. Conclusion

In this paper, we used the criterion of invariance of the equation under the infinitesimal prolonged infinitesimal generators and we found the Lie point symmetries group of the $K(m,n)$-type equations. Also, we have obtained the optimal system of $K(m,n)$ equation. Then, we constructed the classification of group invariant solutions. Exact solutions for a type of $K(m,n)$ equations have been obtained using an algebraic method.

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