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POWER BOUNDED OPERATORS ON HILBERT SPACE AND HELSON SET

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Abstract. Let T be a power bounded operator on a Hilbert space H and assume that local unitary spectrum of T at $x \in H$ is contained in a Helson set. If $\lim_{|m-n| \to \infty} |\langle T^m x, T^n x\rangle| = 0$, then $\lim_{n\to\infty} \|T^n x\| = 0$.

1. Introduction and preliminaries

Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded, linear operators on X. As usual, by $\sigma(T)$ we denote the spectrum of $T \in B(X)$ and by $R(z,T) := (zI - T)^{-1}$ $(z \notin \sigma(T))$, the resolvent of T. The unit circle in the complex plane will be denoted by $\mathbb{T}$, whereas $\mathbb{D}$ indicates the open unit disc.

An operator $T \in B(X)$ is said to be *power bounded* if there exists a constant $C > 0$ such that $\sup_{n\geq 0} \|T^n\| < \infty$. By changing to an equivalent norm given by

$$\|x\|_1 := \sup_{n\geq 0} \|T^n x\| \quad (x \in X)$$

a power bounded operator T can be made contractive, that is, $\|T\| \leq 1$. If T is a contraction on X, then for every $x \in X$, the limit $\lim_{n\to\infty} \|T^n x\|$ exists and is equal to $\inf_{n\geq 0} \|T^n x\|$. If $T \in B(X)$ is power bounded, then $\sigma(T) \subset \overline{\mathbb{D}}$. The set $\sigma_u(T) := \sigma(T) \cap \mathbb{T}$ is called *unitary spectrum* of T.

For an arbitrary $T \in B(X)$ and $x \in X$, we define $\rho_T(x)$ to be the set of all $\lambda \in \mathbb{C}$ for which there exists a neighborhood U_λ of λ with $u(z)$ analytic on U_λ having values in X, such that $(zI - T)u(z) = x$ for all $z \in U_\lambda$. This set is open and contains the resolvent set $\rho(T)$ of T. By definition, the local spectrum of T at x, denoted by $\sigma_T(x)$ is the complement of $\rho_T(x)$, so it is a closed subset of $\sigma(T)$. Notice that local spectrum of an operator may be "very small" with respect to its usual spectrum. To see this, let $T \in B(X)$ and assume that σ is a "very small" clopen part of $\sigma(T)$. Let P_σ be the spectral projection associated with σ and let $X_\sigma := P_\sigma X$. Then, X_σ is a closed T-invariant subspace of X and $\sigma(T|_{X_\sigma}) = \sigma$, where $T|_{X_\sigma}$ is the restriction of T to X_σ. It is easy to check that $\sigma_T(x) \subset \sigma$ for every $x \in X_\sigma$.

The set $\sigma_T(x) \cap \mathbb{T}$ will be called *local unitary spectrum* of $T \in B(X)$ at $x \in X$. Notice that if T is power bounded, then $\sigma_T(x) \cap \mathbb{T}$ consists of all $\xi \in \mathbb{T}$ such that the function $R(z,T)x$ $(|z| > 1)$ has no analytic extension to a neighborhood of ξ. Consider the case where U is a unitary operator on a Hilbert space H. Let

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$E(\cdot)$ be the spectral measure of U. For a given $x \in H$, let μ_x be the vector-measure defined on the Borel subsets of $\mathbb{T}$ by $\mu_x(\Delta) = E(\Delta)x$. One can see that $\sigma_U(x) = \operatorname{supp}\mu_x$.

An operator $T \in B(X)$ is called *strongly stable* if $\lim_{n\to\infty} \|T^n x\| = 0$ for all $x \in X$. Generally speaking, the asymptotic behavior of the orbits $\{T^n x : n = 0, 1, 2, ...\}$ is frequently related to unitary spectrum of underlying operator. This is well illustrated by the following result of Arendt-Batty-Lyubich-Phóng (ABLP) [1, Theorem 5.15]. A power bounded operator T on a Banach space is strongly stable if the unitary spectrum of T is at most countable and T^* has no unitary eigenvalues.

Recall that a contraction on a Hilbert space is said to be *completely nonunitary* if it has no proper reducing subspace on which it acts as a unitary operator. It follows from the Sz.-Nagy-Foiaş theorem [2, Ch.II, Theorem 3.9] that if T is a completely nonunitary contraction on a Hilbert space H, then T is weakly stable, that is, $\lim_{n\to\infty}\langle T^n x, y\rangle = 0$ for all $x, y \in H$. The another result of Sz.-Nagy-Foias [6, Ch.2, Proposition 6.7] asserts that if the unitary spectrum of the completely non-unitary contraction T is of Lebesgue measure zero, then T is strongly stable. For related results see, [2, 4, 5, 6, 8].

Let H be a Hilbert space. In this note, for the individual stability of $T \in B(H)$ at $x \in H$, some sufficient conditions on the local unitary spectrum of T at x will be given.

2. The main result

For a closed subset S of $\mathbb{T}$, we denote by $C(S)$ the space of all continuous functions on S. The classical *Wiener algebra* $A(\mathbb{T})$ is defined by

$$A(\mathbb{T}) = \left\{ f \in C(\mathbb{T}) : \|f\|_1 = \sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right| < \infty \right\},$$

where $\widehat{f}(n)$ is the n'th Fourier coefficient of f. We denote by $A(S)$ the algebra of all functions on S which are the restrictions to S of functions in $A(\mathbb{T})$, with the norm

$$\|f\|_{A(S)} = \inf\{\|g\|_1 : g\mid_S = f, \quad g \in A(\mathbb{T})\}.$$

Recall that S is called a *Helson set* if every continuous function on S can be represented as an absolutely convergent Fourier series. Thus, S is a Helson set if $A(S) = C(S)$. Note that a Helson set is of Lebesgue measure zero. The examples of Helson sets can be found in [3] and [9, Chapter 5]. For example, countable compact independent subset of $\mathbb{T}$ is a Helson set [9, Chapter 5].

Let $M(\mathbb{T})$ denote the space of all finite regular complex Borel measures on $\mathbb{T}$. The n'th Fourier coefficient of $\mu \in M(\mathbb{T})$ is defined by

$$\widehat{\mu}(n) = \int_0^{2\pi} e^{-int} d\mu(t) \quad (n \in \mathbb{Z}).$$

It is well known that if $\widehat{\mu}(n) = 0$ for all $n \in \mathbb{Z}$, then $\mu = 0$.

The Helson Theorem [9, Theorem 5.6.10] asserts the following.

Theorem 2.1. *Assume that the support of the measure $\mu \in M(\mathbb{T})$ is contained in a Helson set. If* $\lim_{|n|\to\infty} |\widehat{\mu}(n)| = 0$*, then* $\mu = 0$.

As an application of the Helson theorem, we have the following.

Theorem 2.2. *Let T be a power bounded operator on a Hilbert space H and assume that $\sigma_T(x) \cap \mathbb{T}$ is contained in a Helson set for some $x \in H$. If*

$$\lim_{|m-n|\to\infty} |\langle T^m x, T^n x\rangle| = 0,$$

then $\lim_{n\to\infty} \|T^n x\| = 0$.

Let S be the unilateral shift operator on the Hardy space $H^2 := H^2(\mathbb{D})$; $Sf = zf$. Then we have

$$|\langle S^m f,\ S^n f\rangle| = \left| \int_0^{2\pi} e^{i|m-n|t} |f(t)|^2 \, dt \right| \quad \text{for all } f \in H^2.$$

Since S is an isometry on H^2, it is not strongly stable. On the other hand, since $|f|^2 \in L^1[0, 2\pi]$, by the Riemann-Lebesgue lemma,

$$\lim_{|m-n|\to\infty} |\langle S^m f,\ S^n f\rangle| = 0.$$

We can say more.

Proposition 2.1. *If V is a completely nonunitary isometry on a Hilbert space H, then*

$$\lim_{|m-n|\to\infty} |\langle V^m x, V^n x\rangle| = 0 \quad \textit{for all } x \in H.$$

Proof. It follows from the Wold's Decomposition Theorem [6, Ch.1, Theorem 1.1] that a completely nonunitary isometry is unitary equivalent to the unilateral shift operator and therefore,

$$\lim_{n\to\infty} \|V^{*n} x\| = 0 \quad \text{for all } x \in H.$$

Now, it follows from the identity

$$|\langle V^m x, V^n x\rangle| = \left| \langle V^{*|m-n|} x, x\rangle \right|$$

that $\lim_{|m-n|\to\infty} |\langle V^m x, V^n x\rangle| = 0$. □

Recall that a unitary operator U on a Hilbert space H is said to be *absolutely continuous* (resp. *singular*) if the spectral measure $E(\cdot)$ of U is absolutely continuous (resp. singular) with respect to the Lebesgue measure on $\mathbb{T}$. There exist direct sum decomposition $H = H_{ac} \oplus H_s$ and $U = U_{ac} \oplus U_s$ of H and U such that H_{ac} and H_s are U-reducing subspaces with $U_{ac} = U\,|_{H_{ac}}$ and $U_s = U\,|_{H_s}$ respectively, absolutely continuous and singular. For $x \in H$, let μ_x be the scalar measure defined on the Borel subsets of $\mathbb{T}$ by

$$\mu_x(\Delta) = \langle E(\Delta) x, x\rangle = \|E(\Delta) x\|^2.$$

If U is absolutely continuous, then for an arbitrary $x \in H$, there is $f_x \in L^1[0, 2\pi]$ such that $d\mu_x(t) = f_x(t)\, dt$. Then, we can write

$$|\langle U^m x,\ U^n x\rangle| = \left| \int_0^{2\pi} e^{i|m-n|t} d\mu_x(t) \right| = \left| \int_0^{2\pi} e^{i|m-n|t} f_x(t)\, dt \right|.$$

From the last identity and from the Riemann-Lebesgue lemma, we have

$$\lim_{|m-n|\to\infty} |\langle U^m x,\ U^n x\rangle| = 0.$$

For the proof of Theorem 2.2, we need some preliminary results.

For convenience, by T_E we will denote the restriction of $T \in B(H)$ to the invariant subspace E of T. The following lemma was proved in [5, Lemma 2.1].

Lemma 2.1. *Let T be a contraction on a Hilbert space H and let E be a (closed) T-invariant subspace of H. Then, for every $x \in E$, we have*

$$\sigma_{T_E}(x) \cap \mathbb{T} = \sigma_T(x) \cap \mathbb{T}.$$

As an illustration of Lemma 2.1, consider the following example. Let K be a Hilbert space and let $H^2(K)$ be the Hardy space of K-valued analytic functions on $\mathbb{D}$. By S_K, we denote the unilateral shift operator on $H^2(K)$;

$$(S_K f)(z) = z f(z), \quad f \in H^2(K).$$

Its adjoint, the backward shift, is given by

$$(S_K^* f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in H^2(K).$$

It is easy to verify that for every $f \in H^2(K)$ and $\lambda \in \mathbb{C}$ with $|\lambda| > 1$,

$$(\lambda I - S_K^*)^{-1} f(z) = \frac{\lambda^{-1} f(\lambda^{-1}) - z f(z)}{1 - \lambda z}.$$

It follows that $\sigma_{S_K^*}(f) \cap \mathbb{T}$ consists of all $\xi \in \mathbb{T}$ such that the function f has no analytic extension to a neighborhood of ξ.

Now, let T be a contraction on a Hilbert space H such that $\lim_{n\to\infty} \|T^n x\| = 0$ for every $x \in H$. Let

$$D := (I - T^*T)^{\frac{1}{2}} \quad \text{and} \quad K := \overline{DH}.$$

By the well-known Model Theorem of Sz.-Nagy-Foias (see [6] and [7]), there exists S_K^*-invariant subspace E of $H^2(K)$ and a unitary operator $U : H \mapsto E$ such that

$$T = U^{-1}(S_K^* |_E) U,$$

where

$$Ux = \sum_{n=0}^{\infty} z^n D T^n x, \quad x \in H.$$

It follows from Lemma 2.1 that if $x \in H$, then

$$\sigma_T(x) \cap \mathbb{T} = \sigma_{S_K^*|_E}(Ux) \cap \mathbb{T} = \sigma_{S_K^*}(Ux) \cap \mathbb{T}.$$

Hence, $\sigma_T(x) \cap \mathbb{T}$ consists of all $\xi \in \mathbb{T}$ such that Ux has no an analytic extension to a neighborhood of ξ.

Recall that $V \in B(X)$ is called an *isometry* if $\|Vx\| = \|x\|$ for all $x \in X$. It is well known that if V is a non-unitary isometry, then $\sigma(V) = \overline{\mathbb{D}}$. A vector $x \in X$ is a *cyclic vector* of $T \in B(X)$ if the smallest closed subspace of X containing $\{T^n x,\ n = 0, 1, 2, ...\}$ is the whole space X.

The following result for the Banach space isometries was proved in [1, Lemma 1.3]. For the Hilbert space isometries we present more elementary proof.

Lemma 2.2. *Let V be an isometry on a Hilbert space H. If $x \in H$ is a cyclic vector of V, then*

$$\sigma_u(V) = \sigma_V(x) \cap \mathbb{T}.$$

Proof. Assume that V is a unitary operator. We must show that $\sigma(V) = \sigma_V(x)$. By Spectral Theorem, there exists a positive measure μ on $\mathbb{T}$ such that the operator M on $L^2(\mathbb{T}, \mu)$ defined by $Mf = e^{it}f$ is unitary equivalent to V. Let χ_Δ denote the characteristic function of any Borel subset Δ of $\mathbb{T}$ and let $\mathbf{1}$ be the constant one function on $\mathbb{T}$. Then, we have $\sigma(V) = \mathrm{supp}\mu$ and $\sigma_V(x) = \mathrm{supp}\nu$, where ν is a vector measure on $\mathbb{T}$ defined by $\nu(\Delta) = \chi_\Delta \mathbf{1}$. Since $\|\nu(\Delta)\| = \mu(\Delta)$, we have $\mathrm{supp}\mu = \mathrm{supp}\nu$ and therefore, $\sigma(V) = \sigma_V(x)$.

Now, assume that $VH \neq H$. In this case, $\sigma(V) = \overline{\mathbb{D}}$. Let us show that $\sigma_V(x) = \overline{\mathbb{D}}$. Let $K = H \ominus VH$. By Wold's Decomposition Theorem [6, Ch.1, Theorem 1.1], there exists a decomposition $H = H_0 \oplus H_1$ such that H_0 and H_1 reduce V, $V_0 = V\mid_{H_0}$ is unitary and $V_1 = V\mid_{H_1}$ is unitary equivalent to the unilateral shift operator S_K on $H^2(K)$. Notice that $\sigma_{S_K}(f) = \overline{\mathbb{D}}$ for every nonzero $f \in H^2(K)$. It follows that if $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1 \setminus \{0\}$, then $\sigma_{V_1}(x_1) = \overline{\mathbb{D}}$. On the other hand, it is easy to verify that $\sigma_{V_1}(x_1) \subset \sigma_V(x)$. So, we have $\sigma_V(x) = \overline{\mathbb{D}}$. □

In the following result we use the method of [1, 4, 8] to construct an isometry on a different Hilbert space.

Lemma 2.3. *If T is a contraction on a Hilbert space H, then there exists a Hilbert space K, a linear contraction $J : H \to K$ with dense range, and an isometry V on K with the following properties:*
(a) $\langle Jx, Jy\rangle = \lim_{n\to\infty}\langle T^n x, T^n y\rangle$ for all $x, y \in H$.
(b) $VJ = JT$.
(c) $\sigma(V) \subset \sigma(T)$.

The triple (K, J, V) will be called *limit isometry* associated with T.

Lemma 2.4. *Let T be a contractions on a Hilbert space H and let $(K, J, \mathbf{V})$ be the limit isometry associated with T. The following assertions hold:*
(a) $\sigma_V(Jx) \subset \sigma_T(x)$ for all $x \in H$.
(b) If $x \in H$ is a cyclic vector of T, then Jx is a cyclic vector of V.

Proof. (a) If $x \in H$ and $\lambda \in \rho_T(x)$, then there is a neighborhood U_λ of λ with $u(z)$ analytic on U_λ having values in H such that $(zI - T)u(z) = x$ for all $z \in U_\lambda$. Since $(zJ - JT)u(z) = Jx$ and by Lemma 2.3 (b), $JT = VJ$, we have $(zI - V)Ju(z) = Jx$. As $Ju(z)$ is a function analytic on U_λ, we get that $\lambda \in \rho_V(Jx)$.

(b) Let $x \in H$ be a cyclic vector of T and let $y \in H$. Then, for any $\varepsilon > 0$ there are constants $c_1, ..., c_k$ and non-negative integers $n_1, ..., n_k$ such that

$$\|y - c_1 T^{n_1} x - ... - c_k T^{n_k} x\| < \varepsilon,$$

which implies

$$\|Jy - c_1 JT^{n_1} x - ... - c_k JT^{n_k} x\| < \varepsilon.$$

By Lemma 2.3 (b), since $JT^n = V^n J$ $(\forall n \in \mathbb{N})$, we have

$$\|Jy - c_1 V^{n_1} Jx - ... - c_k V^{n_k} Jx\| < \varepsilon.$$

Since the operator J has dense range, it follows from the preceding inequality that Jx is a cyclic vector of V. $\square$

Next, we have the following.

Lemma 2.5. *Let U be a unitary operator on a Hilbert space H and assume that $\lim_{|n|\to\infty} |\langle U^n x, x\rangle| = 0$ for some $x \in H$. If $\sigma_U(x)$ is contained in a Helson set, then $x = 0$.*

Proof. Let $E(\cdot)$ be the spectral measure of U. For $x \in H$, let μ_x be the scalar measure defined on the Borel subsets of $\mathbb{T}$ by

$$\mu_x(\Delta) = \langle E(\Delta) x, x\rangle = \|E(\Delta) x\|^2 .$$

Then, $\sigma_U(x) = \operatorname{supp}\mu_x$ and therefore $\operatorname{supp}\mu_x$ is contained in a Helson set. From the spectral decomposition of U, we can write

$$\begin{aligned} \langle U^n x, x\rangle &= \int_0^{2\pi} e^{int} d\langle E_t x, x\rangle \\ &= \int_0^{2\pi} e^{int} d\mu_x(t) = \widehat{\mu_x}(n) \quad (n \in \mathbb{Z}) . \end{aligned}$$

So we have

$$\lim_{|n|\to\infty} |\widehat{\mu_x}(n)| = 0.$$

Since $\operatorname{supp}\mu_x$ is contained in a Helson set, by Theorem 2.1, $\mu_x = 0$. This clearly implies that $x = 0$. $\square$

Now, we are in a position to prove Theorem 2.2.

Proof of Theorem 2.2. It is no restriction to assume that T is a contraction (renorming does not change the spectral assumptions). Let E be the closed linear span of $\{T^n x : n \geq 0\}$. Then, E is a T-invariant subspace of H. Let (K, J, V) be the limit isometry associated with T_E. By Lemma 2.4 (a), $\sigma_V(Jx) \subset \sigma_{T_E}(x)$, which implies

$$\sigma_V(Jx) \cap \mathbb{T} \subset \sigma_{T_E}(x) \cap \mathbb{T}.$$

Taking into account Lemma 2.1, we have

$$\sigma_V(Jx) \cap \mathbb{T} \subset \sigma_T(x) \cap \mathbb{T}.$$

On the other hand, since Jx is a cyclic vector of V (Lemma 2.4 (b)), by Lemma 2.2 we obtain that

$$\sigma(V) \cap \mathbb{T} = \sigma_V(Jx) \cap \mathbb{T} \subset \sigma_T(x) \cap \mathbb{T}.$$

Consequently, V is a unitary operator and $\sigma(V)$ is contained in a Helson set.

From Lemma 2.3 we can write $V^n Jx = JT^n x$ ($\forall n \in \mathbb{N}$), which implies

$$\langle V^m Jx, V^n Jx\rangle = \langle JT^m x, JT^n x\rangle = \lim_{k\to\infty} \langle T^{m+k} x, T^{n+k} x\rangle.$$

By hypothesis, for an arbitrary $\varepsilon > 0$, there is a natural number N such that for all natural numbers m, n with $|m - n| > N$, we have $|\langle T^m x, T^n x\rangle| \leq \varepsilon$. Therefore,

$$\left|\langle T^{m+k} x, T^{n+k} x\rangle\right| \leq \varepsilon \text{ for all } m, n \text{ with } |m-n| > N \text{ and for all } k \in \mathbb{N}.$$

It follows that $|\langle V^m Jx, V^n Jx\rangle| < \varepsilon$ for all m, n with $|m-n| > N$. This means that

$$\lim_{|m-n|\to\infty} \left|\langle V^{|m-n|}Jx, Jx\rangle\right| = 0.$$

By Lemma 2.5, $Jx = 0$. Taking into account Lemma 2.3 (a), finally we obtain

$$\lim_{n\to\infty} \|T^n x\| = 0.$$

□

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