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EXPLORING THE GEOMETRY OF THE COTANGENT BUNDLE ENDOWED WITH BERGER-TYPE DEFORMED SASAKI METRIC OVER A STANDARD KÄHLER MANIFOLD

ABDERRAHIM ZAGANE

Abstract. In this paper, we introduce the Berger-type deformed Sasaki metric on the cotangent bundle T^*M over a standard Kähler manifold (M^{2m}, J, g). Firstly, we compute all forms of the curvature tensors of the cotangent bundle with this metric and present some results concerning curvature properties. Secondly, we construct some almost Hermitian structures on a cotangent bundle and search conditions for these structures to be integrable. Finally, we study some geometric properties of the unit cotangent bundle that is endowed with the Berger-type deformed Sasaki metric.

1. Introduction

In the literature, one of the first works that deal with the cotangent bundles of a manifold as a Riemannian manifold is that of Patterson and Walker [7], who constructed from an affine symmetric connection on a manifold a Riemannian metric on the cotangent bundle, which they call the Riemann extension of the connection. Sekizawa [9] has given a generalization of this metric in his classification of natural transformations of affine connections on manifolds to metrics on their cotangent bundles, obtaining the class of natural Riemann extensions which is a 2-parameter family of metrics, and which had been intensively studied. This situation has prompted many researchers to study other metrics on the cotangent bundle (see [2, 3, 4, 6]), the most famous of which is the Sasaki metric [8]. In this direction, inspired by the concept of g-natural metrics on tangent bundles of Riemannian manifolds, Ağca considered another class of metrics on cotangent bundles of Riemannian manifolds that she called g-natural metrics [1]. On the other hand, Zagane proposed the Berger-type deformed Sasaki metric on the cotangent bundle over an anti-paraKähler manifold [11, 12, 13].

In previous works [14, 15], we proposed the Berger-type deformed Sasaki metric on the cotangent bundle over a standard Kähler manifold, where we studied the geodesics properties and the harmonicity on cotangent bundle with this respectively. After stating the introduction, we describe the preliminary results of the cotangent bundle. In section 3, we present the basic properties of the

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Berger-type deformed Sasaki metric, and we investigate the formulas relating to its Levi-Civita connection. Section 4 investigates the different types of curvature of the Berger-type deformed Sasaki metric, including the Riemannian curvature tensor, the Ricci curvature, the sectional curvature and the scalar curvature. In section 5, we explore and construct some almost Hermitian structures on the cotangent bundle and search for the integrability conditions of these structures. In the last section, we study the geometry of the unit cotangent bundle endowed with the Berger-type deformed Sasaki metric, where we establish the Levi-Civita connection of this metric and all forms of its Riemannian curvature tensors.

2. Preliminary Results

Let (M^m, g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ the natural projection. A local chart $(U, x^i)_{i=1,\ldots,m}$ on M induces a local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i), \bar{i} = m+1, \ldots, 2m$ on T^*M, where p_i is the component of covector p in each cotangent space T^*_xM, $x \in U$ with respect to the natural coframe $\{dx^i\}$. Let $\Im^r_s(M)$ (resp. $\Im^r_s(T^*M)$) the module of C^∞ tensor fields of type (r, s) over the ring of real-valued C^∞ functions on M(resp. T^*M). We denote by ∇ the Levi-Civita connection of g and by Γ^k_{ij} its Christoffel symbols.

The Levi Civita connection ∇ defines a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M \tag{2.1}$$

of TT^*M into two complementary distributions, called the vertical distribution VT^*M and the horizontal distribution HT^*M, respectively defined by:

$$\begin{aligned} V_{(x,p)}T^*M &= \ker(d\pi_{(x,p)}) = \{\omega_i \partial_{\bar{i}}|_{(x,p)},\ \omega_i \in \mathbb{R}\}, \\ H_{(x,p)}T^*M &= \{X^i \partial_i|_{(x,p)} + X^i p_a \Gamma^a_{hi} \partial_{\bar{h}}|_{(x,p)},\ X^i \in \mathbb{R}\}, \end{aligned}$$

for all $(x, p) \in T^*M$, where $\partial_i = \dfrac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \dfrac{\partial}{\partial x^{\bar{i}}}$. Note that V and H represent the vertical and horizontal projections on T^*M induces by ∇.

Let $X = X^i \partial_i$ and $\omega = \omega_i dx^i$ be local expressions in $(U, x^i), i = 1, \ldots, m$, of a vector field and covector field $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively. Then the horizontal lift ${}^HX \in \Im^1_0(T^*M)$ of $X \in \Im^1_0(M)$ and the vertical lift ${}^V\omega \in \Im^1_0(T^*M)$ of $\omega \in \Im^0_1(M)$ are defined, respectively by

$$\begin{aligned} {}^HX &= X^i \partial_i + p_h \Gamma^h_{ij} X^j \partial_{\bar{i}}, \\ {}^V\omega &= \omega_i \partial_{\bar{i}}, \end{aligned}$$

with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}\}$, (see [10] for more details).

In particular, we have the vertical distribution ${}^Vp$ on T^*M defined by

$${}^Vp = p_i {}^V(dx^i) = p_i \xi_{\bar{i}},$$

${}^Vp$ is also called the canonical or Liouville vector field on T^*M.

The bracket operation of vertical and horizontal vector fields on T^*M is given by the formulas: [10]

$$\begin{cases} [{}^V\omega, {}^V\theta] = 0, \\ [{}^HX, {}^V\theta] = {}^V(\nabla_X\theta), \\ [{}^HX, {}^HY] = {}^H[X,Y] + {}^V(pR(X,Y)), \end{cases} \tag{2.2}$$

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, such that $pR(X,Y) = p_a R^a_{ijk} X^i Y^j dx^k$, where R^a_{ijk} are local components of R on M.

Let (M^m, g) be a Riemannian manifold, the maps

$$\begin{array}{cccc} \sharp: & \Im^0_1(M) & \to & \Im^1_0(M) \\ & \omega & \mapsto & \sharp(\omega) \end{array} \quad and \quad \begin{array}{cccc} \flat: & \Im^1_0(M) & \to & \Im^0_1(M) \\ & X & \mapsto & \flat(X) \end{array}$$

defined by

$$g(\sharp(\omega), Y) = \omega(Y) \ and \ \flat(X)(Y) = g(X,Y)$$

respectively for all $Y \in \Im^1_0(M)$ are $C^\infty(M)$-linear isomorphism and one is the inverse of the other.

Locally for all $\omega = \omega_i dx^i \in \Im^0_1(M)$ and $X = X^j\partial_j \in \Im^1_0(M)$, we have

$$\sharp(\omega) = g^{ij}\omega_i\partial_j \ \ and \ \ \flat(X) = g_{ij}X^j dx^i$$

where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

In the following, we denote $\sharp(\omega)$ and $\flat(X)$ by $\widetilde{\omega}$ and $\widetilde{X}$ respectively. for all $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$.

For each $x \in M$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $T^*_x M$ by, for all $\omega, \theta \in \Im^0_1(M)$

$$g^{-1}(\omega, \theta) = g(\widetilde{\omega}, \widetilde{\theta}) = g^{ij}\omega_i\theta_j.$$

In this case we have $\widetilde{\omega} = g^{-1} \circ \omega$ and $\widetilde{X} = g \circ X$.

Lemma 2.1. *Let (M^m, g) be a Riemannian manifold, we have the following.*

$$\widetilde{\widetilde{\omega}} = \omega \quad , \quad \widetilde{\widetilde{X}} = X, \tag{2.3}$$

$$g^{-1}(\omega, \theta J) = g(J\widetilde{\omega}, \widetilde{\theta}), \tag{2.4}$$

$$\nabla_X\widetilde{\omega} = \widetilde{\nabla_X\omega}, \tag{2.5}$$

$$\widetilde{\omega R(X,Y)} = -R(X,Y)\widetilde{\omega} \tag{2.6}$$

$$\omega(\nabla_X J) = \nabla_X(\omega J) - (\nabla_X\omega)J, \tag{2.7}$$

$$\omega(\nabla_X R)(Y,Z) = \nabla_X(\omega R(Y,Z)) - (\nabla_X\omega)R(Y,Z) - \omega R(\nabla_X Y, Z) - \omega R(Y, \nabla_X Z), \tag{2.8}$$

for all $X, Y \in \Im^1_0(M)$, $\omega, \theta \in \Im^0_1(M)$ and $J \in \Im^1_1(M)$, where ∇ is the Levi-Civita connection of g (see [12]*).*

3. Berger-type deformed Sasaki metric

Let M^n be an n-dimensional differentiable manifold. An almost complex structure J on M is a $(1,1)$-tensor field on M such that $J^2 = -I$, (I is the $(1,1)$-identity tensor field on M). The pair (M^n, J) is called an almost complex manifold. Since every almost complex manifold is of even dimensional, We will take

$n = 2m$ in the following (in this paper). Also, note that every complex manifold (topological space endowed with a holomorphic atlas) carries a natural almost complex structure [5].

The integrability of a complex structure J on M is equivalent to the vanishing of the Nijenhuis tensor N_J:

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

for all vector fields X, Y on M. On an almost complex manifold (M^n, J), a Hermitian metric is a Riemannian metric g on M such that

$$g(JX, Y) = -g(X, JY) \Leftrightarrow g(JX, JY) = g(X, Y), \tag{3.1}$$

or from (2.4) equivalently

$$g^{-1}(\omega J, \theta) = -g^{-1}(\omega, \theta J) \Leftrightarrow g^{-1}(\omega J, \theta J) = g^{-1}(\omega, \theta), \tag{3.2}$$

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

The almost complex manifold (M^n, J) having the Hermitian metric g is called an almost Hermitian manifold. Let (M^n, J, g) be an almost Hermitian manifold. We define the fundamental or Kähler 2-form Ω on M by

$$\Omega(X, Y) = g(X, JY)$$

for any vector fields X and Y on M. A Hermitian metric g on an almost Hermitian manifold M^n is called a standard Kähler metric if the fundamental 2-form Ω is closed, i.e., $d\Omega = 0$. In the case, the triple (M^n, J, g) is called an almost standard Kähler manifold. If the almost complex structure is integrable, then the triple (M^n, J, g) is called a standard Kähler manifold. Moreover, the following conditions are equivalent:

(1) $\nabla J = 0$, (∇ is the Levi-Civita connection of g)
(2) $\nabla \Omega = 0$,
(3) $N_J = 0$ and $d\Omega = 0$ [5].

As a result, the almost Hermitian manifold (M^n, J, g) is a standard Kähler manifold if and only if $\nabla J = 0$. Using (2.7), we also the almost Hermitian manifold (M^n, J, g) is a standard Kähler manifold if and only if

$$\nabla_X(\omega J) = (\nabla_X \omega)J.$$

for all $X \in \Im^1_0(M)$, $\omega \in \Im^0_1(M)$. The Riemannian curvature tensor R of a standard Kähler manifold possess the following properties:

$$\begin{cases} R(Y, Z)J &= JR(Y, Z), \\ R(JY, JZ) &= R(Y, Z), \\ R(JY, Z) &= -R(Y, JZ), \end{cases} \tag{3.3}$$

for all vector fields Y, Z on M.

Lemma 3.1. *Given an almost Hermitian manifold (M^n, J, g), we have the following:*

$$\widetilde{\omega J} = -J\widetilde{\omega}, \tag{3.4}$$

$$\widetilde{JX} = -\widetilde{X}J, \tag{3.5}$$

for all $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$.

Proof. Let $J = J_h^j \partial_j \otimes dx^h$, $\omega = \omega_k dx^k$, $\theta = \theta_i dx^i$ and $\omega J = \omega_k J_h^k dx^h$ then,

$$\begin{aligned}\widetilde{\omega J} &= \widetilde{\omega J}^j \partial_j = \omega_k J_h^k g^{hj} \partial_j = g^{-1}(\omega_k J_h^k dx^h, dx^j)\partial_j = g^{-1}(\omega J, dx^j)\partial_j,\\ J\widetilde{\omega} &= J_h^j \omega_k g^{hk} \partial_j = g^{-1}(\omega_k dx^k, J_h^j dx^h)\partial_j = g^{-1}(\omega, dx^j J)\partial_j,\end{aligned}$$

from (3.2), we find (3.4). □

Definition 3.1. [14] Given an almost Hermitian manifold (M^n, J, g) and its cotangent bundle T^*M. A fiber-wise Berger-type deformation of the Sasaki metric noted ${}^{BS}g$ is defined on T^*M by:

$$\begin{aligned}{}^{BS}g({}^HX, {}^HY) &= g(X,Y),\\ {}^{BS}g({}^HX, {}^V\theta) &= 0,\\ {}^{BS}g({}^V\omega, {}^V\theta) &= g^{-1}(\omega,\theta) + \delta^2 g^{-1}(\omega, pJ) g^{-1}(\theta, pJ),\end{aligned}$$

for all $X, Y \in \Im_0^1(M)$, $\omega, \theta \in \Im_1^0(M)$, where δ is some constant, (for anti-paraKähler manifold, see [12]).

In the following, we put $\lambda = 1 + \delta^2\alpha$ and $\alpha = g^{-1}(p,p) = |p|^2$. where $|.|$ denote the norm with respect to g^{-1}.

The Levi-Civita connection ${}^{BS}\nabla$ of T^*M with Berger-type deformed Sasaki metric ${}^{BS}g$ is given by the following theorem:

Theorem 3.1. [14] *Given a standard Kähler manifold (M^n, J, g) and its cotangent bundle $(T^*M, {}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. Then we have:*

$$\begin{aligned}{}^{BS}\nabla_{{}^HX}{}^HY &= {}^H(\nabla_X Y) + \frac{1}{2}{}^V(pR(X,Y)),\\ {}^{BS}\nabla_{{}^HX}{}^V\theta &= {}^V(\nabla_X\theta) + \frac{1}{2}\big({}^H(R(\tilde p, \widetilde{\theta})X) + \delta^2 g^{-1}(\theta, pJ)\,{}^H(R(J\tilde p, \tilde p)X)\big),\\ {}^{BS}\nabla_{{}^V\omega}{}^HY &= \frac{1}{2}\big({}^H(R(\tilde p, \widetilde{\omega})Y) + \delta^2 g^{-1}(\omega, pJ)\,{}^H(R(J\tilde p, \tilde p)Y)\big),\\ {}^{BS}\nabla_{{}^V\omega}{}^V\theta &= \delta^2\big(g^{-1}(\omega, pJ)\,{}^V(\theta J) + g^{-1}(\theta, pJ)\,{}^V(\omega J)\big)\\ &\quad -\frac{\delta^4}{\lambda}\big(g^{-1}(\omega, pJ) g^{-1}(\theta, p) + g^{-1}(\omega, p) g^{-1}(\theta, pJ)\big)\,{}^V(pJ),\end{aligned}$$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, where ∇ is the Levi-Civita connection of (M^n, J, g) and R is its curvature tensor.

Lemma 3.2. *Given a standard Kähler manifold (M^n, J, g) and its cotangent bundle $(T^*M, {}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. Then*

$$\begin{aligned}{}^{BS}\nabla_{{}^HX}{}^Vp &= 0,\\ {}^{BS}\nabla_{{}^Vp}{}^HX &= 0,\\ {}^{BS}\nabla_{{}^V\omega}{}^Vp &= {}^V\omega + \frac{\delta^2}{\lambda} g^{-1}(\omega, pJ)\,{}^V(pJ),\\ {}^{BS}\nabla_{{}^Vp}{}^V\omega &= \frac{\delta^2}{\lambda} g^{-1}(\omega, pJ)\,{}^V(pJ),\\ {}^{BS}\nabla_{{}^Vp}{}^Vp &= {}^Vp,\end{aligned}$$

for all $X \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$.

As a direct consequence of Theorem 3.1, we get the following lemma.

Lemma 3.3. *Given a standard Kähler manifold (M^n, J, g) and its cotangent bundle $(T^*M, {}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. Then*

$$\begin{aligned}
{}^{BS}\nabla_{{}^HX}{}^V(pJ) &= \frac{\lambda}{2}{}^H(R(J\tilde{p},\tilde{p})X),\\
{}^{BS}\nabla_{{}^V\omega}{}^V(pJ) &= \lambda^V(\omega J) - \delta^2 g^{-1}(\omega, pJ)^V p - \frac{\delta^2(\lambda-1)}{\lambda}g^{-1}(\omega,p)^V(pJ),\\
{}^{BS}\nabla_{{}^HX}{}^V(\theta J) &= {}^V((\nabla_X\theta)J) + \frac{1}{2}\big({}^H(R(J\tilde{p},\widetilde{\theta})X) + \delta^2 g^{-1}(\theta,p)^H(R(J\tilde{p},\tilde{p})X)\big),\\
{}^{BS}\nabla_{{}^V\omega}{}^V(\theta J) &= \delta^2\big(g^{-1}(\theta,p)^V(\omega J) - g^{-1}(\omega,pJ)^V\theta\big)\\
&\quad -\frac{\delta^4}{\lambda}\big(g^{-1}(\omega,p)g^{-1}(\theta,p) - g^{-1}(\omega,pJ)g^{-1}(\theta,pJ)\big)^V(pJ),
\end{aligned}$$

for all $X \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

Definition 3.2. Given a standard Kähler manifold (M^n, J, g) and its cotangent bundle $(T^*M, {}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. Let F be a $(1,1)$-tensor field on M and K be a $(1,2)$-tensor field on M. Then we define the vertical and horizontal lifts ${}^VF$, ${}^HF$ and ${}^HK$ respectively on T^*M as follows:

$$\begin{aligned}
{}^VF: \; T^*M &\to TT^*M\\
(x,p) &\mapsto {}^V(pF_x)\\
{}^HF: \; T^*M &\to TT^*M\\
(x,p) &\mapsto {}^H(F_x(\tilde{p}))\\
{}^HK: \; T^*M &\to TT^*M\\
(x,p) &\mapsto {}^H(K_x(J\tilde{p},\tilde{p})).
\end{aligned}$$

Locally, we have:

$$\begin{aligned}
{}^V(pF) &= p_i{}^V(dx^iF),\\
{}^H(F(\tilde{p})) &= \tilde{p}^i{}^H(F\partial_i),\\
{}^H(K(J\tilde{p},\tilde{p})) &= \tilde{p}^i\tilde{p}^j{}^H(K(J\partial_i,\partial_j)).
\end{aligned}$$

Proposition 3.1. *Given a standard Kähler manifold (M^n, J, g) and its cotangent bundle $(T^*M, {}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. Let F be a $(1,1)$-tensor field on M and K be a $(1,2)$-tensor field on M, then:*

$$\begin{aligned}
{}^{BS}\nabla_{{}^HX}{}^H(F(\tilde{p})) &= {}^H((\nabla_XF)(\tilde{p})) + \frac{1}{2}{}^V(pR(X,F(\tilde{p}))),\\
{}^{BS}\nabla_{{}^HX}{}^V(pF) &= {}^V(p(\nabla_XF)) + \frac{1}{2}{}^H(R(\tilde{p},\widetilde{pF})X) + \frac{\delta^2}{2}g^{-1}(pF,pJ)^H(R(J\tilde{p},\tilde{p})X),\\
{}^{BS}\nabla_{{}^V\omega}{}^H(F(\tilde{p})) &= {}^H(F(\widetilde{\omega})) + \frac{1}{2}{}^H(R(\tilde{p},\widetilde{\omega})F(\tilde{p})) + \frac{\delta^2}{2}g^{-1}(\omega,pJ)^H(R(J\tilde{p},\tilde{p})F(\tilde{p})),\\
{}^{BS}\nabla_{{}^V\omega}{}^V(pF) &= {}^V(\omega F) + \delta^2\big(g^{-1}(\omega,pJ)^V(pFJ) + g^{-1}(pF,pJ)^V(\omega J)\big)\\
&\quad - \frac{\delta^4}{\lambda}\big(g^{-1}(\omega,pJ)g^{-1}(pF,p) + g^{-1}(\omega,p)g^{-1}(pF,pJ)\big)^V(pJ),
\end{aligned}$$

$$
\begin{aligned}
{}^{BS}\nabla_{{}^HX}{}^H(K(J\tilde{p},\tilde{p})) &={}^H((\nabla_X K)(J\tilde{p},\tilde{p})) + \frac{1}{2}{}^V(pR(X,K(J\tilde{p},\tilde{p}))),\\
{}^{BS}\nabla_{{}^V\omega}{}^H(K(J\tilde{p},\tilde{p})) &={}^H(K(J\widetilde{\omega},\tilde{p})) + {}^H(K(J\tilde{p},\widetilde{\omega})) + \frac{1}{2}({}^H(R(\tilde{p},\widetilde{\omega})K(J\tilde{p},\tilde{p}))\\
&\quad + \delta^2 g^{-1}(\omega,pJ){}^H(R(J\tilde{p},\tilde{p})K(J\tilde{p},\tilde{p}))),
\end{aligned}
$$

for all $X\in\Im^1_0(M)$ *and* $\omega\in\Im^0_1(M)$*, where* $\widetilde{pF}=g^{-1}\circ(pF)$*,* $pFJ=p_iF^i_jJ^j_t dx^t$ *and* $p(\nabla_X F)=\nabla_X(pF)-(\nabla_X p)F$*.*

Proof. By the Definition 3.2, Theorem 3.1, (2.5) and (2.7) we have:

$$
\begin{aligned}
{}^{BS}\nabla_{{}^HX}{}^H(F(\tilde{p})) &= {}^{BS}\nabla_{{}^HX}(\tilde{p}^k{}^H(F(\partial_k)))\\
&= X^i\big(\partial_i(\tilde{p}^k)+p_a\Gamma^a_{hi}\partial_{\bar{h}}(\tilde{p}^k)\big){}^H(F(\partial_k)) + \tilde{p}^k{}^H(\nabla_X F(\partial_k))\\
&\quad +\frac{\tilde{p}^k}{2}{}^V\big(pR(X,F(\partial_k))\big)\\
&= {}^H\big(X(\tilde{p}^k)F(\partial_k)+\tilde{p}^k\nabla_X F(\partial_k)\big) + X^ip_a\Gamma^a_{hi}\partial_{\bar{h}}(p_jg^{jk}){}^H(F(\partial_k))\\
&\quad +\frac{1}{2}{}^V\big(pR(X,F(\tilde{p}))\big)\\
&= {}^H(\nabla_X F(\tilde{p})) + X^ip_a\Gamma^a_{ji}g^{jk}{}^H(F(\partial_k)) + \frac{1}{2}{}^V\big(pR(X,F(\tilde{p}))\big).
\end{aligned}
$$

Since the $(\nabla_X p)_j=-X^ip_a\Gamma^a_{ji}$, $\widetilde{dx_j}=g^{jk}\partial_k$, $\widetilde{\nabla_X p}=(\nabla_X p)_j\widetilde{dx_j}$, then:

$$
\begin{aligned}
{}^{BS}\nabla_{{}^HX}{}^H(F(\tilde{p})) &= {}^H(\nabla_X F(\tilde{p})) - {}^H(F(\widetilde{\nabla_X p})) + \frac{1}{2}{}^V\big(pR(X,F(\tilde{p}))\big)\\
&= {}^H(\nabla_X F(\tilde{p})) - {}^H(F(\nabla_X \tilde{p})) + \frac{1}{2}{}^V\big(pR(X,F(\tilde{p}))\big)\\
&= {}^H((\nabla_X F)(\tilde{p})) + \frac{1}{2}{}^V\big(pR(X,F(\tilde{p}))\big).
\end{aligned}
$$

$$
\begin{aligned}
{}^{BS}\nabla_{{}^HX}{}^V(pF) &= {}^{BS}\nabla_{{}^HX}(p_k{}^V(dx^kF))\\
&= X^i\big(\partial_i(p_k)+p_a\Gamma^a_{hi}\partial_{\bar{h}}(p_k)\big){}^V(dx^kF) + p_k{}^V(\nabla_X(dx^kF))\\
&\quad +\frac{p_k}{2}\big({}^H(R(\tilde{p},\widetilde{dx^kF})X) + \delta^2g^{-1}(dx^kF,pJ){}^H(R(J\tilde{p},\tilde{p})X)\big)\\
&= X(p_k){}^V(dx^kF) + p_k{}^V(\nabla_X(dx^kF)) + X^ip_a\Gamma^a_{ki}{}^V(dx^kF)\\
&\quad +\frac{1}{2}\big({}^H(R(\tilde{p},\widetilde{pF})X) + \delta^2g^{-1}(pF,pJ){}^H(R(J\tilde{p},\tilde{p})X)\big)\\
&= {}^V\big((\nabla_X(pF)-(\nabla_Xp)F\big) + \frac{1}{2}{}^H(R(\tilde{p},\widetilde{pF})X)\\
&\quad +\frac{\delta^2}{2}g^{-1}(pF,pJ){}^H(R(J\tilde{p},\tilde{p})X)\\
&= {}^V\big(p(\nabla_XF)\big) + \frac{1}{2}{}^H(R(\tilde{p},\widetilde{pF})X) + \frac{\delta^2}{2}g^{-1}(pF,pJ){}^H(R(J\tilde{p},\tilde{p})X)
\end{aligned}
$$

The other formulas are obtained by a similar calculation. □

4. Curvatures of Berger-type deformed Sasaki metric

Theorem 4.1. *Given a standard Kähler manifold (M^n, J, g) and its cotangent bundle $(T^*M, {}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. The Riemannian curvature tensor ${}^{BS}R$ of $(T^*M, {}^{BS}g)$ is expressed by:*

$$\begin{aligned}{}^{BS}R({}^HX, {}^HY)^HZ =& \frac{1}{2}{}^H(R(\tilde{p}, R(X,Y)\tilde{p})Z) - \frac{\delta^2}{2}g^{-1}(pR(X,Y), pJ)^H(R(J\tilde{p},\tilde{p})Z) \\ &+ \frac{1}{4}{}^H(R(\tilde{p}, R(X,Z)\tilde{p})Y) - \frac{\delta^2}{4}g^{-1}(pR(X,Z), pJ)^H(R(J\tilde{p},\tilde{p})Y) \\ &- \frac{1}{4}{}^H(R(\tilde{p}, R(Y,Z)\tilde{p})X) + \frac{\delta^2}{4}g^{-1}(pR(Y,Z), pJ)^H(R(J\tilde{p},\tilde{p})X) \\ &+ {}^H(R(X,Y)Z) - \frac{1}{2}{}^V\big(p(\nabla_Z R)(X,Y)\big), \end{aligned} \tag{4.1}$$

$$\begin{aligned}{}^{BS}R({}^HX, {}^V\theta)^HZ =& \frac{1}{2}{}^H((\nabla_X R)(\tilde{p},\widetilde{\theta})Z) + \frac{\delta^2}{2}g^{-1}(\theta, pJ)^H((\nabla_X R)(J\tilde{p},\tilde{p})Z) \\ &+ \frac{1}{4}{}^V(pR(X, R(\tilde{p},\widetilde{\theta})Z)) + \frac{\delta^2}{4}g^{-1}(\theta, pJ)^V(pR(X, R(J\tilde{p},\tilde{p})Z)) \\ &- \frac{\delta^2}{2}g^{-1}(\theta, pJ)^V(pR(X,Z)J) - \frac{\delta^2}{2}g^{-1}(pR(X,Z), pJ)^V(\theta J) \\ &+ \frac{\delta^4}{2\lambda}g^{-1}(\theta, p)g^{-1}(pR(X,Z), pJ)^V(pJ) - \frac{1}{2}{}^V(\theta R(X,Z)), \end{aligned} \tag{4.2}$$

$$\begin{aligned}{}^{BS}R({}^HX, {}^HY)^V\eta =& \frac{1}{2}{}^H((\nabla_X R)(\tilde{p},\tilde{\eta})Y) - \frac{1}{2}{}^H((\nabla_Y R)(\tilde{p},\tilde{\eta})X) \\ &+ \frac{\delta^2}{2}g^{-1}(\eta, pJ)\big({}^H((\nabla_X R)(J\tilde{p},\tilde{p})Y) - {}^H((\nabla_Y R)(J\tilde{p},\tilde{p})X)\big) \\ &+ \frac{\delta^2}{4}g^{-1}(\eta, pJ)\big({}^V(pR(X, R(J\tilde{p},\tilde{p})Y) - {}^V(pR(Y, R(J\tilde{p},\tilde{p})X)\big) \\ &- \delta^2\big(g^{-1}(pR(X,Y), pJ)^V(\eta J) + g^{-1}(\eta, pJ)^V(pR(X,Y)J)\big) \\ &+ \frac{\delta^4}{\lambda}g^{-1}(\eta, p)g^{-1}(pR(X,Y), pJ)^V(pJ) - {}^V(\eta R(X,Y)) \\ &+ \frac{1}{4}{}^V(pR(X, R(\tilde{p},\tilde{\eta})Y)) - \frac{1}{4}{}^V(pR(Y, R(\tilde{p},\tilde{\eta})X)), \end{aligned} \tag{4.3}$$

$$\begin{aligned}{}^{BS}R({}^HX, {}^V\theta)^V\eta =& \frac{-1}{2}{}^H(R(\widetilde{\theta},\tilde{\eta})X) - \frac{1}{4}{}^H(R(\tilde{p},\widetilde{\theta})R(\tilde{p},\tilde{\eta})X) \\ &+ \frac{\delta^2}{4}g^{-1}(\theta, pJ)\big(2{}^H(R(J\tilde{p},\tilde{\eta})X) - {}^H(R(J\tilde{p},\tilde{p})R(\tilde{p},\tilde{\eta})X)\big) \\ &- \frac{\delta^2}{4}g^{-1}(\eta, pJ)\big(2{}^H(R(J\tilde{p},\widetilde{\theta})X) + {}^H(R(\tilde{p},\widetilde{\theta})R(J\tilde{p},\tilde{p})X)\big) \\ &- \frac{\delta^4}{4}g^{-1}(\theta, pJ)g^{-1}(\eta, pJ)^H(R(J\tilde{p},\tilde{p})R(J\tilde{p},\tilde{p})X) \\ &- \frac{\delta^2}{2}g^{-1}(\eta, \theta J)^H(R(J\tilde{p},\tilde{p})X), \end{aligned} \tag{4.4}$$

$$\begin{aligned}
{}^{BS}R({}^V\omega,{}^V\theta){}^HZ ={}& {}^H(R(\widetilde{\omega},\widetilde{\theta})Z)+\frac{1}{4}{}^H(R(\tilde{p},\widetilde{\omega})R(\tilde{p},\widetilde{\theta})Z)-\frac{1}{4}{}^H(R(\tilde{p},\widetilde{\theta})R(\tilde{p},\widetilde{\omega})Z)\\
&+\frac{\delta^2}{4}g^{-1}(\theta,pJ)\big({}^H(R(\tilde{p},\widetilde{\omega})R(J\tilde{p},\tilde{p})Z)-{}^H(R(J\tilde{p},\tilde{p})R(\tilde{p},\widetilde{\omega})Z)\big)\\
&-\frac{\delta^2}{4}g^{-1}(\omega,pJ)\big({}^H(R(\tilde{p},\widetilde{\theta})R(J\tilde{p},\tilde{p})Z)-{}^H(R(J\tilde{p},\tilde{p})R(\tilde{p},\widetilde{\theta})Z)\big)\\
&+\delta^2g^{-1}(\theta,pJ){}^H(R(J\tilde{p},\widetilde{\omega})Z)-\delta^2g^{-1}(\omega,pJ){}^H(R(J\tilde{p},\widetilde{\theta})Z)\\
&+\delta^2g^{-1}(\theta,\omega J){}^H(R(J\tilde{p},\tilde{p})Z), \qquad (4.5)
\end{aligned}$$

$$\begin{aligned}
{}^{BS}R({}^V\omega,{}^V\theta){}^V\eta ={}& \delta^4g^{-1}(\eta,pJ)\big(g^{-1}(\theta,pJ){}^V\omega-g^{-1}(\omega,pJ){}^V\theta\big)\\
&+\delta^2\big(g^{-1}(\theta,\eta J){}^V(\omega J)-g^{-1}(\omega,\eta J){}^V(\theta J)-2g^{-1}(\omega,\theta J){}^V(\eta J)\big)\\
&+\frac{\delta^6}{\lambda}g^{-1}(\eta,pJ)\big(g^{-1}(\omega,pJ)g^{-1}(\theta,p)-g^{-1}(\omega,p)g^{-1}(\theta,pJ)\big){}^Vp\\
&+\Big(\frac{\delta^6}{\lambda^2}g^{-1}(\eta,p)\big(g^{-1}(\omega,p)g^{-1}(\theta,pJ)-g^{-1}(\omega,pJ)g^{-1}(\theta,p)\big)\\
&+\frac{\delta^4}{\lambda}\big(g^{-1}(\omega,pJ)g^{-1}(\theta,\eta)-g^{-1}(\theta,pJ)g^{-1}(\omega,\eta)\big)\\
&+\frac{\delta^4}{\lambda}\big(g^{-1}(\omega,\eta J)^{-1}g(\theta,p)-g^{-1}(\theta,\eta J)g^{-1}(\omega,p)\big)\\
&+\frac{2\delta^4}{\lambda}g^{-1}(\omega,\theta J)g^{-1}(\eta,p)\Big){}^V(pJ), \qquad (4.6)
\end{aligned}$$

for all $X,Y,Z\in\Im^1_0(M)$ *and* $\omega,\theta,\eta\in\Im^0_1(M)$.

Proof. In the proof, we will use Theorem 3.1, Lemma 3.3 and Proposition 3.1.

(1) ${}^{BS}R({}^HX,{}^HY){}^HZ={}^{BS}\nabla_{{}^HX}{}^{BS}\nabla_{{}^HY}{}^HZ-{}^{BS}\nabla_{{}^HY}{}^{BS}\nabla_{{}^HX}{}^HZ-{}^{BS}\nabla_{[{}^HX,{}^HY]}{}^HZ$.

Let $F:TM\to TM$ be the bundle endomorphism given by $pF=pR(Y,Z)$. Using (2.6) and direct calculations, we have:

$$\begin{aligned}
{}^{BS}\nabla_{{}^HX}{}^{BS}\nabla_{{}^HY}{}^HZ &= {}^{BS}\nabla_{{}^HX}\big({}^H(\nabla_YZ)+\frac{1}{2}{}^V(pF)\big)\\
&= {}^H(\nabla_X\nabla_YZ)+\frac{1}{2}{}^V(pR(X,\nabla_YZ))+\frac{1}{2}{}^V(\nabla_X(pR(Y,Z)))\\
&\quad -\frac{1}{2}{}^V((\nabla_Xp)R(Y,Z))-\frac{1}{4}{}^H(R(\tilde{p},R(Y,Z)\tilde{p})X)\\
&\quad +\frac{\delta^2}{4}g^{-1}(pR(Y,Z),pJ){}^H(R(J\tilde{p},\tilde{p})X). \qquad (4.7)
\end{aligned}$$

From which, with permutation of X by Y, we get:

$$\begin{aligned}
{}^{BS}\nabla_{{}^HY}{}^{BS}\nabla_{{}^HX}{}^HZ &= {}^H(\nabla_Y\nabla_XZ)+\frac{1}{2}{}^V(pR(Y,\nabla_XZ))+\frac{1}{2}{}^V(\nabla_Y(pR(X,Z)))\\
&\quad -\frac{1}{2}{}^V((\nabla_Yp)R(X,Z))-\frac{1}{4}{}^H(R(\tilde{p},R(X,Z)\tilde{p})Y)\\
&\quad +\frac{\delta^2}{4}g^{-1}(pR(X,Z),pJ){}^H(R(J\tilde{p},\tilde{p})Y). \qquad (4.8)
\end{aligned}$$

Also, we find:

$$\begin{aligned} {}^{BS}\nabla_{[{}^HX,{}^HY]}{}^HZ &= {}^{BS}\nabla_{{}^H[X,Y]}{}^HZ + {}^{BS}\nabla_{{}^V(pR(X,Y))}{}^HZ \\ &= {}^H(\nabla_{[X,Y]}Z) + \frac{1}{2}{}^V(pR([X,Y],Z)) - \frac{1}{2}{}^H(R(\tilde{p},R(X,Y)\tilde{p})Z) \\ &\quad + \frac{\delta^2}{2}g^{-1}(pR(X,Y),pJ)^H(R(J\tilde{p},\tilde{p})Z). \end{aligned} \tag{4.9}$$

From the formulas (4.7), (4.8), (4.9), and using (2.8) we get:

$$\begin{aligned} {}^{BS}R({}^HX,{}^HY)^HZ =& \frac{1}{2}{}^H(R(\tilde{p},R(X,Y)\tilde{p})Z) - \frac{\delta^2}{2}g^{-1}(pR(X,Y),pJ)^H(R(J\tilde{p},\tilde{p})Z) \\ &+ \frac{1}{4}{}^H(R(\tilde{p},R(X,Z)\tilde{p})Y) - \frac{\delta^2}{4}g^{-1}(pR(X,Z),pJ)^H(R(J\tilde{p},\tilde{p})Y) \\ &- \frac{1}{4}{}^H(R(\tilde{p},R(Y,Z)\tilde{p})X) + \frac{\delta^2}{4}g^{-1}(pR(Y,Z),pJ)^H(R(J\tilde{p},\tilde{p})X) \\ &+ {}^H(R(X,Y)Z) + \frac{1}{2}{}^V\big(p(\nabla_XR)(Y,Z)\big) - \frac{1}{2}{}^V\big(p(\nabla_YR)(X,Z)\big). \end{aligned}$$

Using the second Bianchi identity, we obtain the formula (4.1).

(2) ${}^{BS}R({}^HX,{}^V\theta)^HZ = {}^{BS}\nabla_{{}^HX}{}^{BS}\nabla_{{}^V\theta}{}^HZ - {}^{BS}\nabla_{{}^V\theta}{}^{BS}\nabla_{{}^HX}{}^HZ - {}^{BS}\nabla_{[{}^HX,{}^V\theta]}{}^HZ.$

Let $F : TM \to TM$ be the bundle endomorphism given by $F(u) = R(u,\widetilde{\theta})Z$ and $K : TM \times TM \to TM$ given by $K(u,v) = R(u,v)Z$. Hence, we obtain:

$$\begin{aligned} {}^{BS}\nabla_{{}^HX}{}^{BS}\nabla_{{}^V\theta}{}^HZ =& {}^{BS}\nabla_{{}^HX}\big(\frac{1}{2}{}^HF(\tilde{p}) + \frac{\delta^2}{2}g^{-1}(\theta,pJ)^H(K(J\tilde{p},\tilde{p}))\big) \\ =& \frac{1}{2}{}^H(\nabla_X(R(\tilde{p},\widetilde{\theta})Z)) - \frac{1}{2}{}^H(R(\nabla_X\tilde{p},\widetilde{\theta})Z) \\ &+ \frac{1}{4}{}^V(pR(X,R(\tilde{p},\widetilde{\theta})Z)) + \frac{\delta^2}{2}g^{-1}(\nabla_X\theta,pJ)^H(R(J\tilde{p},\tilde{p})Z) \\ &+ \frac{\delta^2}{2}g^{-1}(\theta,pJ)^H(\nabla_X(R(J\tilde{p},\tilde{p})Z)) - \frac{\delta^2}{2}g^{-1}(\theta,pJ)^H(R(J\nabla_X\tilde{p},\tilde{p})Z) \\ &- \frac{\delta^2}{2}g^{-1}(\theta,pJ)^H(R(J\tilde{p},\nabla_X\tilde{p})Z) + \frac{\delta^2}{4}g^{-1}(\theta,pJ)^V(p(R(X,R(J\tilde{p},\tilde{p})Z)). \end{aligned} \tag{4.10}$$

Let $F : TM \to TM$ be the bundle endomorphism given by $pF = pR(X,Z)$, then we get:

$$\begin{aligned} {}^{BS}\nabla_{{}^V\theta}{}^{BS}\nabla_{{}^HX}{}^HZ =& {}^{BS}\nabla_{{}^V\theta}\big({}^H\big(\nabla_XZ\big) + \frac{1}{2}{}^V(pF)\big) \\ =& \frac{1}{2}{}^H(R(\tilde{p},\widetilde{\theta})(\nabla_XZ)) + \frac{\delta^2}{2}g^{-1}(\theta,pJ)^H(R(J\tilde{p},\tilde{p})(\nabla_XZ)) \\ &+ \frac{\delta^2}{2}g^{-1}(pR(X,Z)),pJ)^V(\theta J) + \frac{\delta^2}{2}g^{-1}(\theta,pJ)^V(pR(X,Z)J) \\ &+ \frac{1}{2}{}^V(\theta R(X,Z)) - \frac{\delta^4}{2\lambda}g^{-1}(\theta,p)g^{-1}(pR(X,Z),pJ)^V(pJ). \end{aligned} \tag{4.11}$$

Also, we find:

$$\begin{aligned}{}^{BS}\nabla_{[{}^HX,{}^V\theta]}{}^HZ =& {}^{BS}\nabla_{{}^V(\nabla_X\theta)}{}^HZ \\ =& \frac{1}{2}{}^H(R(\tilde{p},\nabla_X\widetilde{\theta})Z)+\frac{\delta^2}{2}g^{-1}(\nabla_X\theta,pJ){}^H(R(J\tilde{p},\tilde{p})Z). \qquad (4.12)\end{aligned}$$

From the formulas (4.10), (4.11) and (4.12), we obtain the formula (4.2).

(3) Applying formula (4.2) and 1^{st} Bianchi identity, we get:

$$ {}^{BS}R({}^HX,{}^HY)\eta^V \;=\; {}^{BS}R({}^HX,\eta^V){}^HY-{}^{BS}R({}^HY,\eta^V){}^HX. $$

With direct calculations, we obtain the formula (4.3).
The other formulas (4.4)-(4.6) are obtained by a similar calculation. We omit them to avoid repetition. □

Proposition 4.1. *Given a standard Kähler manifold (M^n,J,g) and its cotangent bundle $(T^*M,{}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. If $(T^*M,{}^{BS}g)$ is flat, then (M^n,J,g) is flat.*

Proof. It is easy to see from (4.1) If we assume that ${}^{BS}R=0$ and calculate the Riemann curvature tensor for three horizontal vector fields at $(x,0)$ we get

$$ {}^{BS}R_{(x,0)}({}^HX,{}^HY){}^HZ={}^H(R_x(X,Y)Z)=0. $$

□

Let $(x,p)\in T^*M$ with $p\neq 0$, $\{E_i\}_{i=\overline{1,n}}$ and $\{\omega^i\}_{i=\overline{1,n}}$ be a local orthonormal frame and coframe on M, respectively, such that $\omega^n=\dfrac{pJ}{|pJ|}=\dfrac{pJ}{|p|}$, then

$$ \big\{F_i={}^HE_i,F_{n+j}={}^V\omega^j,F_{2n}=\frac{1}{\sqrt{\lambda}}{}^V\omega^n\big\}_{i=\overline{1,n},j=\overline{1,n-1}} \qquad (4.13)$$

is a local orthonormal frame on T^*M.

Theorem 4.2. *Given a standard Kähler manifold (M^n,J,g) and its cotangent bundle $(T^*M,{}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. If Ric (resp. ${}^{BS}Ric$) denote the Ricci curvature of (M^n,J,g) (resp. $(T^*M,{}^{BS}g)$), then ${}^{BS}Ric$ is expressed by:*

$$\begin{aligned}{}^{BS}Ric({}^HX,{}^HY) \;=\;& Ric(X,Y)-\frac{1}{2}\sum_{a=1}^{n}g(R(E_a,X)\tilde{p},R(E_a,Y)\tilde{p}) \\ & -\frac{\delta^2}{2}g(R(J\tilde{p},\tilde{p})X,R(J\tilde{p},\tilde{p})Y),\end{aligned}$$

$$\begin{aligned}{}^{BS}Ric({}^HX,{}^V\theta) \;=\;& \frac{1}{2}\sum_{a=1}^{n}g((\nabla_{E_a}R)(\tilde{p},\widetilde{\theta})X,E_a) \\ & +\frac{\delta^2}{2}g^{-1}(\theta,pJ)\sum_{a=1}^{n}g((\nabla_{E_a}R)(J\tilde{p},\tilde{p})X,E_a),\end{aligned}$$

$$
\begin{aligned}
{}^{BS}Ric({}^{V}\omega,{}^{V}\theta) &= \frac{1}{4}\sum_{a=1}^{n} g(R(\tilde{p},\widetilde{\omega}))E_a, R(\tilde{p},\widetilde{\theta})E_a) \\
&+\frac{\delta^2}{4}g^{-1}(\omega,pJ)\sum_{a=1}^{n} g(R(\tilde{p},\widetilde{\theta})E_a, R(J\tilde{p},\tilde{p})E_a) \\
&+\frac{\delta^2}{4}g^{-1}(\theta,pJ)\sum_{a=1}^{n} g(R(\tilde{p},\widetilde{\omega})E_a, R(J\tilde{p},\tilde{p})E_a) \\
&+\frac{\delta^4}{4}g^{-1}(\omega,pJ)g^{-1}(\theta,pJ)\sum_{a=1}^{n} |R(J\tilde{p},\tilde{p})E_a|^2 \\
&+\frac{\delta^4(2\lambda+1)}{\lambda^2}g^{-1}(\omega,p)g^{-1}(\theta,p) - \frac{\delta^2(2\lambda+1)}{\lambda}g^{-1}(\omega,\theta) \\
&+\delta^4(n-2)g^{-1}(\omega,pJ)g^{-1}(\theta,pJ),
\end{aligned}
$$

for all $X, Y \in \Im^1_0(M)$ *and* $\omega, \theta \in \Im^0_1(M)$.

Proof. Using the local orthonormal frame (4.13) and from (4.1) and (4.2), we have:

$$
\begin{aligned}
{}^{BS}Ric({}^{H}X,{}^{H}Y) &= \sum_{a=1}^{n} {}^{BS}g({}^{BS}R({}^{H}E_a,{}^{H}X){}^{H}Y,{}^{H}E_a) \\
&+\sum_{a=1}^{n-1} {}^{BS}g({}^{BS}R({}^{V}\omega^a,{}^{H}X){}^{H}Y,{}^{V}\omega^a) \\
&+\frac{1}{\lambda}{}^{BS}g({}^{BS}R({}^{V}\omega^n,{}^{H}X){}^{H}Y,{}^{V}\omega^n),
\end{aligned}
$$

by simple calculation, we get:

$$
\begin{aligned}
{}^{BS}Ric({}^{H}X,{}^{H}Y) =& \sum_{a=1}^{n} g(R(E_a,X)Y,E_a) - \frac{3}{4}\sum_{a=1}^{n} g(R(E_a,X)\tilde{p}, R(E_a,Y)\tilde{p}) \\
&- \frac{3\delta^2}{4}\sum_{a=1}^{n} g(R(X,E_a)\tilde{p}, J\tilde{p})g(R(Y,E_a)\tilde{p}, J\tilde{p}) \\
&+ \frac{1}{4}\sum_{a=1}^{n-1} g(R(\tilde{p},\widetilde{\omega^a})X, R(\tilde{p},\widetilde{\omega^a})Y) + \frac{\lambda}{4}g(R(\tilde{p},\widetilde{\omega^n})X, R(\tilde{p},\widetilde{\omega^n})Y).
\end{aligned}
$$

After doing some calculations, we find:

$$
\begin{aligned}
{}^{BS}Ric({}^{H}X,{}^{H}Y) =& Ric(X,Y) - \frac{3}{4}\sum_{a=1}^{n} g(R(E_a,X)\tilde{p}, R(E_a,Y)\tilde{p}) \\
&- \frac{3\delta^2}{4}g(R(J\tilde{p},\tilde{p})X, R(J\tilde{p},\tilde{p})Y) + \frac{1}{4}\sum_{a=1}^{n} g(R(\tilde{p},\widetilde{\omega^a})X, R(\tilde{p},\widetilde{\omega^a})Y) \\
&- \frac{1}{4}g(R(\tilde{p},\widetilde{\omega^n})X, R(\tilde{p},\widetilde{\omega^n})Y) + \frac{\lambda}{4}g(R(\tilde{p},\widetilde{\omega^n})X, R(\tilde{p},\widetilde{\omega^n})Y).
\end{aligned}
$$

In order to simplify this last expression, we have:

$$\begin{aligned}\widetilde{\omega^a} &= \sum_{i=1}^{n} g(\widetilde{\omega^a}, E_i)E_i = \sum_{i,h,k=1}^{n} g_{hk}\widetilde{\omega^a}^h E_i^k E_i = \sum_{i,h,k,j=1}^{n} g_{hk}g^{jh}\omega_j^a E_i^k E_i \\ &= \sum_{i,k,j=1}^{n} \delta_k^j \omega_j^a E_i^k E_i = \sum_{i,j=1}^{n} \omega_j^a E_i^j E_i = \sum_{i=1}^{n} \omega^a(E_i)E_i \\ &= \sum_{i=1}^{n} \delta_i^a E_i = E_a, \end{aligned} \tag{4.14}$$

which gives

$$\frac{1}{4}\sum_{a=1}^{n} g(R(\tilde{p},\widetilde{\omega^a})X, R(\tilde{p},\widetilde{\omega^a})Y) = \frac{1}{4}\sum_{a=1}^{n} g(R(\tilde{p},E_a)X, R(\tilde{p},E_a)Y). \tag{4.15}$$

On the other hand, we have:

$$\begin{aligned}\sum_{a=1}^{n} g(R(\tilde{p},E_a)X, R(\tilde{p},E_a)Y) &= \sum_{a,b=1}^{n} g(R(\tilde{p},E_a)X, E_b)g(R(\tilde{p},E_a)Y, E_b) \\ &= \sum_{a,b=1}^{n} g(R(X,E_b)\tilde{p}, E_a)g(R(Y,E_b)\tilde{p}, E_a) \\ &= \sum_{a,b=1}^{n} g(R(E_b,X)\tilde{p}, E_a)g(R(E_b,Y)\tilde{p}, E_a) \\ &= \sum_{b=1}^{n} g(R(E_b,X)\tilde{p}, R(E_b,Y)\tilde{p}) \\ &= \sum_{a=1}^{n} g(R(E_a,X)\tilde{p}, R(E_a,Y)\tilde{p}). \end{aligned} \tag{4.16}$$

From,(4.14), (4.15) and (4.16) we get the result.
The other formulas are obtained by a similar calculation. □

It is known that the sectional curvature ${}^{BS}K$ on $(T^*M, {}^{BS}g)$ for a plane P is given by:

$${}^{BS}K(V,W) = \frac{{}^{BS}g({}^{BS}R(V,W)W,V)}{{}^{BS}g(V,V){}^{BS}g(W,W) - {}^{BS}g(V,W)^2}, \tag{4.17}$$

where $P = P(V,W)$ denotes the plane spanned by $\{V,W\}$, for all, linearly independent vector fields $V,W \in \Im_0^1(T^*M)$. Let ${}^{BS}K({}^HX, {}^HY)$, ${}^{BS}K({}^HX, {}^V\theta)$ and ${}^{BS}K({}^V\omega, \theta^V)$ denote the sectional curvature of the plane spanned by $\{{}^HX, {}^HY\}$, $\{{}^HX, {}^V\theta\}$ and $\{{}^V\omega, {}^V\theta\}$ on $(T^*M, {}^{BS}g)$ respectively, where X,Y orthonormal vector fields and ω,θ orthonormal covector fields on M.

Proposition 4.2. *Given a standard Kähler manifold (M^n, J, g) and its cotangent bundle $(T^*M, {}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. Then* *we have the following:*

$$
\begin{aligned}
{}^{BS}g({}^{BS}R({}^{H}X,{}^{H}Y){}^{H}Y,{}^{H}X) &= g(R(X,Y)Y,X)-\frac{3}{4}|R(X,Y)\tilde{p}|^2\\
&\quad -\frac{3\delta^2}{4}g(R(X,Y)\tilde{p},J\tilde{p})^2,\\
{}^{BS}g({}^{BS}R({}^{H}X,{}^{V}\theta){}^{V}\theta,{}^{H}X) &= \frac{1}{4}|R(\tilde{p},\widetilde{\theta})X|^2+\frac{\delta^4}{4}g^{-1}(\theta,pJ)^2|R(J\tilde{p},\tilde{p})X|^2\\
&\quad +\frac{\delta^2}{2}g^{-1}(\theta,pJ)g(R(\tilde{p},\widetilde{\theta})X,R(J\tilde{p},\tilde{p})X),\\
{}^{BS}g({}^{BS}R({}^{V}\omega,{}^{V}\theta){}^{V}\theta,{}^{V}\omega) &= -3\delta^2g^{-1}(\omega,\theta J)^2+\delta^4(g^{-1}(\omega,pJ)^2+g^{-1}(\theta,pJ)^2)\\
&\quad -\frac{\delta^6}{\lambda}\big(g^{-1}(\omega,p)g^{-1}(\theta,pJ)-g^{-1}(\omega,pJ)g^{-1}(\theta,p)\big)^2.
\end{aligned}
$$

From the Proposition 4.2 and the formula (4.17), we obtain the following result.

Theorem 4.3. *Given a standard Kähler manifold (M^n,J,g) and its cotangent bundle $(T^*M,{}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. Then the sectional curvature ${}^{BS}K$ is expressed by:*

$$
\begin{aligned}
{}^{BS}K({}^{H}X,{}^{H}Y) &= K(X,Y)-\frac{3}{4}|R(X,Y)\tilde{p}|^2-\frac{3\delta^2}{4}g(R(X,Y)\tilde{p},J\tilde{p})^2,\\
{}^{BS}K({}^{H}X,{}^{V}\theta) &= \frac{1}{1+\delta^2g^{-1}(\theta,pJ)^2}\Big(\frac{1}{4}|R(\tilde{p},\widetilde{\theta})X|^2+\frac{\delta^4}{4}g^{-1}(\theta,pJ)^2|R(J\tilde{p},\tilde{p})X|^2\\
&\quad +\frac{\delta^2}{2}g^{-1}(\theta,pJ)g(R(\tilde{p},\widetilde{\theta})X,R(J\tilde{p},\tilde{p})X)\Big),\\
{}^{BS}K({}^{V}\omega,{}^{V}\theta) &= \frac{1}{1+\delta^2g^{-1}(\omega,pJ)^2+\delta^2g^{-1}(\theta,pJ)^2}\Big(-3\delta^2g^{-1}(\omega,\theta J)^2\\
&\quad -\frac{\delta^6}{\lambda}\big(g^{-1}(\omega,p)g^{-1}(\theta,pJ)-g^{-1}(\omega,pJ)g^{-1}(\theta,p)\big)^2\\
&\quad +\delta^4(g^{-1}(\omega,pJ)^2+g^{-1}(\theta,pJ)^2)\Big).
\end{aligned}
$$

where K denote the sectional curvature tensor of (M^n,J,g).

Theorem 4.4. *Given a standard Kähler manifold (M^n,J,g) and its cotangent bundle $(T^*M,{}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. If σ (resp., ${}^{BS}\sigma$) denote the scalar curvature of (M^n,J,g) (resp., $(T^*M,{}^{BS}g)$). Then ${}^{BS}\sigma$ is expressed by:*

$$
\begin{aligned}
{}^{BS}\sigma &= \sigma-\frac{1}{4}\sum_{a,b=1}^{n}|R(E_a,E_b)\tilde{p}|^2-\frac{\delta^2}{4}\sum_{a=1}^{n}|R(J\tilde{p},\tilde{p})E_a|^2\\
&\quad -\frac{\delta^2}{\lambda^2}((n-2)\lambda^2+2n\lambda+2),
\end{aligned}
\tag{4.18}
$$

where $(E_a)_{a=\overline{1,n}}$ is a local orthonormal frame on M.

Proof. Let $(F_k)_{k=\overline{1,2n}}$ be a local orthonormal frame on $(T^*M, {}^{BS}g)$ defined by (4.13). Using Theorem 4.2 and definition of scalar curvature, we have

$$ {}^{BS}\sigma = \sum_{b=1}^{n} {}^{BS}Ric(F_b, F_b) + \sum_{b=1}^{n-1} {}^{BS}Ric(F_{n+b}, F_{n+b}) + {}^{BS}Ric(F_{2n}, F_{2n}), $$

Through direct calculations, we get:

$$ \begin{aligned} \sum_{b=1}^{n} {}^{BS}Ric(F_b, F_b) &= \sum_{b=1}^{n} {}^{BS}Ric({}^{H}E_b, {}^{H}E_b) \\ &= \sigma - \frac{1}{2}\sum_{a,b=1}^{n} |R(E_a, E_b)\tilde{p}|^2 - \frac{\delta^2}{2}\sum_{b=1}^{n} |R(J\tilde{p}, \tilde{p})E_b|^2, \end{aligned} $$

$$ \begin{aligned} \sum_{b=1}^{n-1} {}^{BS}Ric(F_{n+b}, F_{n+b}) &= \sum_{b=1}^{n-1} {}^{BS}Ric({}^{V}\omega^b, {}^{V}\omega^b) \\ &= \frac{1}{4}\sum_{b=1}^{n-1}\sum_{a=1}^{n} |R(\tilde{p}, E_b)E_a|^2 + \frac{\delta^2(2\lambda^2 - \lambda - 1)}{\lambda^2} \\ &\quad - \frac{\delta^2(n-1)(2\lambda+1)}{\lambda}, \end{aligned} $$

and

$$ \begin{aligned} {}^{BS}Ric(F_{2n}, F_{2n}) &= \frac{1}{\lambda}\, {}^{BS}Ric({}^{V}\omega^n, {}^{V}\omega^n) \\ &= \frac{\delta^2\lambda}{4(\lambda-1)}\sum_{a=1}^{n} |R(J\tilde{p}, \tilde{p})E_a|^2 - \frac{\delta^2(2\lambda+1)}{\lambda^2} \\ &\quad - \frac{\delta^2(n-2)(\lambda-1)}{\lambda}. \end{aligned} $$

From this, we deduce:

$$ \begin{aligned} {}^{BS}\sigma &= \sigma - \frac{1}{2}\sum_{a,b=1}^{n} |R(E_a, E_b)\tilde{p}|^2 - \frac{\delta^2}{2}\sum_{b=1}^{n} |R(J\tilde{p}, \tilde{p})E_b|^2 \\ &\quad + \frac{1}{4}\sum_{b=1}^{n-1}\sum_{a=1}^{n} |R(\tilde{p}, E_b)E_a|^2 + \frac{\delta^2(2\lambda^2 - \lambda - 1)}{\lambda^2} - \frac{\delta^2(n-1)(2\lambda+1)}{\lambda} \\ &\quad + \frac{\delta^2\lambda}{4(\lambda-1)}\sum_{a=1}^{n} |R(J\tilde{p}, \tilde{p})E_a|^2 - \frac{\delta^2(2\lambda+1)}{\lambda^2} - \frac{\delta^2(n-2)(\lambda-1)}{\lambda}. \end{aligned} $$

In order to simplify this last expression, we have

$$ \sum_{a,b=1}^{n} |R(\tilde{p}, E_b)E_a|^2 = \sum_{a,b=1}^{n} |R(E_a, E_b)\tilde{p}|^2. $$

This completes the proof. □

From Theorem (4.4), we deduce the result.

Proposition 4.3. *Given a standard Kähler manifold* (M^n, J, g) *of constant sectional curvature* κ *and its cotangent bundle* $(T^*M, {}^{BS}g)$ *endowed with the Berger-type deformed Sasaki metric. The scalar curvature* ${}^{BS}\sigma$ *of* $(T^*M, {}^{BS}g)$*, is expressed by:*

$$ {}^{BS}\sigma = n(n-1)\kappa - \frac{(n+\lambda-2)(\lambda-1)}{2\delta^2}\kappa^2 - \frac{\delta^2}{\lambda^2}((n-2)\lambda^2 + 2n\lambda + 2). $$

Proof. Since M has constant curvature κ, then $\sigma = n(n-1)\kappa$. Through direct calculations, we get the following:

$$ \begin{aligned} \sum_{a,b=1}^{n} |R(E_a, E_b)\tilde{p}|^2 &= 2(n-1)\kappa^2\frac{\lambda-1}{\delta^2}, \\ \sum_{a=1}^{n} |R(J\tilde{p}, \tilde{p})E_a|^2 &= 2\kappa^2\frac{(\lambda-1)^2}{\delta^4}. \end{aligned} $$

This completes the proof. □

5. Some almost complex structures with Hermitian metrics on the cotangent bundle

Given an almost Hermitian manifold (M^n, J, g), we consider the tensor field ϕ on T^*M defined by:

$$ \begin{cases} \phi\,{}^HX = {}^V\widetilde{X} + \eta g(X, J\tilde{p})\,{}^V(pJ) \\ \phi\,{}^V\omega = -{}^H\widetilde{\omega} + \mu g^{-1}(\omega, pJ)\,{}^H(J\tilde{p}) \end{cases} \tag{5.1} $$

for all $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, where $\eta, \mu : \mathbb{R} \to \mathbb{R}$ are smooth functions.

$I)$ First we start by studying the case $\eta \neq 0$ and $\mu \neq 0$.

Note that, from (3.4) and (3.5), we have:

$$ \begin{cases} \phi\,{}^H(J\tilde{p}) = (-1 + \eta\alpha)\,{}^V(pJ) \\ \phi\,{}^V(pJ) = (1 + \mu\alpha)\,{}^H(J\tilde{p}) \end{cases} $$

Lemma 5.1. *Given a standard Kähler manifold* (M^n, J, g) *and its cotangent bundle* $(T^*M, {}^{BS}g)$ *endowed with the Berger-type deformed Sasaki metric. Then the tensor field* ϕ *defined by* (5.1) *is an almost complex structure if and only if*

$$ \eta - \mu + \eta\mu\alpha = 0. $$

Proof. According to (2.3), (2.4), (3.1) and (3.2), we obtain:

$$ \begin{aligned} \phi^2({}^HX) &= \phi(\phi({}^HX)) \\ &= \phi({}^V\widetilde{X}) + \eta g(X, J\tilde{p})\phi({}^V(pJ)) \\ &= -{}^H\widetilde{\widetilde{X}} + \mu g^{-1}(\widetilde{X}, pJ)\,{}^H(J\tilde{p}) + \eta g(X, J\tilde{p})(1 + \mu\alpha)\,{}^H(J\tilde{p}) \\ &= -{}^HX - \mu g(X, J\tilde{p})\,{}^H(J\tilde{p}) + g(X, J\tilde{p})(\eta + \eta\mu\alpha)\,{}^H(J\tilde{p}) \\ &= -{}^HX + (\eta - \mu + \eta\mu\alpha)g(X, J\tilde{p})\,{}^H(J\tilde{p}). \end{aligned} \tag{5.2} $$

$$\begin{aligned}
\phi^2({}^V\omega) &= \phi(\phi({}^V\omega)) \\
&= -\phi({}^H\widetilde{\omega}) + \mu g^{-1}(\omega, pJ)\phi({}^H(J\tilde{p})) \\
&= -{}^V\widetilde{\widetilde{\omega}} - \eta g(\widetilde{\omega}, J\tilde{p})^V(pJ) + \mu g^{-1}(\omega, pJ)(-1+\eta\alpha)^V(pJ) \\
&= -{}^V\omega + \eta g^{-1}(\omega, pJ)^V(pJ) + g^{-1}(\omega, pJ)(-\mu+\eta\mu\alpha)^V(pJ) \\
&= -{}^V\omega + (\eta - \mu + \eta\mu\alpha) g^{-1}(\omega, pJ)^V(pJ). \qquad (5.3)
\end{aligned}$$

From (5.2) and (5.3), then $\phi^2 = Id_{T^*M}$ equivalent to $\eta - \mu + \eta\mu\alpha = 0$. □

Theorem 5.1. *Given a standard Kähler manifold (M^n, J, g), whose cotangent bundle $(T^*M, {}^{BS}g)$ is endowed with the Berger-type deformed Sasaki metric and an almost complex structure ϕ defined by (5.1). Then the triple $(T^*M, \phi, {}^{BS}g)$ is an almost Hermitian manifold if and only if*

$$\begin{cases} \eta - \mu + \eta\mu\alpha = 0, \\ \mu + \lambda\eta - \delta^2 = 0, \end{cases} \qquad (5.4)$$

where $\lambda = 1 + \delta^2\alpha$.

Proof. For Hermiticity condition, we put for all $U, V \in \Im^1_0(M)$:

$$A(U, V) = {}^{BS}g(\phi U, V) + {}^{BS}g(U, \phi V).$$

$$\begin{aligned}
(i)\ A({}^HX, {}^HY) &= {}^{BS}g(\phi^H X, {}^HY) + {}^{BS}g({}^HX, \phi^H Y) = 0, \\
(ii)\ A({}^V\omega, {}^V\theta) &= {}^{BS}g(\phi^V\omega, {}^V\theta) + {}^{BS}g({}^V\omega, \phi^V\theta) = 0, \\
(iii)\ A({}^V\omega, {}^HY) &= {}^{BS}g(\phi^V\omega, {}^HY) + {}^{BS}g({}^V\omega, \phi^H Y) \\
&= {}^{BS}g(-{}^H\widetilde{\omega} + \mu g^{-1}(\omega, pJ)^H(J\tilde{p}), {}^HY) \\
&\quad + {}^{BS}g({}^V\omega, {}^V\widetilde{Y} + \eta g(Y, J\tilde{p})^V(pJ)) \\
&= -{}^{BS}g({}^H\widetilde{\omega}, {}^HY) + \mu g^{-1}(\omega, pJ){}^{BS}g({}^H(J\tilde{p}), {}^HY) \\
&\quad + {}^{BS}g({}^V\omega, {}^V\widetilde{Y}) + \eta g(Y, J\tilde{p}){}^{BS}g({}^V\omega, {}^V(pJ)) \\
&= -g(\widetilde{\omega}, Y) + \mu g^{-1}(\omega, pJ) g(J\tilde{p}, Y) + g^{-1}(\omega, \widetilde{Y}) \\
&\quad + \delta^2 g^{-1}(\omega, pJ) g^{-1}(\widetilde{Y}, pJ) + \eta\lambda g(Y, J\tilde{p}) g^{-1}(\omega, pJ) \\
&= (\mu + \eta\lambda - \delta^2) g^{-1}(\omega, pJ) g(Y, J\tilde{p}),
\end{aligned}$$

then ${}^{BS}g$ is Hermitian on T^*M if and only if $A({}^V\omega, {}^HY) = 0$ i.e. $\mu + \eta\lambda - \delta^2 = 0$. □

By (5.4), we have

$$\eta = \frac{\varepsilon + \sqrt{\lambda}}{\alpha\sqrt{\lambda}} \quad and \quad \mu = \frac{\varepsilon\sqrt{\lambda} - 1}{\alpha} \qquad (5.5)$$

where $\varepsilon = \pm 1$. By substituting them in (5.1), we get:

$$\begin{cases} \phi^H X = {}^V\widetilde{X} + \dfrac{\varepsilon + \sqrt{\lambda}}{\alpha\sqrt{\lambda}} g(X, J\tilde{p})^V(pJ) \\ \phi^V\omega = -{}^H\widetilde{\omega} + \dfrac{\varepsilon\sqrt{\lambda} - 1}{\alpha} g^{-1}(\omega, pJ)^H(J\tilde{p}) \end{cases} \qquad (5.6)$$

We shall study integrability of ϕ. As we know that the integrability of ϕ is equivalent to the vanishing of the Nijenhuis tensor. The Nijenhuis tensor of ϕ is given by

$$N_\phi(U,V) = [\phi U, \phi V] - \phi[\phi U, V] - \phi[U, \phi V] - [U,V].$$

where $U, V \in \Im^1_0(T^*M)$.

Proposition 5.1. *Given a standard Kähler manifold (M^n, J, g) and its cotangent bundle $(T^*M, {}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. The almost complex structure ϕ defined by* (5.1) *is integrable if and only if $N_\phi({}^HX, {}^HY) = 0$, for all $X, Y \in \Im^1_0(M)$.*

Proof. We put $\phi{}^V\omega = {}^HW$ and $\phi{}^V\theta = {}^HZ$, then we have:

$$\begin{aligned} N_\phi({}^V\omega, {}^V\theta) &= [\phi{}^V\omega, \phi{}^V\theta] - \phi[\phi{}^V\omega, {}^V\theta] - \phi[{}^V\omega, \phi{}^V\theta] - [{}^V\omega, {}^V\theta] \\ &= [{}^HW, {}^HZ] + \phi[{}^HW, \phi{}^HZ] + \phi[\phi{}^HW, {}^HZ] - [\phi{}^HW, \phi{}^HZ] \\ &= -\big([\phi{}^HW, \phi{}^HZ] - \phi[\phi{}^HW, {}^HZ] - \phi[{}^HW, \phi{}^HZ] - [{}^HW, {}^HZ]\big) \\ &= -N_\phi({}^HW, {}^HZ). \end{aligned}$$

$$\begin{aligned} N_\phi({}^V\omega, {}^HY) &= [\phi{}^V\omega, \phi{}^HY] - \phi[\phi{}^V\omega, {}^HY] - \phi[{}^V\omega, \phi{}^HY] - [{}^V\omega, {}^HY] \\ &= [{}^HW, \phi{}^HY] - \phi[{}^HW, {}^HY] + \phi[\phi{}^HW, \phi{}^HY] + [\phi{}^HW, {}^HY] \\ &= \phi\big([\phi{}^HW, \phi{}^HY] - \phi[\phi{}^HW, {}^HY] - \phi[{}^HW, \phi{}^HY] - [{}^HW, {}^HY]\big) \\ &= \phi N_\phi({}^HW, {}^HY). \end{aligned}$$

□

Lemma 5.2. *Given a standard Kähler manifold (M^n, J, g), we have the following:*

(1) ${}^HX(\eta) = 0$,
(2) ${}^V\widetilde{X}(\eta) = 2\eta' g(X, \tilde{p})$,
(3) ${}^V(pJ)(\eta) = 0$,
(4) ${}^HX(g(Y, J\tilde{p})) = g(\nabla_Y X, J\tilde{p})$,
(5) ${}^V\widetilde{X}(g(Y, J\tilde{p})) = g(Y, JX)$,
(6) ${}^V(pJ)(g(Y, J\tilde{p})) = g(Y, \tilde{p})$,
(7) $[{}^HX, {}^V(pJ)] = 0$,
(8) $[{}^V\widetilde{X}, {}^V(pJ)] = {}^V(\widetilde{X}J)$,
(9) $[{}^V(pJ), {}^V(pJ)] = 0$.

for all $X, Y \in \Im^1_0(M)$, where η is defined by (5.5).

Proposition 5.2. *Given a standard Kähler manifold (M^n, J, g), whose cotangent bundle $(T^*M, {}^{BS}g)$ is endowed with the Berger-type deformed Sasaki metric and an almost complex structure ϕ defined by* (5.6)*, then*

$$\begin{aligned} N_\phi(X^H, Y^H) &= \eta\big(g(Y, J\tilde{p}){}^V(\widetilde{X}J) - g(X, J\tilde{p}){}^V(\widetilde{Y}J) - g(X, JY){}^V(pJ)\big) \\ &\quad + 2\eta'\big(g(X, \tilde{p})g(Y, J\tilde{p}) - g(X, J\tilde{p})g(Y, \tilde{p})\big){}^V(pJ) - {}^V(pR(X,Y)). \end{aligned}$$

for all $X, Y \in \Im^1_0(M)$, where η is defined by (5.5).

Proof. We have:

$$N_\phi({}^HX, {}^HY) = [\phi^HX, \phi^HY] - \phi[\phi^HX, {}^HY] - \phi[{}^HX, \phi^HY] - [{}^HX, {}^HY].$$

Using Lemma 5.2 and direct calculations, we get the following formulas:

$$\begin{aligned}
[\phi^HX, \phi^HY] &= \eta\big(g(Y, J\tilde{p})^V(\widetilde{X}J) - g(X, J\tilde{p})^V(\widetilde{Y}J) - g(X, JY)^V(pJ)\big) \\
&\quad +2\eta'\big(g(X,\tilde{p})g(Y, J\tilde{p}) - g(X, J\tilde{p})g(Y,\tilde{p})\big)^V(pJ), \\
\phi[\phi^HX, {}^HY] &= {}^H(\nabla_Y X), \\
\phi[{}^HX, \phi^HY] &= -{}^H(\nabla_X Y), \\
[{}^HX, {}^HY] &= {}^H[X, Y] + {}^V(pR(X, Y)).
\end{aligned}$$

This completes the proof. □

Theorem 5.2. *Given a standard Kähler manifold* (M^n, J, g) *and its cotangent bundle* $(T^*M, {}^{BS}g)$ *endowed with the Berger-type deformed Sasaki metric. The almost complex structure* ϕ *defined by* (5.6) *is integrable if and only if*

$$\begin{aligned}
pR(X, Y) &= \eta\big(g(Y, J\tilde{p})(\widetilde{X}J) - g(X, J\tilde{p})(\widetilde{Y}J) - g(X, JY)(pJ)\big) \\
&\quad +2\eta'\big(g(X,\tilde{p})g(Y, J\tilde{p}) - g(X, J\tilde{p})g(Y,\tilde{p})\big)(pJ). \qquad (5.7)
\end{aligned}$$

It is known that if the base manifold (M^n, J, g) is a standard Kähler manifold, then the Riemannian curvature tensor R of the base manifold satisfies the equalities (3.3). Then, according to (5.7), this identity is never satisfied. This shows that the almost complex structure ϕ is never integrable. Hence the triple$(T^*M, \phi, {}^{BS}g)$ is never a standard Kähler manifold.

II) Secondly, we study the case: $\eta = \mu = 0$. the tensor field ϕ on T^*M expressed by:

$$\begin{cases} \phi^HX = {}^V\widetilde{X} \\ \phi^V\omega = -{}^H\widetilde{\omega} \end{cases} \qquad (5.8)$$

for all $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$. We easily see that the tensor field ϕ is an almost complex structure on T^*M.

Theorem 5.3. *Given a standard Kähler manifold* (M^n, J, g) *and its cotangent bundle* $(T^*M, {}^{BS}g)$ *endowed with the Berger-type deformed Sasaki metric and the almost complex structure* ϕ *defined by* (5.8)*, then*
(i) The Berger-type deformed Sasaki metric is Hermitian with respect to ϕ *if and only if* $\delta = 0$ *i.e. the triple* $(T^*M, \phi, {}^{BS}g)$ *is an almost Hermitian manifold, then* ${}^{BS}g$ *reduces to the Sasaki metric.*
(ii) In the case of $\delta \neq 0$ *The Berger-type deformed Sasaki metric is never Hermitian with respect to* ϕ*.*

Proof. For Hermitity condition, we put for all $U, V \in \Im^1_0(M)$:

$$A(U, V) = {}^{BS}g(\phi U, V) + {}^{BS}g(U, \phi V).$$

$$\begin{aligned}
(i)\ A({}^HX,{}^HY) &= {}^{BS}g(\phi{}^HX,{}^HY)+{}^{BS}g({}^HX,\phi{}^HY)=0,\\
(ii)\ A({}^V\omega,{}^V\theta) &= {}^{BS}g(\phi{}^V\omega,{}^V\theta)+{}^{BS}g({}^V\omega,\phi{}^V\theta)=0,\\
(iii)\ A({}^V\omega,{}^HY) &= {}^{BS}g(\phi{}^V\omega,{}^HY)+{}^{BS}g({}^V\omega,\phi{}^HY)\\
&= {}^{BS}g(-{}^H\widetilde{\omega},{}^HY)+{}^{BS}g({}^V\omega,{}^V\widetilde{Y})\\
&= -g(\widetilde{\omega},Y)+g^{-1}(\omega,\widetilde{Y})+\delta^2g^{-1}(\omega,pJ)g^{-1}(\widetilde{Y},pJ)\\
&= \delta^2g^{-1}(\omega,pJ)g(Y,J\tilde{p}),
\end{aligned}$$

then ${}^{BS}g$ is Hermitian with respect to ϕ if and only if $\delta=0$. □

Theorem 5.4. *Given a standard Kähler manifold (M^n,J,g) and its cotangent bundle $(T^*M,{}^{BS}g)$ endowed with the Berger-type deformed Sasaki metric. The almost complex structure ϕ defined by* (5.8) *is integrable if and only if M is flat.*

Proof. Using (2.2), (2.5) and Proposition 5.1, we have:

$$\begin{aligned}
N_\phi({}^HX,{}^HY) &= [\phi{}^HX,\phi{}^HY]-\phi[\phi{}^HX,{}^HY]-\phi[{}^HX,\phi{}^HY]-[{}^HX,{}^HY]\\
&= [{}^V\widetilde{X},{}^V\widetilde{Y}]-\phi[{}^V\widetilde{X},{}^HY]-\phi[{}^HX,{}^V\widetilde{Y}]-{}^H[X,Y]-{}^V(pR(X,Y))\\
&= \phi{}^V(\nabla_Y\widetilde{X})-\phi{}^V(\nabla_X\widetilde{Y})-{}^H[X,Y]-{}^V(pR(X,Y))\\
&= -{}^H\widetilde{(\nabla_Y\widetilde{X})}+{}^H\widetilde{(\nabla_X\widetilde{Y})}-{}^H[X,Y]-{}^V(pR(X,Y))\\
&= {}^H(\nabla_XY)-{}^H(\nabla_YX)-{}^H[X,Y]-{}^V(pR(X,Y))\\
&= -{}^V(pR(X,Y)).
\end{aligned}$$

□

6. Berger-type deformed Sasaki metric on unit cotangent bundle T_1^*M

The unit cotangent (sphere) bundle over a standard Kähler manifold (M^n,J,g), is the hyper-surface

$$T_1^*M \;=\; \{(x,p)\in T^*M,\ g^{-1}(p,p)=1\}.$$

The unit normal vector field $\mathcal{N}$ to T_1^*M is given by $\mathcal{N}={}^Vp$.
The tangential lift ${}^T\omega$ with respect to ${}^{BS}g$ of a covector $\omega\in T_x^*M$ to $(x,p)\in T_1^*M$ as the tangential projection of the vertical lift of ω to (x,p) with respect to $\mathcal{N}$, that is

$${}^T\omega={}^V\omega-{}^{BS}g_{(x,p)}({}^V\omega,\mathcal{N}_{(x,p)})\mathcal{N}_{(x,p)}={}^V\omega-g_x^{-1}(\omega,p){}^Vp_{(x,p)}.$$

From the above, T^*M decomposes into the direct sum as follows:

$$T_{(x,p)}T^*M=T_{(x,p)}T_1^*M\oplus span\{\mathcal{N}_{(x,p)}\}=T_{(x,p)}T_1^*M\oplus span\{{}^Vp_{(x,p)}\},\tag{6.1}$$

where $(x,p)\in T_1^*M$.

The Levi-Civita connection ${}^{BS}\widehat{\nabla}$ on T_1^*M induced by ${}^{BS}g$ is given by the following theorem:

Theorem 6.1. *Let (M^n, J, g) be a standard Kähler manifold and T_1^*M its unit cotangent bundle equipped with the Berger-type deformed Sasaki metric. Then we have:*

$$\begin{array}{rcl} {}^{BS}\widehat{\nabla}_{{}^HX}{}^HY &=& {}^H(\nabla_X Y) + \frac{1}{2}{}^T(pR(X,Y)), \\ {}^{BS}\widehat{\nabla}_{{}^HX}{}^T\theta &=& {}^T(\nabla_X \theta) + \frac{1}{2}\big({}^H(R(\tilde{p},\widetilde{\theta})X) + \delta^2 g^{-1}(\theta, pJ)^H(R(J\tilde{p},\tilde{p})X)\big), \\ {}^{BS}\widehat{\nabla}_{{}^T\omega}{}^HY &=& \frac{1}{2}\big({}^H(R(\tilde{p},\tilde{\omega})Y) + \delta^2 g^{-1}(\omega, pJ)^H(R(J\tilde{p},\tilde{p})Y)\big), \\ {}^{BS}\widehat{\nabla}_{{}^T\omega}{}^T\theta &=& -g^{-1}(\theta,p)^T\omega + \delta^2\big(g^{-1}(\omega,pJ)^T(\theta J) + g^{-1}(\theta,pJ)^T(\omega J)\big) \\ && -\delta^2\big(g^{-1}(\omega,pJ)g^{-1}(\theta,p) + g^{-1}(\omega,p)g^{-1}(\theta,pJ)\big)^T(pJ), \end{array}$$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, where ∇ is the Levi-Civita connection and R is its curvature tensor.

Now, we shall calculate the Riemannian curvature tensor on T_1^*M induced by the Berger-type deformed Sasaki metric ${}^{BS}g$.

Denoting by ${}^{BS}\widehat{R}$ the Riemannian curvature tensors on T_1^*M induced by ${}^{BS}g$, from the Gauss equation for hypersurfaces we deduce that ${}^{BS}\widehat{R}(U,V)W$ satisfies

$$ {}^{BS}\widehat{R}(U,V)W = {}^t({}^{BS}R(U,V)W) - B(U,W).A_{\mathcal{N}}V + B(V,W).A_{\mathcal{N}}U, \qquad (6.2)$$

for all $U, V, W \in \Im_0^1(T^*M)$, where ${}^t(R^f(U,V)W)$ is the tangential component of $R^f(U,V)W$ with respect to the direct sum decomposition (6.1), $A_{\mathcal{N}}$ is the shape operator of T_1^*M in $(T^*M, {}^{BS}g)$ derived from $\mathcal{N}$, and B is the second fundamental form of T_1^*M (T_1^*M as a hypersurface immersed in T^*M), associated to $\mathcal{N}$ on T_1^*M.

$A_{\mathcal{N}}U$ is the tangential component of $(-{}^{BS}\nabla_U\mathcal{N})$ i.e.

$$A_{\mathcal{N}}U = -{}^t({}^{BS}\nabla_U\mathcal{N}), \qquad (6.3)$$

$B(U,V)$ is given by Gauss's formula, ${}^{BS}\nabla_U V = {}^{BS}\widehat{\nabla}_U V + B(U,V).\mathcal{N}$, so

$$B(U,V) = {}^{BS}g({}^{BS}\nabla_U V, \mathcal{N}). \qquad (6.4)$$

Lemma 6.1. *Given a standard Kähler manifold (M^n, J, g) and its unit cotangent bundle T_1^*M endowed with the Berger-type deformed Sasaki metric. Then we have:*

$$A_{\mathcal{N}}{}^HX = 0\ , \quad A_{\mathcal{N}}{}^T\omega = -{}^T\omega - \frac{\delta^2}{1+\delta^2}g^{-1}(\omega,pJ)^T(pJ),$$

$$B({}^HX, {}^HY) = B({}^HX, {}^T\theta) = B({}^T\omega, {}^HY) = 0,$$

and

$$B({}^T\omega, {}^T\theta) \;=\; g^{-1}(\omega,p)g^{-1}(\theta,p) - g^{-1}(\omega,\theta) - 2\delta^2 g^{-1}(\omega,Jp)g^{-1}(\theta,Jp),$$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$.

Proof. Using Theorem 3.1, Lemma 3.2, (6.3) and (6.4) we get the following:

$$
\begin{aligned}
(i)\ A_{\mathcal{N}}{}^{H}X &= -{}^{t}\big({}^{BS}\nabla_{{}^{H}X}\mathcal{N}\big) = -{}^{t}\big({}^{BS}\nabla_{{}^{H}X}{}^{V}p\big) = 0,\\
(ii)\ A_{\mathcal{N}}{}^{T}\omega &= -{}^{t}\big({}^{BS}\nabla_{{}^{T}\omega}\mathcal{N}\big) = -{}^{t}\big({}^{BS}\nabla_{{}^{V}\omega - g^{-1}(\omega,p){}^{V}p}{}^{V}p\big)\\
&= -{}^{t}\big({}^{BS}\nabla_{{}^{V}\omega}{}^{V}p - g^{-1}(\omega,p){}^{BS}\nabla_{{}^{V}p}{}^{V}p\big)\\
&= -{}^{t}\big({}^{V}\omega + \frac{\delta^2}{\lambda}g^{-1}(\omega,pJ){}^{V}(pJ) - g^{-1}(\omega,p){}^{V}p\big)\\
&= -{}^{T}\omega - \frac{\delta^2}{1+\delta^2}g^{-1}(\omega,pJ){}^{T}(pJ),\\
(iii)\ B({}^{H}X,{}^{H}Y) &= {}^{BS}g({}^{BS}\nabla_{{}^{H}X}{}^{H}Y,\mathcal{N})\\
&= {}^{BS}g({}^{H}(\nabla_X Y),{}^{V}p) + \frac{1}{2}{}^{BS}g({}^{V}(pR(X,Y)),{}^{V}p) = 0,
\end{aligned}
$$

$$
\begin{aligned}
(iv) B({}^{H}X,{}^{T}\theta) &= {}^{BS}g({}^{BS}\nabla_{{}^{H}X}{}^{T}\theta,\mathcal{N})\\
&= {}^{BS}g({}^{BS}\nabla_{{}^{H}X}({}^{V}\theta - g^{-1}(\theta,p){}^{V}p),\mathcal{N})\\
&= {}^{BS}g({}^{BS}\nabla_{{}^{H}X}{}^{V}\theta - {}^{H}X(g^{-1}(\theta,p)){}^{V}p - g^{-1}(\theta,p)\nabla_{{}^{H}X}{}^{V}p,\mathcal{N})\\
&= {}^{BS}g({}^{V}(\nabla_X\theta) + \frac{1}{2}{}^{H}(R(\tilde{p},\widetilde{\theta})X) + \frac{\delta^2}{2}g^{-1}(\theta,pJ){}^{H}(R(J\tilde{p},\tilde{p})X)\\
&\quad - g^{-1}(\nabla_X\theta,p){}^{V}p,\mathcal{N})\\
&= {}^{BS}g({}^{T}(\nabla_X\theta) + \frac{1}{2}{}^{H}(R(\tilde{p},\widetilde{\theta})X) + \frac{\delta^2}{2}g^{-1}(\theta,pJ){}^{H}(R(J\tilde{p},\tilde{p})X),\mathcal{N})\\
&= 0.
\end{aligned}
$$

The other formulas are obtained by a similar calculation. □

Theorem 6.2. *Let (M^n, J, g) be a standard Kähler manifold and $(T_1^*M, {}^{BS}\hat{g})$ its unit cotangent bundle equipped with the Berger-type deformed Sasaki metric, then we have the following formulas.*

$$
\begin{aligned}
{}^{BS}\widehat{R}({}^{H}X,{}^{H}Y){}^{H}Z =& \frac{1}{2}{}^{H}(R(\tilde{p},R(X,Y)\tilde{p})Z) - \frac{\delta^2}{2}g^{-1}(pR(X,Y),pJ){}^{H}(R(J\tilde{p},\tilde{p})Z)\\
&+ \frac{1}{4}{}^{H}(R(\tilde{p},R(X,Z)\tilde{p})Y) - \frac{\delta^2}{4}g^{-1}(pR(X,Z),pJ){}^{H}(R(J\tilde{p},\tilde{p})Y)\\
&- \frac{1}{4}{}^{H}(R(\tilde{p},R(Y,Z)\tilde{p})X) + \frac{\delta^2}{4}g^{-1}(pR(Y,Z),pJ){}^{H}(R(J\tilde{p},\tilde{p})X)\\
&+ {}^{H}(R(X,Y)Z) - \frac{1}{2}{}^{T}\big(p(\nabla_Z R)(X,Y)\big),
\end{aligned}
\tag{6.5}
$$

$$
\begin{aligned}
{}^{BS}\widehat{R}({}^{H}X,{}^{T}\theta){}^{H}Z =& \frac{1}{2}{}^{H}((\nabla_X R)(\tilde{p},\widetilde{\theta})Z) + \frac{\delta^2}{2}g^{-1}(\theta,pJ){}^{H}((\nabla_X R)(J\tilde{p},\tilde{p})Z)\\
&+ \frac{1}{4}{}^{T}(pR(X,R(\tilde{p},\widetilde{\theta})Z)) + \frac{\delta^2}{4}g^{-1}(\theta,pJ){}^{T}(pR(X,R(J\tilde{p},\tilde{p})Z))\\
&- \frac{\delta^2}{2}g^{-1}(\theta,pJ){}^{T}(pR(X,Z)J) - \frac{\delta^2}{2}g^{-1}(pR(X,Z),pJ){}^{T}(\overline{\theta}J)\\
&- \frac{1}{2}{}^{T}(\overline{\theta}R(X,Z)),
\end{aligned}
\tag{6.6}
$$

$$
\begin{aligned}
{}^{BS}\widehat{R}({}^{H}X,{}^{H}Y)^{T}\eta =& \frac{1}{2}({}^{H}(\nabla_X R)(\tilde{p},\tilde{\eta})Y) - \frac{1}{2}({}^{H}(\nabla_Y R)(\tilde{p},\tilde{\eta})X) \\
&+ \frac{\delta^2}{2} g^{-1}(\eta, pJ)\big({}^{H}((\nabla_X R)(J\tilde{p},\tilde{p})Y) - {}^{H}((\nabla_Y R)(J\tilde{p},\tilde{p})X)\big) \\
&+ \frac{\delta^2}{4} g^{-1}(\eta, pJ)\big({}^{T}(pR(X,R(J\tilde{p},\tilde{p})Y) - {}^{T}(pR(Y,R(J\tilde{p},\tilde{p})X)\big) \\
&- \delta^2\big(g^{-1}(pR(X,Y),pJ)^{T}(\overline{\eta}J) + g^{-1}(\eta,pJ)^{T}(pR(X,Y)J)\big) \\
&+ \frac{1}{4}{}^{T}(pR(X,R(\tilde{p},\tilde{\eta})Y)) - \frac{1}{4}{}^{T}(pR(Y,R(\tilde{p},\tilde{\eta})X)) \\
&- {}^{T}(\overline{\eta}R(X,Y)),
\end{aligned} \tag{6.7}
$$

$$
\begin{aligned}
{}^{BS}\widehat{R}({}^{H}X,{}^{T}\theta)^{T}\eta =& \frac{-1}{2}{}^{H}(R(\tilde{\overline{\theta}},\tilde{\overline{\eta}})X) - \frac{1}{4}{}^{H}(R(\tilde{p},\widetilde{\theta})R(\tilde{p},\tilde{\eta})X) \\
&+ \frac{\delta^2}{4} g^{-1}(\theta,pJ)\big(2^{H}(R(J\tilde{p},\tilde{\overline{\eta}})X) - {}^{H}(R(J\tilde{p},\tilde{p})R(\tilde{p},\tilde{\eta})X)\big) \\
&- \frac{\delta^2}{4} g^{-1}(\eta,pJ)\big(2^{H}(R(J\tilde{p},\tilde{\overline{\theta}})X) + {}^{H}(R(\tilde{p},\widetilde{\theta})R(J\tilde{p},\tilde{p})X)\big) \\
&- \frac{\delta^4}{4} g^{-1}(\theta,pJ)g^{-1}(\eta,pJ)^{H}(R(J\tilde{p},\tilde{p})R(J\tilde{p},\tilde{p})X) \\
&- \frac{\delta^2}{2} g^{-1}(\overline{\eta},\overline{\theta}J)^{H}(R(J\tilde{p},\tilde{p})X),
\end{aligned} \tag{6.8}
$$

$$
\begin{aligned}
{}^{BS}R({}^{T}\omega,{}^{T}\theta)^{H}Z =& {}^{H}(R(\tilde{\overline{\omega}},\tilde{\overline{\theta}})Z) + \frac{1}{4}{}^{H}(R(\tilde{p},\widetilde{\omega})R(\tilde{p},\widetilde{\theta})Z) - \frac{1}{4}{}^{H}(R(\tilde{p},\widetilde{\theta})R(\tilde{p},\widetilde{\omega})Z) \\
&+ \frac{\delta^2}{4} g^{-1}(\theta,pJ)\big({}^{H}(R(\tilde{p},\widetilde{\omega})R(J\tilde{p},\tilde{p})Z) - {}^{H}(R(J\tilde{p},\tilde{p})R(\tilde{p},\widetilde{\omega})Z)\big) \\
&- \frac{\delta^2}{4} g^{-1}(\omega,pJ)\big({}^{H}(R(\tilde{p},\widetilde{\theta})R(J\tilde{p},\tilde{p})Z) - {}^{H}(R(J\tilde{p},\tilde{p})R(\tilde{p},\widetilde{\theta})Z)\big) \\
&+ \delta^2 g^{-1}(\theta,pJ)^{H}(R(J\tilde{p},\tilde{\overline{\omega}})Z) - \delta^2 g^{-1}(\omega,pJ)^{H}(R(J\tilde{p},\tilde{\overline{\theta}})Z) \\
&+ \delta^2 g^{-1}(\theta,\omega J)^{H}(R(J\tilde{p},\tilde{p})Z),
\end{aligned} \tag{6.9}
$$

$$
\begin{aligned}
{}^{BS}\widehat{R}({}^{T}\omega,{}^{T}\theta)^{T}\eta =& \big(g^{-1}(\overline{\theta},\overline{\eta}) + \delta^2(1+2\delta^2)g^{-1}(\theta,pJ)g^{-1}(\eta,pJ)\big)^{T}\omega \\
&- \big(g^{-1}(\overline{\omega},\overline{\eta}) + \delta^2(1+2\delta^2)g^{-1}(\omega,pJ)g^{-1}(\eta,pJ)\big)^{T}\theta \\
&+ \delta^2\big(g^{-1}(\overline{\theta},\overline{\eta}J)^{T}(\overline{\omega}J) - g^{-1}(\overline{\omega},\overline{\eta}J)^{T}(\overline{\theta}J) - 2g^{-1}(\overline{\omega},\overline{\theta}J)^{T}(\overline{\eta}J)\big) \\
&+ \delta^2\big(g^{-1}(\omega,pJ)g^{-1}(\overline{\theta},\overline{\eta}) - g^{-1}(\theta,pJ)g^{-1}(\overline{\omega},\overline{\eta})\big)^{T}(pJ),
\end{aligned} \tag{6.10}
$$

for all $X, Y \in \Im^1_0(M)$ *and* $\omega, \theta \in \Im^0_1(M)$, *where* $\overline{\omega} = \omega - g^{-1}(\omega,p)p$ *and* $\tilde{\overline{\omega}} = g^{-1} \circ \overline{\omega}$.

Proof. Using Gauss's equation (6.2), Theorem 4.1 and Lemma 6.1, we directly obtain the formulas (6.5) - (6.9) for the curvature tensor. As for the last formula (6.10), using simple calculations, we find:

$$
{}^{BS}\widehat{R}({}^{T}\omega,{}^{T}\theta)^{T}\eta = {}^{t}({}^{BS}R({}^{T}\omega,{}^{T}\theta)^{T}\eta) - B({}^{T}\omega,{}^{T}\eta).A_{\mathcal{N}}{}^{T}\theta + B({}^{T}\theta,{}^{T}\eta).A_{\mathcal{N}}{}^{T}\omega, \tag{6.11}
$$

$$
\begin{aligned}
{}^{t}\big({}^{BS}R({}^{T}\omega,{}^{T}\theta){}^{T}\eta\big) =& {}^{t}\big({}^{BS}R({}^{V}\overline{\omega},{}^{V}\overline{\theta}){}^{V}\overline{\eta}\big)\\
=& \delta^4 g^{-1}(\eta,pJ)\big(g^{-1}(\theta,pJ)^{T}\omega - g^{-1}(\omega,pJ)^{T}\theta\big)\\
&+\delta^2\big(g^{-1}(\overline{\theta},\overline{\eta}J)^{T}(\overline{\omega}J) - g^{-1}(\overline{\omega},\overline{\eta}J)^{T}(\overline{\theta}J) - 2g^{-1}(\overline{\omega},\overline{\theta}J)^{T}(\overline{\eta}J)\big)\\
&+\frac{\delta^4}{\lambda}\big(g^{-1}(\omega,pJ)g^{-1}(\overline{\theta},\overline{\eta}) - g^{-1}(\theta,pJ)g^{-1}(\overline{\omega},\overline{\eta})\big)^{T}(pJ),
\end{aligned}
$$

$$
\begin{aligned}
B({}^{T}\omega,{}^{T}\eta).A_{\mathcal{N}}{}^{T}\theta =& \big(g^{-1}(\overline{\theta},\overline{\eta}) + 2\delta^2 g^{-1}(\omega,pJ)g^{-1}(\eta,pJ)\big)^{T}\theta\\
&+\frac{\delta^2}{\lambda}g^{-1}(\theta,pJ)\big(g^{-1}(\overline{\theta},\overline{\eta}) + 2\delta^2 g^{-1}(\omega,pJ)g^{-1}(\eta,pJ)\big)^{T}(pJ),
\end{aligned}
$$

and

$$
\begin{aligned}
B({}^{T}\theta,{}^{T}\eta).A_{\mathcal{N}}{}^{T}\omega =& \big(g^{-1}(\overline{\omega},\overline{\eta}) + 2\delta^2 g^{-1}(\theta,pJ)g^{-1}(\eta,pJ)\big)^{T}\omega\\
&+\frac{\delta^2}{\lambda}g^{-1}(\omega,pJ)\big(g^{-1}(\overline{\omega},\overline{\eta}) + 2\delta^2 g^{-1}(\theta,pJ)g^{-1}(\eta,pJ)\big)^{T}(pJ),
\end{aligned}
$$

By substituting them in (6.11), we get (6.10). □

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Abderrahim Zagane
Analysis, geometry and its applications Laboratory,
Department of Mathematics, Relizane University, 48000, Relizane-Algeria
E-mail address: Zaganeabr2018@gmail.com

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