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SPECTRAL PROBLEMS FOR STURM-LIOUVILLE OPERATOR WITH NON-SEPARATED BOUNDARY CONDITION LINEARLY DEPENDENT ON THE EIGENPARAMETER

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Abstract. The paper considers the Sturm-Liouville operator with separated and non-separated boundary conditions. The non-separated boundary condition contains a linear function of the spectral parameter. The properties of eigenvalues are studied, and a theorem of uniqueness of the solution of the inverse problem of recovering the corresponding boundary value problems from two spectra is proved.

1. Introduction

The theory of direct and inverse spectral problems has gained considerable popularity and importance in the last few decades, mainly due to their applications in numerous fields of science and technology (see, for example, [1, 7, 19, 25, 26]). Various aspects of such problems and methods for solving them have been studied by many authors, a large number of works have been published on this topic (see, for example, [20, 26] and the literature therein). Since 1970, specialists have actively begun to study direct and inverse spectral problems for differential operators with non-separated (including periodic, antiperiodic, quasiperiodic and generalized periodic) boundary conditions. A review of results related to solutions of these problems can be found in [12, 18, 22, 27].

From the perspective of physical applications, boundary value problems with a spectral parameter in the boundary conditions are of great interest. Many applied problems from the fields of geophysics, electronics, meteorology, ecology and other sections of modern science also lead to the consideration of such problems (see monograph [1] and the literature cited therein).

Consider a boundary value problem generated on the interval $[0, \pi]$ by the Sturm-Liouville differential equation

$$-y'' + q(x)\, y = \lambda^2 y \tag{1.1}$$

and boundary conditions

$$\begin{aligned} &y(0) = 0, \\ &y'(0) = (\alpha\lambda + \beta)\, y(\pi), \end{aligned} \tag{1.2}$$

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where $q(x) \in L_2[0,\pi]$ is a real function, λ- spectral parameter, $\alpha > 1$, $\beta \neq 0$ are real numbers. Hereafter, we will denote this problem by B.

Direct and inverse problems for equation (1.1) under separated boundary conditions containing a spectral parameter were considered in the works of a number of authors (see [3–5, 7, 10–12, 16, 17, 21, 22], etc.). The problems of recovering operators with a polynomial inclusion of the spectral parameter in non-separated boundary conditions have been studied in [2, 9, 14, 19, 23, 24].

It should be noted that in the work [16] a problem close to B in its formulation is considered, in which the asymptotics of the eigenvalues and eigenfunctions are found in the case when λ is present in the differential equation and boundary condition.

In this paper some properties of the eigenvalues of problem B are studied, and the formulation and proof of a uniqueness theorem for the inverse problem of recovering the corresponding boundary value problems from spectral data are provided. The spectra of two boundary value problems are used as spectral data.

2. Properties of the spectrum of boundary value problems

Definition 2.1. A complex number λ_0 is called an eigenvalue of a boundary value problem B, if the equation (1.1) has a nontrivial solution $y_0(x)$ for $\lambda = \lambda_0$ that satisfies boundary conditions (1.2); in this case $y_0(x)$ is called the eigenfunction of the problem B which corresponds to the eigenvalue λ_0. The set of eigenvalues is called the spectrum of the problem B.

We denote by $s(x,\lambda)$ the solution of equation (1.1) satisfying the initial conditions

$$s(0,\lambda) = 0, s'(0,\lambda) = 1.$$

For any x, functions $s(x,\lambda)$ and $s'(x,\lambda)$ are entire functions (of exponential type) of the variable λ. The eigenvalues of problem B are the zeros of the characteristic function

$$\delta(\lambda) = (\alpha\lambda + \beta)\, s(\pi,\lambda) - 1. \tag{2.1}$$

Using the well-known formula [20]

$$s(\pi,\lambda) = \frac{\sin \pi\lambda}{\lambda} - Q\frac{\cos \pi\lambda}{\lambda^2} + \frac{f(\lambda)}{\lambda^2}$$

and the Paley–Wiener theorem from (2.1) for function $\delta(\lambda)$ we obtain the following representation:

$$\delta(\lambda) = \alpha \sin \pi\lambda - 1 - Q\alpha\frac{\cos \pi\lambda}{\lambda} + \beta\frac{\sin \pi\lambda}{\lambda} + \frac{g(\lambda)}{\lambda}, \tag{2.2}$$

where $Q = \frac{1}{2}\int_0^\pi q(x)\,dx$, $f(\lambda) = \int_{-\pi}^{\pi} \tilde{f}(x) \cos \lambda x dx$, $g(\lambda) = \int_{-\pi}^{\pi} \tilde{g}(x) e^{i\lambda x} dx$,

$$\tilde{f}(x) \in L_2[0,\pi], \tilde{g}(x) \in L_2[-\pi,\pi].$$

Theorem 2.1. *The following statements are true:*

1) The boundary value problem B has a countable set of eigenvalues.

2) The number $\lambda = -\frac{\alpha}{\beta}$ is not an eigenvalue of problem B.

3) If the number $\lambda = 0$ is an eigenvalue of the problem, then it is simple, i.e. if $\delta(0) = 0$, then $\delta'(0) \neq 0$.

4) If for some $\lambda_0 \neq 0$, $\delta(\lambda_0) = 0$ holds, then $\delta(-\lambda_0) \neq 0$, i.e. no two eigenvalues of problem B can be symmetric with respect to the origin.

Proof. Statement 1) is established similarly to the proof of theorem 1 from [18]. The number $\lambda = -\frac{\alpha}{\beta}$ is not a zero of the characteristic function $\delta(\lambda)$, since by virtue of (2.1), $\delta\left(-\frac{\alpha}{\beta}\right) = -1 \neq 0$ holds. Therefore, statement 2) is true.

If $\lambda = 0$ is an eigenvalue of the problem, then from relation (2.1) it follows that

$$\delta(0) = \beta s(\pi, 0) - 1 = 0.$$

From here

$$s(\pi, 0) = \frac{1}{\beta}. \tag{2.3}$$

Since function $s(\pi, \lambda)$ is even, its derivative is an odd function. It is clear that an odd function (defined at zero) is equal to zero at zero.

Therefore

$$\dot{s}(\pi, 0) = 0, \tag{2.4}$$

where the dot above the function denotes differentiation with respect to the parameter λ. Differentiating equality (2.1) with respect to λ, we have

$$\dot{\delta}(\lambda) = \alpha s(\pi, \lambda) + (\alpha\lambda + \beta)\dot{s}(\pi, \lambda).$$

By substituting $\lambda = 0$ into this equation and taking into account relations (2.3) and (2.4), we obtain

$$\dot{\delta}(0) = \alpha s(\pi, 0) + \beta\dot{s}(\pi, 0) = \alpha s(\pi, 0) = \alpha \cdot \frac{1}{\beta} \neq 0.$$

Thus, we have shown that if $\lambda = 0$ is an eigenvalue of problem B, then it is simple and we have thus established the validity of statement 3). Let us finally prove statement 4). Let's assume the opposite. Let $\delta(\lambda_0) = \delta(-\lambda_0) = 0$ $(\lambda_0 \neq 0)$. Then from relation (2.1), due to the parity of function $s(\pi, \lambda)$, we obtain

$$\begin{gathered}
0 = \delta(\lambda_0) - \delta(-\lambda_0) = (\alpha\lambda_0 + \beta) s(\pi, \lambda_0) - (-\alpha\lambda_0 + \beta) s(\pi, -\lambda_0) = \\
= (\alpha\lambda_0 + \beta) s(\pi, \lambda_0) - (-\alpha\lambda_0 + \beta) s(\pi, \lambda_0) = \\
= \alpha\lambda_0 s(\pi, \lambda_0) + \alpha\lambda_0 s(\pi, \lambda_0) = 2\alpha\lambda_0 s(\pi, \lambda_0); \\
0 = \delta(\lambda_0) + \delta(-\lambda_0) = (\alpha\lambda_0 + \beta) s(\pi, \lambda_0) - 1 + (-\alpha\lambda_0 + \beta) s(\pi, \lambda_0) - 1 = \\
= 2\beta s(\pi, \lambda_0) - 2 = 2\beta \cdot 0 - 2 = -2.
\end{gathered}$$

We got the wrong equality $0 = -2$. This means that our assumption is incorrect. Thus, if the number $\lambda_0 \neq 0$ is an eigenvalue, then $-\lambda_0$ cannot be an eigenvalue of problem B. The theorem is proved. □

Along with problem B, boundary value problem B_1 is also considered, generated by the same equation (1.1) and boundary conditions

$$\begin{aligned}
&y(0) = 0, \\
&y'(0) = (\alpha\lambda + \beta) y'(\pi).
\end{aligned}$$

Spectrum $\{\mu_k\}$ $(k = 0, \pm1, \pm2, ...)$ of problem B_1 coincides with the sequence of zeros of the characteristic function

$$\delta_1(\lambda) = (\alpha\lambda + \beta) s'(\pi, \lambda) - 1. \tag{2.5}$$

All the statements of theorem 2.1 are also valid for problem B_1.

Theorem 2.2. *Eigenvalues γ_k and μ_k $(k = 0, \pm 1, \pm 2, ...)$ of boundary value problems B and B_1 respectively, for $|k| \to \infty$ satisfy the following asymptotic formulas:*

$$\gamma_k = k + a_k + \frac{b_k}{\pi k} + \frac{r_k}{k}, \tag{2.6}$$

$$\mu_k = k + \frac{1}{2} + \frac{d_k}{\pi k} + \frac{\xi_k}{k}, \tag{2.7}$$

where $a_k = \frac{(-1)^k}{\pi} \arcsin \frac{1}{\alpha}$, $Q = \frac{1}{2} \int_0^\pi q(x)\, dx$,

$$b_k = Q + \frac{(-1)^{k+1} \beta}{\alpha \sqrt{\alpha^2 - 1}}, \tag{2.8}$$

$$d_k = Q + \frac{(-1)^{k+1}}{\alpha}, \{r_k\}, \ \{\xi_k\} \in l_2.$$

Proof. Using representation (2.2) and Rouché's theorem, it is easy to establish that the roots γ_k of the characteristic equation

$$\delta(\lambda) = 0 \tag{2.9}$$

at $|k| \to \infty$ obey the asymptotics

$$\gamma_k = k + (-1)^k \arcsin \frac{1}{\alpha} + \theta_k, \tag{2.10}$$

where $\theta_k = O\left(\frac{1}{k}\right)$. By substituting the right-hand side of equality (2.10) into equation (2.9), we can obtain a more accurate asymptotic formula for γ_k. Indeed, taking into account (2.10) and expansions $\cos x = 1 + O\left(x^2\right)$, $\sin x = x + O\left(x^3\right)$ $(x \to 0)$, we have

$$\sin \gamma_k \pi = (-1)^k \sin \left[(-1)^k \arcsin \frac{1}{\alpha} + \theta_k \pi \right] =$$

$$= (-1)^k \left[\sin (-1)^k \arcsin \frac{1}{\alpha} \cos \theta_k \pi + \right.$$

$$\left. + \cos (-1)^k \arcsin \frac{1}{\alpha} \sin \theta_k \pi \right] + O\left(\frac{1}{k}\right) =$$

$$= \frac{1}{\alpha} + \sqrt{1 - \frac{1}{\alpha^2}} \cdot \theta_k \pi \cdot (-1)^k + O\left(\frac{1}{k}\right);$$

$$\cos \gamma_k \pi = (-1)^k \cos \left[(-1)^k \arcsin \frac{1}{\alpha} + \theta_k \pi \right] =$$

$$= (-1)^k \sqrt{1 - \frac{1}{\alpha^2}} - \frac{1}{\alpha} \theta_k \pi + O\left(\frac{1}{k}\right).$$

Moreover, $\{g(\gamma_k)\} \in l_2$ by virtue of Lemma 1.4.3 of the book [20]. Then

$$\delta(\gamma_k) = \sqrt{\alpha^2 - 1}\, \theta_k \pi (-1)^k - \frac{Q (-1)^k \sqrt{\alpha^2 - 1}}{k} + \frac{\beta}{\alpha k} + \frac{r_k}{k} = 0, \ \{r_k\} \in l_2.$$

From here

$$\theta_k = \frac{Q}{\pi k} + \frac{(-1)^{k+1} \beta}{\alpha \pi \sqrt{\alpha^2 - 1}} \cdot \frac{1}{k} + \frac{\tilde{r}_k}{k}, \ \{\tilde{r}_k\} \in l_2.$$

Taking this relation into account, from (2.10) we obtain the asymptotic formula (2.6). The validity of the asymptotics (2.7) is established in a completely similar way. It is only necessary to use the relation (2.5) and the representation [20]

$$s'(\pi, \lambda) = \cos\lambda\pi + Q\frac{\sin\lambda\pi}{\lambda} + \frac{f_1(\lambda)}{\lambda},$$

where $f_1(\lambda)$ is an entire function of exponential type no greater than π, square summable on the real axis. The theorem is proved. □

3. Inverse problem

Consider the following inverse problem: given spectra $\{\gamma_k\}$ and $\{\mu_k\}$ of boundary value problems B and B_1, construct function $q(x)$ in the Sturm-Liouville equation (1.1) and coefficients α, β in the boundary conditions (1.2).

The following uniqueness theorem is true.

Theorem 3.1. *Boundary value problems B and B_1 are uniquely recovered from their spectra, with exception of some arbitrary eigenvalue.*

Proof. According to the asymptotic formula (2.6)

$$\gamma_{2k} = 2k + \frac{1}{\pi}\arcsin\frac{1}{\alpha} + \frac{b_{2k}}{2\pi k} + \frac{r_{2k}}{2k}. \tag{3.1}$$

From here we find parameter α of the boundary condition (1.2) using formula

$$\alpha = \frac{1}{\lim\limits_{k\to\infty}\sin\pi\left(\gamma_{2k} - 2k\right)}.$$

Due to the asymptotics (2.7)

$$\mu_{2k+1} + \mu_{2k} = 2k + 1 + \frac{1}{2} + \frac{1}{(2k+1)\pi}\left(Q + \frac{1}{\alpha}\right) +$$
$$+2k + \frac{1}{2} + \frac{1}{2k\pi}\left(Q - \frac{1}{\alpha}\right) + \frac{\omega_k}{k} = 4k + 2 + \frac{Q}{k\pi} + \frac{l_k}{k},$$

where $\{\omega_k\}$, $\{l_k\} \in l_2$. Using this relation we define Q as follows:

$$Q = \pi\lim_{k\to\infty} k\left(\mu_{2k+1} + \mu_{2k} - 4k - 2\right).$$

Knowing the values of α and Q, it is possible to uniquely recover the parameter β using (2.6) and (2.8). Indeed, due to the relation (2.8), we have

$$b_{2k} = Q - \frac{\beta}{\alpha\sqrt{\alpha^2 - 1}}.$$

On the other hand, from (3.1) it easily follows that

$$b_{2k} = 2\pi k\left(\gamma_{2k} - 2k - \frac{1}{\pi}\arcsin\frac{1}{\alpha}\right) + \pi r_{2k}.$$

According to the last two equalities we find

$$\beta = \alpha\sqrt{\alpha^2 - 1}\left[Q - 2\pi\lim_{k\to\infty} k\left(\gamma_{2k} - 2k - \frac{1}{\pi}\arcsin\frac{1}{\alpha}\right)\right].$$

Now, using Theorem 5 of article [6], we will show that characteristic functions $\delta(\lambda)$ and $\delta_1(\lambda)$ of boundary value problems B and B_1 can be uniquely represented as an infinite product over spectra $\{\gamma_k\}$ and $\{\mu_k\}$ with exception of any one eigenvalue. Indeed, let us consider the function

$$\theta(\lambda) = \lambda\delta(\lambda) = \lambda S(\lambda) + \int_{-\pi}^{\pi} w(x)\, e^{i\lambda x} dx,$$

where

$$S(\lambda) = \alpha \sin \pi\lambda - 1 + \frac{Q\alpha}{\lambda}\left(\frac{\sin \pi\lambda}{\pi\lambda} - \cos \pi\lambda\right) + \beta\frac{\sin \pi\lambda}{\lambda},$$

$$w(x) = \tilde{g}(x) - \frac{Q\alpha}{2\pi}$$

(see (2.2)). The function $S(\lambda)$ is a sine-type function, and the theorem mentioned above states that $\theta(\lambda)$ is representable as an infinite product over its zeros and uniquely determined by them with exception of any one. The same fact can be obtained with respect to function $\delta_1(\lambda)$ in a similar way.

From relations (2.1) and (2.5) we determine the characteristic functions $s(\pi, \lambda)$ and $s'(\pi, \lambda)$ of the boundary value problems generated by the same equation (1.1) and boundary conditions $y(0) = y(\pi) = 0$ and $y(0) = y'(\pi) = 0$ respectively, using formulas

$$s(\pi, \lambda) = \frac{\delta(\lambda) + 1}{\alpha\lambda + \beta}, s'(\pi, \lambda) = \frac{\delta_1(\lambda) + 1}{\alpha\lambda + \beta}.$$

Finally, from the sequences of zeros of functions $s(\pi, \lambda)$ and $s'(\pi, \lambda)$, the coefficient $q(x)$ of equation (1.1) is recovered using a well-known procedure (see, for example, [15]). The theorem is proved. □

It is easy to see that the proof of theorem 3.1 also contains an algorithm for recovering boundary value problems from two spectra.

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References

[1] A. M Akhtyamov, *Theory of identification of boundary conditions and its applications*, Fizmatlit, Moscow, 2009 (in Russian).

[2] A.M. Akhtyamov, Kh.R. Mamedov, Ya.T. Sultanaev, Inverse Sturm-Liouville problems with polynomials in nonseparated boundary conditions, *Baku Math. Journal* **1(2)** (2022), 179-194.

[3] V. Ala, Kh. R. Mamedov, On a discontinuous Sturm-Liouville problem with eigenvalue parameter in the boundary conditions, Dynamic Systems and Applications, 29 (2020), 182-191.

[4] N.M. Aslanova, M. Bayramoglu, Kh.M. Aslanov, On One Class Eigenvalue Problem with Eigenvalue Parameter in the Boundary Condition at One End-Point, *Filomat*, **32(19)** (2018), 6667–6674.

[5] N.P. Bondarenko, E.E. Chitorkin, Inverse Sturm-Liouville Problem with Spectral Parameter in the Boundary Conditions, *Mathematics*, **11(5)** (2023) article 1138.

[6] S.A. Buterin, On the uniform stability of recovering sine-type functions with asymptotically separated zeros, *Math. Notes*, **111(3)** (2022), 343–355.
[7] S. Buterin, M. Kuznetsova, On Borg's method for non-selfadjoint Sturm-Liouville operators, *Anal. Math. Phys.*, **9** (2019), 2133-2150.
[8] L. Collatz, *Eigenwertaufgaben mit technischen Anwendungen*, Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1949.
[9] G. Freiling, V. Yurko, Recovering nonselfadjoint differential pencils with nonseparated boundary conditions, *Applicable Anal.*, **94(8)** (2015), 1649-1661.
[10] Y. Güldü, R. Kh. Amirov, N. Topsakal, On impulsive Sturm–Liouville operators with singularity and spectral parameter in boundary conditions, *Ukrainian Math. J.* **64(12)** (2013), 1816-1838.
[11] N. J. Guliyev, Schrödinger operators with distributional potentials and boundary conditions dependent on the eigenvalue parameter, *J. Math. Phys*, **60(6)** (2019), 23 pp.
[12] N. J. Guliyev, Inverse square singularities and eigenparameter dependent boundary conditions are two sides of the same coin, *Q. J. Math.*, **74(3)** (2023), 889–910.
[13] I. M. Guseinov, I. M. Nabiev, The inverse spectral problem for pencils of differential operators, *Sb. Math.*, **198(11)** (2007), 1579–1598.
[14] Ch.G. Ibadzadeh, I.M. Nabiev, Reconstruction of the Sturm–Liouville operator with nonseparated boundary conditions and a spectral parameter in the boundary condition, *Ukrainian Math. J.*, **69(9)** (2018), 1416-1423.
[15] T. M. Karaseva, On the inverse Sturm-Liouville problem for a non-Hermitian operator, *Matem. Sb.*, **32(74)**, 2, (1953), 477-483 (in Russian).
[16] T. F. Kasimov, Asymptotics of the Eigenvalues and Eigenfunctions of a Non-Self-Adjoint Problem with a Spectral Parameter in the Boundary Condition, *J. Contemporary Applied Math.*, **11(2)** (2021), 11-22.
[17] N. B. Kerimov, Kh. R. Mamedov, On a boundary value problem with a spectral parameter in boundary conditions, *Siberian Math. J.*, **40(2)** (1999), 281-290.
[18] L. I. Mammadova, I. M. Nabiev, Spectral properties of the Sturm–Liouville operator with a spectral parameter quadratically included in the boundary condition, *Vestnik Udmurtskogo Universiteta. Matematika Mekhanika Komp'yuternye Nauki*, **30(2)** (2020), 237—248.
[19] L. I. Mammadova, I. M. Nabiev, Solvability of the inverse spectral problem for Sturm-Liouville operator with spectral parameter in the boundary condition, *Baku Math. Journal*, 2024, **3(1)**, 12-23.
[20] V. A. Marchenko, *Sturm–Liouville operators and applications*, AMS Chelsea Publishing, Providence, RI, 2011.
[21] M. Möller, V. Pivovarchik, *Spectral Theory of Operator Pencils, Hermite-Biehler Functions, and their Applications*, Birkhauser: Cham, 2015.
[22] A. A. Nabiev, On a boundary value problem for a polynomial pencil of the Sturm-Liouville equation with spectral parameter in boundary conditions, *Applied Mathematics*, **7(18)** (2016), 2418–2423.
[23] I. M. Nabiev, Reconstruction of the differential operator with spectral parameter in the boundary condition, *Mediterr. J Math.*, **19(3)** (2022), 1–14.
[24] I.M. Nabiev, L.I. Mammadova, G.S. Mammedzadeh, Algorithms for recovery of diffusion and Sturm-Liouville operators with semi-separated boundary conditions, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, **49(2)** (2023), 219–227.
[25] V. Pivovarchik, Inverse Sturm-Liouville problem with a boundary condition linear in the spectral parameter, *J. Nonlinear Funct. Anal.*, **2018** (2018), Article ID 21, 1-10.

[26] A. N. Tikhonov, A. A. Samarskii, *Equations of Mathematical Physics*, Publishing House of Moscow State University, Moscow, 1999.

[27] V. A. Yurko, *Introduction to the theory of inverse spectral problems*, Fizmatlit, Moscow, 2007 (in Russian).

[28] V. A. Yurko, Inverse spectral problems for differential operators with non-separated boundary conditions, *J. Inverse Ill-Posed Probl.*, **28(4)** (2020), 567- 616.

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