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ON THE APPLICATION OF ANALOGUES OF THE KOROVKIN THEOREM TO THE CONVERGENCE OF CERTAIN CLASSES OF SEQUENCES OF CONVOLUTION OPERATORS

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Abstract. This paper establishes conditions on the Young function Φ and the summability exponent $p(\cdot)$ that ensure the validity of analogues of Korovkin's theorem for sequences of positive convolution operators in Orlicz spaces L^{Φ} and Lebesgue spaces $L^{p}(\cdot)$ with a variable summability exponent. The results obtained in the space L^{Φ} are applied to the convergence of a sequence of Fejer and Poisson operators. Also, using the obtained result, the convergence of the family of Fejer and Steklov operators to the unit operator in the space $L^{p}(\cdot)$ is established.

1. Introduction

It is well known that one of the important tools in approximation theory is Korovkin's theorem ([19]) on the convergence of a sequence of linear positive operators. This theorem states that if a sequence of linear positive operators $L_n : C([0,1]) \to C([0,1])$, $n \in N$, satisfies the condition

$$\lim_{n\to\infty} \|L_n(g_i) - g_i\|_\infty = 0, g_i(t) = t^i, i = 0, 1, 2,$$

then for any $f \in C([0,1])$ it holds

$$\lim_{n\to\infty} \|L_n(f) - f\|_\infty = 0.$$

Note that a linear operator $L : F(X) \to F(Y)$ is called positive if for $\forall f \in F(X)$ satisfying $f \geq 0$, we have $L(f) \geq 0$, where X and Y are metric spaces, $F(X)$ is a linear space of functions $f : X \to R$. In [2], applications of Korovkin's theorem are given for a sequence of linear positive operators generated by the polynomials of Bernstein, Kantorovich, and others. These results contribute to obtaining analogues of theorems of Korovkin type and their statistical variants in L^p spaces. Note that the concept of statistical convergence was introduced by J.A. Fridy in [11]. Statistical convergence in arbitrary metric and uniformly topological spaces was studied by B.T. Bilalov, T. Nazarova in [4]. Korovkin-type theorems in Lebesgue spaces have been studied in [9, 12, 23]. The convergence

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of a sequence of operators generated by Kantorovich polynomials in Morrey-type spaces was studied in [8]. Korovkin-type theorems and their statistical variants in grand Lebesgue spaces were considered in [29]. Convergence of sequences of operators to an identity operator in Lebesgue spaces with a variable summability exponent was studied in [14, 26]. It should be noted that in [26] the uniform boundedness of the family of convolution operators is proved and its applications to the convergence of the sequence of Fejer and Poisson operators to the identity operator are given. In [14], it is proved that a sequence of convolution operators with an approximative kernel converges to an identity operator in the Lebesgue space with a variable summability exponent. Analogues of Korovkin's theorems in general function spaces were studied in [3, 13, 30].

This paper is devoted to the study of the convergence of a sequence of positive convolution operators to an identity operator in Orlicz spaces and in spaces with a variable summability exponent. It is established that in the reflexive Orlicz space, the family of convolution operators with a positive kernel is uniformly bounded. The obtained result is used to prove an analogue of Korovkin's theorem in Orlicz spaces. This question is also studied in the Lebesgue space with a variable summability exponent. The results obtained are applied to the convergence of convolution operators with Fejer and Poisson kernels in Orlicz spaces, Fejer and Steklov operators in Lebesgue spaces with variable summability exponent.

2. Preliminary concepts and facts

Let us give some standard notations: R is the set of real numbers; N is the set of natural numbers; X^* is the conjugate space to the Banach space X; $B(X)$ is the Banach space of linear bounded operators acting from X to X. Let $\Phi(t)$: $[0,+\infty) \to R$ be the Young function, i.e. is a convex, continuous function such that $\Phi(0) = 0, \Phi(t) > 0$, $t > 0$, and the following conditions are satisfied

$$\lim_{t\to+0} \frac{\Phi(t)}{t} = 0, \lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty.$$

By specifying the Young function $\Phi(t)$, the function is determined

$$\Psi(t) = \sup_{s\geq 0} \left\{ts - \Phi(s)\right\}, t \geq 0.$$

Function $\Psi(t)$ is also a Young function and is called a complementary function to function $\Phi(t)$. Complementary to $\Psi(t)$ is the function $\Phi(t)$.

Definition 2.1. The Young function $\Phi(t)$ is said to satisfy the Δ_2-condition, and write $\Phi \in \Delta_2$, if there exist $k > 0$ and $t_0 \geq 0$ such that

$$\Phi(2t) \leq k\Phi(t), t \geq t_0.$$

Denote by $L^{\Phi}(-\pi,\pi)$ the Orlicz space of measurable on $[-\pi,\pi]$ functions f : $[-\pi,\pi] \to C$ for which there exists a number $\lambda > 0$ such that

$$\int_{-\pi}^{\pi} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx < +\infty.$$

$L^{\Phi}(-\pi,\pi)$ is a Banach space with Luxembourg norm

$$\|f\|_{\Phi}=\inf\left\{\lambda>0:\int\limits_{-\pi}^{\pi}\Phi\left(\frac{|f(x)|}{\lambda}\right)dx\le 1\right\}.$$

In particular, for $\Phi(t)=t^p$ the space $L^{\Phi}(-\pi,\pi)$ coincides with the ordinary Lebesgue space $L^p(-\pi,\pi)$. If $\Phi\in\Delta_2$, then the dual space of the space $L^{\Phi}(-\pi,\pi)$ is isometric to the space $L^{\Psi}(-\pi,\pi)$. Moreover, if $\Psi\in\Delta_2$, then the space $L^{\Phi}(-\pi,\pi)$ is reflexive. The space $L^{\Phi}(-\pi,\pi)$ is complete. In the space $L^{\Phi}(-\pi,\pi)$, the equivalent norm is the following Orlicz norm

$$\|f\|_{\Phi}^{*}=\sup_{g\in S_{\Psi}}\int\limits_{-\pi}^{\pi}|f(x)g(x)|\,dx,$$

where $S_{\Psi}=\left\{g\in L^{\Psi}(-\pi,\pi):\|g\|_{\Psi}\le 1\right\}$. The following Hölder inequality holds: if $f\in L^{\Phi}(-\pi,\pi)$ and $g\in L^{\Psi}(-\pi,\pi)$, then $fg\in L^{1}(-\pi,\pi)$ and there exists $c>0$ such that

$$\int\limits_{-\pi}^{\pi}|f(x)g(x)|\,dx\le c\,\|f\|_{\Phi}\,\|g\|_{\Psi}\,.$$

We will need the following fact ([25, Theorem 3, p. 1375]).

Theorem 2.1. *Suppose that T is a bounded linear operator on $L^p(-\pi,\pi)$ into $L^p(-\pi,\pi)$, for $1<p<+\infty$. If $L^{\Phi}(-\pi,\pi)$ is reflexive then T is defined and bounded on $L^{\Phi}(-\pi,\pi)$ into $L^{\Phi}(-\pi,\pi)$.*

Let $\Phi^{-1}(t)$ be the inverse of the function $\Phi(t)$. Let

$$h(t)=\limsup_{x\to+\infty}\frac{\Phi^{-1}(t)}{\Phi^{-}(tx)}.$$

The numbers

$$\alpha_{\Phi}=-\lim_{t\to+\infty}\frac{\ln h(t)}{\ln t},\beta_{\Phi}=-\lim_{t\to+0}\frac{\ln h(t)}{\ln t}$$

are the Boyd indices (see [7]) of the Orlicz space $L^{\Phi}(-\pi,\pi)$. The following properties are valid

$$0\le\alpha_{\Phi}\le\beta_{\Phi}\le 1;$$

$$\alpha_{\Phi}+\beta_{\Psi}=1.$$

Condition $0<\alpha_{\Phi}\le\beta_{\Phi}<1$ is equivalent to the reflexivity of the space $L^{\Phi}(-\pi,\pi)$. Moreover, if

$$1\le q<\frac{1}{\beta_{\Phi}}\le\frac{1}{\alpha_{\Phi}}<p\le+\infty,$$

then the following continuous embedding holds:

$$L^{p}(-\pi,\pi)\subset L^{\Phi}(-\pi,\pi)\subset L^{q}(-\pi,\pi).$$

More general information about Orlicz spaces can be obtained from [21, 22, 24].

Now we present the necessary information from the theory of Lebesgue spaces with a variable summability exponent.

Let a function $p(\cdot) : [-\pi, \pi] \to [1, +\infty)$ be given and $p_+ = \underset{x\in[-\pi,\pi]}{ess\sup}\, p(x)$. Denote by $L^{p(\cdot)}(-\pi, \pi)$ the Lebesgue space with a variable exponent $p(\cdot)$ of measurable on $[-\pi, \pi]$ functions $f : [-\pi, \pi] \to C$ for which there exists a number $\lambda > 0$ such that

$$\int_{-\pi}^{\pi} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < +\infty.$$

$L^{p(\cdot)}(-\pi, \pi)$ is a Banach space ([10, 17, 18, 20, 27]) with norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{-\pi}^{\pi} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\}.$$

For $p(x) = p$, the space $L^{p(\cdot)}(-\pi, \pi)$ coincides with $L^p(-\pi, \pi)$.

Definition 2.2. A function $p(x)$ is called locally log-Holder continuous and is written $p \in \mathrm{P}^{\log}(-\pi, \pi)$, if $\exists c > 0$ such that

$$|p(x_1) - p(x_2)| \le -\frac{c}{\ln |x_1 - x_2|}, \forall x_1, x_2 \in [-\pi, \pi], |x_1 - x_2| \le \frac{1}{2}.$$

Let $L^{p(\cdot)}_{2\pi}(-\pi, \pi)$ denote the space of functions $f \in L^{p(\cdot)}(-\pi, \pi)$ that are 2π-periodically extended to the entire line R. It is known [20] that if $p_+ < +\infty$, then the set $C_0^\infty(-\pi, \pi)$ is dense in $L^{p(\cdot)}(-\pi, \pi)$. Concerning approximation in the space $L^{p(\cdot)}(-\pi, \pi)$, the works [1, 5, 6, 15, 16, 27, 28] and others are known. In [16] it is also established that if $p \in \mathrm{P}^{\log}(-\pi, \pi)$ and $p_+ < +\infty$, then there exists a number $c_{p(\cdot)} > 0$ such that for $\forall f \in L^{p(\cdot)}(-\pi, \pi)$ and $\forall g \in L^1(-\pi, \pi)$ the relation

$$\|f * g\|_{p(\cdot)} \le c_{p(\cdot)} \|f\|_{p(\cdot)} \|g\|_1 \tag{2.1}$$

holds.

3. Main Results

Let $C[-\pi, \pi]$ be the set of continuous functions $f : [-\pi, \pi] \to R$ on $[-\pi, \pi]$ with norm

$$\|f\|_\infty = \sup_{x\in[-\pi,\pi]} |f(x)|.$$

$L^p(-\pi, \pi)$, $1 \le p < +\infty$, is the space of measurable functions $f : [-\pi, \pi] \to C$ on $[-\pi, \pi]$ with norm

$$\|f\|_p = \left(\int_{-\pi}^{\pi} |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

We denote by $C_{2\pi}(R)$ and $L^p_{2\pi}(R)$, the space of functions from $C[-\pi, \pi]$ and $L^p(-\pi, \pi)$, respectively, which are 2π-periodically extended to the entire line R.

Let $\{\varphi_n\}_{n\in N} \subset L^1_{2\pi}(R)$ be a positive periodic kernel, i.e. $\varphi_n \geq 0$ almost everywhere on R and such that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(t)dt = 1.$$

A positive periodic kernel $\{\varphi_n\}_{n\in N}$ is called approximatively identical if

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{\delta\leq|t|\leq\pi} \varphi_n(t)dt = 0$$

holds for $\forall\delta \in (0, \pi)$. Consider the sequence of convolution operators

$$L_n(f)(x) = (f * \varphi_n)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\varphi_n(t)dt, n \in N \tag{3.1}$$

for $f \in L^p_{2\pi}(R)$. In the work [2] equivalent conditions of convergence in $L^p_{2\pi}(R)$ the sequence of operators (3.1) to the identity operator are established. Namely ([2, Theorem 4.4, p. 108]), the following is proved

Theorem 3.1. *The following conditions are equivalent:*
a) for every $f \in L^p_{2\pi}(R)$

$$\lim_{n\to\infty} \|L_n f - f\|_p = 0$$

and for every $f \in C_{2\pi}(R)$

$$\lim_{n\to\infty} \|L_n f - f\|_\infty = 0;$$

b) $\lim\limits_{n\to\infty} \beta_n = \lim\limits_{n\to\infty} \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \varphi_n(t)\sin^2 \frac{t}{2}dt = 0$*;*
c) $\{\varphi_n\}_{n\in N}$ *is approximately identical.*

In the proposed work this result is proved in Orlicz spaces and in Lebesgue spaces with variable summability exponent. Let $\Phi(t) : [0, +\infty) \to R$ be a Young function, $L^\Phi_{2\pi}(R)$ be the space of functions $f \in L^\Phi(-\pi, \pi)$ that are 2π-periodically extended to the entire line R.

Theorem 3.2. *Let* $0 < \alpha_\Phi \leq \beta_\Phi < 1$*. The following conditions are equivalent:*
a) for every $f \in L^\Phi_{2\pi}(R)$

$$\lim_{n\to\infty} \|L_n f - f\|_\Phi = 0$$

and for every $f \in C_{2\pi}(R)$

$$\lim_{n\to\infty} \|L_n f - f\|_\infty = 0;$$

b) $\lim\limits_{n\to\infty} \beta_n = 0$*;*
c) $\{\varphi_n\}_{n\in N}$ *is approximately identical.*

Proof. Let condition a) be satisfied. It is known that

$$\|L_n f\|_p \leq \|\varphi_n\|_1 \cdot \|f\|_p, n \in N.$$

Consequently, $\|L_n\|_{B(L^p)} \le c$, $n \in N$, where $c = \sup_n \int_{-\pi}^{\pi} \varphi_n(t)dt$. By virtue of the continuous embedding of $C_{2\pi}(R)$ in $L^p_{2\pi}(R)$ there exists a number $c_0 > 0$ such that $\|f\|_p \le c_0 \|f\|_\infty$, $\forall f \in C_{2\pi}(R)$. Then for $\forall f \in C_{2\pi}(R)$ we have $\lim_{n\to\infty} \|L_n f - f\|_p = 0$. Taking into account $\|L_n\|_{B(L^p)} \le c$, $n \in N$, using the principle of uniform boundedness, for $\forall f \in L^p_{2\pi}(R)$ we obtain that $\lim_{n\to\infty} \|L_n f - f\|_p = 0$. By virtue of Theorem 3.1, conditions b) and c) are valid.

Now let condition b) be satisfied. By Theorem 3.1, for $\forall f \in L^p_{2\pi}(R)$ the conditions are true

$$\lim_{n\to\infty} \|L_n f - f\|_p = 0$$

and for $\forall f \in C_{2\pi}(R)$

$$\lim_{n\to\infty} \|L_n f - f\|_\infty = 0.$$

By virtue of Theorem 3.1, we obtain that the operator L_n acts boundedly in L^Φ, and there exists a number $k > 0$ such that $\|L_n f\|_\Phi \le k \|f\|_\Phi$, i.e.

$$\|L_n\|_{B(L^\Phi)} \le k, n \in N. \tag{3.2}$$

Then a) holds. Indeed, take an arbitrary function $f \in L^\Phi_{2\pi}(R)$. Since $C_{2\pi}(R)$ is dense in $L^\Phi_{2\pi}(R)$, there exists a sequence of functions $g_m \in C_{2\pi}(R), m \in N$, such that

$$\lim_{m\to\infty} \|g_m - f\|_\Phi = 0. \tag{3.3}$$

By the triangle inequality, we obtain

$$\|L_n f - f\|_\Phi \le \|L_n(g_m - f)\|_\Phi + \|L_n g_m - g_m\|_\Phi + \|g_m - f\|_\Phi . \tag{3.4}$$

Taking into account (3.2) in (3.4), we obtain

$$\|L_n f - f\|_\Phi \le \|L_n g_m - g_m\|_\Phi + (k+1) \|g_m - f\|_\Phi . \tag{3.5}$$

Therefore, passing to the limit in (3.5) at $n \to \infty$, we obtain

$$\overline{\lim_{n\to\infty}} \|L_n f - f\|_\Phi \le (k+1) \|g_m - f\|_\Phi .$$

On the other hand, taking into account (3.3), it follows from the last relation that

$$\lim_{n\to\infty} \|L_n f - f\|_\Phi = 0.$$

The equivalence of b) and c) follows from Theorem 3.1. The theorem is proved.

Let us apply Theorem 3.2 to the convergence of the Fejer and Poisson operators to the identity operator.

Consider the Fejer operator

$$F_n(f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\varphi_n(x-t)dt, n \in N, f \in L^\Phi_{2\pi}(R),$$

where

$$\varphi_n(t) = \begin{cases} \frac{\sin^2((n+1)t/2)}{(n+1)\sin^2(t/2)}, t \ne 2\pi k \\ n+1, t = 2\pi k. \end{cases}$$

If $0 < \alpha_\Phi \le \beta_\Phi < 1$, then $F_n(f) \to f$ in the space $L^\Phi_{2\pi}(R)$ for $n \to \infty$.

In fact, for $n \to \infty$ we have

$$\beta_n = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \varphi_n(t) \sin^2 \frac{t}{2} dt = \frac{1}{\pi(n+1)} \int\limits_{0}^{\pi} \sin^2 \frac{(n+1)t}{2} dt = \frac{1}{2(n+1)} \to 0.$$

It remains to apply Theorem 3.2.

Consider the Poisson operator

$$P_r(f)(x) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(t) P_r(x-t) dt, 0 \le r < 1, f \in L_{2\pi}^{\Phi}(R),$$

where $P_r(t) = \frac{1-r^2}{1-2r\cos t + t^2}$. If $0 < \alpha_\Phi \le \beta_\Phi < 1$, then $P_r(f) \to f$ in the space $L_{2\pi}^{\Phi}(R)$ for $r \to 1$.

Indeed, let $r_n < 1$ be any sequence: $r_n \to 1$. We have

$$\beta_r = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} P_r(t) \sin^2 \frac{t}{2} dt = \frac{1-r}{2}.$$

Thus, $\beta_n = \frac{1-r_n}{2} \to 0$. Therefore, by Theorem 3.2, we have $P_r(f) \to f$ in the space $L_{2\pi}^{\Phi}(R)$ for $r \to 1$.

Let us proceed to study the analogue of Korovkin's theorem for a sequence of operators (3.1) in spaces $L_{2\pi}^{p(\cdot)}(-\pi,\pi)$. Recall that the following theorem ([14, 16]) is known regarding the convergence of a sequence of operators (3.1) in spaces $L_{2\pi}^{p(\cdot)}(-\pi,\pi)$.

Theorem 3.3. *Let $p \in \mathrm{P}^{\log}(-\pi,\pi)$ and $p_+ < +\infty$. If $\{\varphi_n\}_{n\in N}$ is an approximately identical kernel, then $\forall f \in L_{2\pi}^{p(\cdot)}(R)$*

$$\lim_{n\to\infty} \|L_n f - f\|_{p(\cdot)} = 0.$$

We obtain this result from the following analogue of Korovkin's theorem in the space $L_{2\pi}^{p(\cdot)}(-\pi,\pi)$.

Theorem 3.4. *Let $p \in \mathrm{P}^{\log}(-\pi,\pi)$ and $p_+ < +\infty$. The following conditions are equivalent:*

a) for every $\forall f \in L_{2\pi}^{p(\cdot)}(R)$

$$\lim_{n\to\infty} \|L_n f - f\|_{p(\cdot)} = 0$$

and for every $f \in C_{2\pi}(R)$

$$\lim_{n\to\infty} \|L_n f - f\|_{\infty} = 0;$$

b) $\lim\limits_{n\to\infty} \beta_n = 0$;

c) $\{\varphi_n\}_{n\in N}$ is approximately identical.

Proof. According to Theorem 3.1, conditions b) and c) are equivalent. Let us prove the equivalence of conditions a) and b). Let condition b) be satisfied. Then, by Theorem 3.1, $\forall g \in C_{2\pi}(R)$ we have

$$\lim_{n\to\infty} \|L_n g - g\|_{\infty} = 0. \tag{3.6}$$

Take an arbitrary $\forall f \in L_{2\pi}^{p(\cdot)}(R)$. It follows from the conditions of the theorem that the set $C_{2\pi}(R)$ is dense in $L_{2\pi}^{p(\cdot)}(R)$. Then for $\forall\varepsilon > 0$ there exists $g \in C_{2\pi}(R)$ such that $\|f-g\|_{p(\cdot)} < \varepsilon$. Applying (2.1), for $\forall g \in C_{2\pi}(R)$ we obtain that

$$\|L_n f - L_n g\|_{p(\cdot)} < c_{p(\cdot)}\varepsilon.$$

Therefore, by the triangle inequality, we have

$$\|L_n f - f\|_{p(\cdot)} \le \|L_n f - L_n g\|_{p(\cdot)} +$$

$$+ \|L_n g - g\|_{p(\cdot)} + \|f-g\|_{p(\cdot)} \le$$
$$\le (1 + c_{p(\cdot)}) \|f-g\|_{p(\cdot)} + c_1 \|L_n g - g\|_\infty .$$

Hence, taking into account (3.6), we obtain that

$$\overline{\lim_{n\to\infty}} \|L_n f - f\|_{p(\cdot)} \le (1 + c_{p(\cdot)})\varepsilon.$$

Then, due to the arbitrariness of ε, we obtain that

$$\lim_{n\to\infty} \|L_n f - f\|_{p(\cdot)} = 0.$$

The proof that a) implies b) is similar to the proof of this part in Theorem 3.2. The theorem is proved.

From Theorem 3.4 it immediately follows that the families of Fejer and Steklov operators converge to the identity operator (see [26, §3]).

Let $p \in \mathrm{P}^{\log}(-\pi,\pi)$ and $p_+ < +\infty$, assume that

$$F_\lambda(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)k_\lambda(x-t)dt, \lambda \ge 1, f \in L_{2\pi}^{p(\cdot)}(R),$$

is a Fejer operator, i.e. the kernel is given by the formula

$$k_\lambda(t) = \frac{2}{n+1}\left(\frac{\sin((n+1)t/2)}{2\sin(t/2)}\right)^2, n \le \lambda < n+1.$$

Then $F_\lambda(f) \to f$ in the space $L_{2\pi}^{p(\cdot)}(R)$ for $\lambda \to \infty$.

Let us calculate the number β_λ for the Fejer kernel $k_\lambda(t)$. We have

$$\beta_\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_\lambda(t)\sin^2\frac{t}{2}dt = \frac{1}{2\pi(n+1)}\int_{0}^{\pi} \sin^2\frac{(n+1)t}{2}dt = \frac{1}{4(n+1)} \to 0,$$

as $\lambda \to \infty$. By Theorem 3.4 we obtain that $F_\lambda(f) \to f$ in $L_{2\pi}^{p(\cdot)}(R)$ when $\lambda \to \infty$.

Let $\lambda \ge 1$, $\Delta_\lambda = [-\frac{1}{2\lambda}, \frac{1}{2\lambda}]$. Assume that

$$k_\lambda(t) = \begin{cases} 2\pi\lambda, t \in \Delta_\lambda \\ 0, t \in [-\pi,\pi]\backslash\Delta_\lambda \end{cases}$$

Extend $k_\lambda(t)$ 2π-periodically to the entire line R. Consider the Steklov operator

$$S_\lambda(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)k_\lambda(x-t)dt, \lambda \ge 1, f \in L_{2\pi}^{p(\cdot)}(R).$$

Then $S_\lambda(f) \to f$ in $L_{2\pi}^{p(\cdot)}(R)$ for $\lambda \to \infty$.

For any sequence $\lambda_n \geq 1$, $\lim\limits_{n\to\infty} \lambda_n = +\infty$, the kernel $k_n(t) = k_{\lambda_n}(t)$ is approximately identical. In fact, for $\forall\delta \in (0,\pi)$ there exists n_0 such that for $\forall n > n_0$ we have $\delta > \frac{1}{2\lambda_n}$. Therefore, for $\forall n > n_0$ we have

$$\int\limits_{\delta\leq|t|\leq\pi} k_n(t)dt = 0.$$

Using Theorem 3.4, we obtain $S_\lambda(f) \to f$ in the space $L_{2\pi}^{p(\cdot)}(R)$ for $\lambda \to \infty$.

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