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THE BEREZIN TRANSFORM AND COMMUTATIVITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE OF THE UPPER HALF-PLANE

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Abstract. The aim of this paper is twofold. First, we explore the role of the pioneering concept of the Berezin transform and its fixed points in the framework of the Bergman space on the complex upper half-plane. Second, in the setting of this Bergman space, we characterize commuting pairs of Toeplitz operators with bounded symbols that are either harmonic, of bi-harmonic type, or of bi-analytic type. In particular, this allows us to characterize normal Toeplitz operators.

1. Introduction

Let Π^+ be the complex upper half-plane in $\mathbb{C}$, which is defined by

$$\Pi^+ = \{z \in \mathbb{C}, \text{ such that } \operatorname{Im} z > 0\}.$$

The Cayley transform $\varphi(z) := \dfrac{z-i}{z+i}$ provides a conformal bijection between the upper half-plane Π^+ and the unit disk $\mathbb{D}$. As we will see repeatedly in the sequel, this Möbius transformation enables us to carry certain properties of holomorphic and harmonic functions from the upper half-plane to the unit disk and vice versa. If $dA(z) := \frac{1}{\pi} dz d\overline{z} = \frac{1}{\pi} dx dy$ denotes the Lebesgue area measure over Π^+, then for $1 \leq p < \infty$, denote by $\mathbb{L}^p(\Pi^+, dA)$ the Lebesgue space of p-integrable functions over Π^+, and by $\mathbb{L}^\infty(\Pi^+)$ the algebra of essentially bounded functions over Π^+. Let $\mathcal{A}^p(\Pi^+)$ be the Bergman space over the upper half-plane Π^+, which is the closed subspace of $\mathbb{L}^p(\Pi^+, dA)$ consisting of all holomorphic functions over Π^+. It is therefore a Banach space with induced norm given by: $\|f\|_{\mathcal{A}^p(\Pi^+)} := \left(\int\limits_{\Pi^+} |f(z)|^p dA(z) \right)^{\frac{1}{p}}$. In particular, for $p = 2$, the space $\mathcal{A}^2(\Pi^+)$ is a Hilbert space with induced inner product given by $\langle f, g \rangle := \int\limits_{\Pi^+} f(z)\overline{g}(z) dA(z)$. While $\mathcal{A}^\infty(\Pi^+)$ denotes the algebra of bounded holomorphic functions on Π^+.

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The set of vectors $(e_n(z))_{n\in\mathbb{N}}$ from $\mathcal{A}^2(\Pi^+)$ defined by:

$$e_n(z) = 2\sqrt{n+1}\Big(\frac{z-i}{z+i}\Big)^n \frac{1}{(z+i)^2}, \qquad z \in \Pi^+,$$

constitutes an orthonormal basis of $\mathcal{A}^2(\Pi^+)$. This space is in fact a reproducing kernel Hilbert space with kernel function given by $K_z(w) = \dfrac{-1}{(\overline{z}-w)^2}$, for any $z \in \Pi^+$. For any $f \in \mathcal{A}^2(\Pi^+)$ the following reproducing property holds:

$$f(z) = \langle f, K_z\rangle = \int\limits_{\Pi^+} \frac{-f(w)}{(z-\overline{w})^2}\, dA(w). \tag{1.1}$$

Since $\mathcal{A}^2(\Pi^+)$ is a closed subspace of $\mathbb{L}^2(\Pi^+, dA)$, there exists a unique orthogonal projection $\mathcal{P}$ of $\mathbb{L}^2(\Pi^+, dA)$ onto $\mathcal{A}^2(\Pi^+)$. The reproducing property (1.1) provides an elegant integral representation to $\mathcal{P}$ as follows:

$$\mathcal{P}f(z) = \int\limits_{\Pi^+} \frac{-f(w)}{(z-\overline{w})^2} dA(w), \ \ \forall f \in \mathbb{L}^2(\Pi^+, dA), \ \forall z \in \Pi^+. \tag{1.2}$$

Moreover, for $f \in \mathbb{L}^\infty(\Pi^+)$, the Toeplitz operator with symbol f is defined by

$$\begin{aligned} T_f :\ \mathcal{A}^2(\Pi^+) &\longrightarrow \mathcal{A}^2(\Pi^+) \\ h &\longrightarrow T_f h = \mathcal{P}(fh) = \int\limits_{\Pi^+} f(w)h(w)\overline{K_z(w)}dA(w). \end{aligned}$$

It is in fact a bounded operator on $\mathcal{A}^2(\Pi^+)$ with norm $\|T_f\| \leq \|f\|_\infty$. In a standard way, we see also that $T_f^* = T_{\overline{f}}$, and that T_f is self-adjoint if and only if the symbol f is real-valued. Later on, we will characterize also normal Toeplitz operators.
The class of Toeplitz operators on "weighted" Bergman spaces over the upper half-plane and over related unbounded domains has been studied by many authors from various points of view, such as boundedness, compactness and algebras generated by them, see for instance [2, 3, 12, 19, 20, 26].

The Berezin transform (symbol) of an operator $A \in \mathscr{B}\left(\mathcal{A}^2(\Pi^+)\right)$ is defined by $\widetilde{A}(z) := \langle Ak_z, k_z\rangle$, where $k_z = \dfrac{-2\mathrm{Im}z}{(\overline{z}-w)^2}$ is the normalized reproducing kernel of $\mathcal{A}^2(\Pi^+)$, while the Berezin transform of a function $f \in \mathbb{L}^1(\Pi^+, dA)$ is defined by:

$$\mathcal{B}f(z) = \widetilde{f}(z) := \langle fk_z, k_z\rangle = \int\limits_{\Pi^+} \frac{4(\mathrm{Im}z)^2 f(w)}{|z-\overline{w}|^4} dA(w) = \mathcal{B}(T_f)(z). \tag{1.3}$$

The characterization of fixed points of the Berezin transform is a key property of prominent role in operator theory, in function theory and in quantum physics, namely that it is intimately related to harmonicity. It has been proved independently by Ahern and Flores and Rudin [1] and by Engliš [11] in the case of the ball of $\mathbb{C}^n$ and unit disk $\mathbb{D}$ respectively, see also [15, 27] for a detailed exposition. For further interesting developments, and in particular for various applications of Berezin symbols, we refer to [5, 6, 12, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 27].

In this paper we are interested in the problem of when two Toeplitz operators commute. This problem has been considered in many different settings. It was first considered by Brown and Halmos [7], see also [13], in the framework of the Hardy space of the unit circle $\mathcal{H}^2(\mathbb{T})$ where the following characterization has seen the light for the first time: *"a necessary and sufficient condition for two bounded Toeplitz operators on $\mathcal{H}^2(\mathbb{T})$ to commute is that their symbols are either both holomorphic, or they are both conjugate holomorphic or a non-trivial combination of them is constant".* In the setting of the Bergman space of the complex unit disk $L^2_a(\mathbb{D})$, Axler and Čuckovič [4] have extended the latter commutativity Brown and Halmos theorem to Toeplitz operators with bounded harmonic symbols, which fairly represent the generic unit disk counterparts of $\mathbb{L}^\infty(\mathbb{T})$ symbols. Subsequent generalizations to higher-dimensions and to different settings have been considered by many authors [27].

One of our main results in this paper, namely Theorem 3.1, consists in establishing the analog of the above commutativity theorem in the upper half-plane setting. Clearly the sufficient condition follows immediately from elementary properties of Toeplitz operators, whereas the proof of the necessity is more subtle and hinges heavily on certain properties of harmonic functions and uses another one of our main results, namely Theorem 2.3, about the fixed points of the Berezin transform, which is as important as the commutativity characterization stated in Theorem 3.1, and it relies on the idea of carrying certain properties of holomorphic and harmonic functions from the upper half-plane to the unit disk and vice-versa owing to conformal equivalence via the famous aforementioned Cayley transform. This idea has been used in [9], and has been significantly developed by M. Engliš [10].

Notice that our bounded harmonic symbols are not the only class that gives rise to commuting Toeplitz operators, bounded vertical symbols (the ones depending only on $\mathrm{Im} z$, i.e. the ones satisfying $\varphi(x+iy) = \varphi(y)$) also do, namely that: *the product of two Toeplitz operators with bounded vertical symbols on $\mathcal{A}^2(\Pi^+)$ is commutative.* Vertical symbols can be seen as the half-plane substitute of radial symbols in the case of the unit disk. This result is due to Vasilevski see [26]. The same is true for the case of angular symbols, see [3] and the references therein.

Very interesting novel classes of symbols giving rise to commutative Toeplitz operators are introduced and explored in this paper, namely the class of bi-analytic-type symbols, and that of bi-harmonic-type symbols. In Section 3, we develop a thorough analysis to examine commuting pairs of Toeplitz operators in the following cases: one symbol is bi-analytic-type and the other one is harmonic, the two symbols are bi-analytic-type, and the two symbols are bi-harmonic-type. The underlying calculations are stimulating, and may give insight into new practical class of non-harmonic symbols; a fact that has a central importance in the theory of Toeplitz operators.

Our results are presented in the following way: Section 2 is devoted to the exploration of the famous Berezin transform. In particular, we show that the only invariant class of integrable functions under the action of the Berezin transform is the class of harmonic ones. We start Section 3 by characterizing commuting Toeplitz operators with bounded harmonic symbols on Π^+. In particular, we prove the analog of the aforementioned commutativity theorem of Brown and

Halmos in the setting of the Bergman space $\mathcal{A}^2(\Pi^+)$. As an immediate but no less important consequence of the latter, we characterize also normal Toeplitz operators with bounded harmonic symbols. However, the major part of Section 3 is in fact dedicated to the investigation of commuting pairs of Toeplitz operators for bi-analytic-type and bi-harmonic-type symbols.

2. Fixed points of the Berezin transform

Given $z \in \Pi^+$, the mapping $\phi_z : \Pi^+ \longrightarrow \Pi^+$ defined by $\phi_z(w) = (\mathrm{Im} z)w + \mathrm{Re} z$ is a holomorphic bijection and its inverse is given by: $\phi_z^{-1}(w) = \dfrac{w - \mathrm{Re} z}{\mathrm{Im} z}$. Note that $\phi_z(i) = z$ and $\phi_i^{-1}(w) = w$. Also, as $\Delta(f \circ \phi_z) = (\mathrm{Im} z)^2(\Delta f) \circ \phi_z = 0$ for any $z \in \Pi^+$ and for any harmonic function f over Π^+, we see that the transformation $f \longrightarrow f \circ \phi_z$ preserves harmonicity. Now, let $d\nu$ denote the measure on Π^+ given by

$$d\nu(z) = \frac{4}{|z+i|^4} dA(z).$$

It is easy to check that $\nu(\Pi^+) = 1$. Moreover, consider the weighted Lebesgue space $\mathbb{L}^p(\Pi^+, dA)$ and observe the following two elementary properties of it:

Lemma 2.1. *If $f \in \mathbb{L}^p(\Pi^+, dA)$, $1 \leq p \leq \infty$, then $f \in \mathbb{L}^p(\Pi^+, d\nu)$ as well.*

Proof. Since $\left|\dfrac{1}{(z+i)^4}\right| \leq 1$ for all $z \in \Pi^+$, we see that for any $f \in \mathbb{L}^p(\Pi^+, dA)$, $1 \leq p < \infty$, we have:

$$\int_{\Pi^+} |f(z)|^p d\nu(z) = \int_{\Pi^+} \frac{4|f(z)|^p}{|z+i|^4} dA(z) \leq 4\|f\|_{\mathbb{L}^p(\Pi^+, dA)} < \infty.$$

Consequently $\mathbb{L}^p(\Pi^+, dA)$ is contained in $\mathbb{L}^p(\Pi^+, d\nu)$. The result is clear for $p = \infty$ as well. □

Lemma 2.2. *Let φ be the above Cayley transform and $1 \leq p \leq \infty$. Then $f \in \mathbb{L}^p(\Pi^+, dA)$ if and only if $f \circ \varphi^{-1} \in \mathbb{L}^p(\mathbb{D}, dA)$.*

Proof. For all $z \in \Pi^+$ and $w \in \mathbb{D}$ we have $z = \varphi^{-1}(w) = i\frac{1+w}{1-w}$. Then taking into account the fact that $dA(w) = |\varphi'(z)|^2 dA(z) = \dfrac{4}{|z+i|^4} dA(z)$, we see that

$$\int_{\Pi^+} |f(z)|^p d\nu(z) = \int_{\Pi^+} \frac{4|f(z)|^p}{|z+i|^4} dA(z) = \int_{\mathbb{D}} |f \circ \varphi^{-1}(w)|^p dA(w), \ 1 \leq p < \infty.$$

Thus, we conclude that $f \in \mathbb{L}^p(\Pi^+, d\nu)$ if and only if $f \circ \varphi^{-1} \in \mathbb{L}^p(\mathbb{D}, dA)$. The result is valid for the case $p = \infty$ as well. □

Lemma 2.3. *If $f \in \mathbb{L}^p(\Pi^+, dA)$, $1 \leq p \leq \infty$ then, for each $z \in \Pi^+$, the function $f \circ \phi_z$ is in $\mathbb{L}^p(\Pi^+, d\nu)$.*

Proof. Let $f \in \mathbb{L}^p(\Pi^+, dA)$. For $z \in \Pi^+$, put $w = \phi_z^{-1}(\xi)$, and observe that:

$$|\phi_z^{-1}(\xi) + i| = \frac{|\xi - \overline{z}|}{\mathrm{Im} z} \text{ and } \left|\left(\phi_z^{-1}\right)'(\xi)\right|^2 = \frac{1}{(\mathrm{Im} z)^2}.$$

Hence we get $dA(w) = \left|\left(\phi_z^{-1}\right)'(\xi)\right|^2 dA(\xi)$, and we infer that

$$d\nu(w) = \frac{4dA(w)}{|w+i|^4} = \frac{4\left|\left(\phi_z^{-1}\right)'(\xi)\right|^2 dA(\xi)}{\left|\phi_z^{-1}(\xi)+i\right|^4} = \frac{4(\mathrm{Im} z)^2 dA(\xi)}{|\xi-\overline{z}|^4}.$$

Thus, by this variable change, we obtain

$$\|f\circ\phi_z\|_{\mathbb{L}^p(\Pi^+,d\nu)} = \int\limits_{\Pi^+} |(f\circ\phi_z)(w)|^p d\nu(w) = \int\limits_{\Pi^+} \frac{4(\mathrm{Im} z)^2|f(\xi)|^p}{|z-\overline{\xi}|^4} dA(\xi). \tag{2.1}$$

As $\dfrac{4(\mathrm{Im} z)^2}{(\overline{z}-\xi)^4} = k_z^2(\xi)$ is bounded on Π^+ for each $z\in\Pi^+$, we infer therefore that $f\circ\phi_z \in\ \mathbb{L}^p(\Pi^+,d\nu)$. The same is true for $\mathbb{L}^\infty(\Pi^+,dA)$. $\square$

The following property is a variant of the so-called area mean value property on Π^+. In the case of the Euclidean half-space $\mathbb{R}^n_+$, it has been established in [8, 24]:

Theorem 2.1. *For any harmonic function $f\in\mathbb{L}^p(\Pi^+,dA), 1\le p\le\infty$, we have:*

$$f(z) = \int\limits_{\Pi^+} (f\circ\phi_z)(w)d\nu(w). \tag{2.2}$$

Proof. Let $g\in\mathbb{L}^p(\Pi^+,d\nu)$ be a harmonic function on Π^+ and consider the Cayley transform φ. Then, the composite function $g\circ\varphi^{-1}$ is harmonic on $\mathbb{D}$ and lies in $\mathbb{L}^p(\mathbb{D},dA)$ by Lemma 2.2, and thus satisfies the disk area mean value property:

$$\left(g\circ\varphi^{-1}\right)(0) = \int\limits_{\mathbb{D}} \left(g\circ\varphi^{-1}\right)(\xi)dA(\xi).$$

Setting $w=\varphi^{-1}(\xi) = i\frac{1+\xi}{1-\xi}$ in the latter formula, and then taking into account the fact that $dA(\xi) = \dfrac{4}{|w+i|^4}dA(w)$, as in the proof of Lemma 2.3 we see that

$$\begin{aligned} g(i) = \left(g\circ\varphi^{-1}\right)(0) &= \int\limits_{\mathbb{D}} \left(g\circ\varphi^{-1}\right)(\xi)dA(\xi) \\ &= \int\limits_{\Pi^+} \frac{4g(w)}{|w+i|^4}dA(w) = \int\limits_{\Pi^+} g(w)d\nu(w),\ \forall g\in\mathbb{L}^p(\Pi^+,d\nu). \end{aligned}$$

Now, for any harmonic function $f\in\mathbb{L}^p(\Pi^+,dA)$, the composite function $f\circ\phi_z$ is harmonic in Π^+ and belongs to $\mathbb{L}^p(\Pi^+,d\nu)$ by Lemma 2.3. Thus, replacing g by $f\circ\phi_z$ in the latter identity, we obtain:

$$f(z) = (f\circ\phi_z)(i) = \int\limits_{\Pi^+} (f\circ\phi_z)(w)d\nu(w),\ \text{for any}\ \ f\in\mathbb{L}^p(\Pi^+,dA). \qquad \square$$

Theorem 2.2. *The Berezin transform of each $f\in\mathbb{L}^p(\Pi^+,dA), 1\le p\le\infty$, can be rewritten as follows:*

$$\mathcal{B}f(z) = \int\limits_{\Pi^+} (f\circ\phi_z)(w)d\nu(w). \tag{2.3}$$

Proof. Let $f \in \mathbb{L}^p(\Pi^+, dA)$ then $f \circ \phi_z \in \mathbb{L}^p(\Pi^+, d\nu)$ by Lemma 2.3. For fixed $z \in \Pi^+$, set $\phi_z(w) = t$, and then $w = \phi_z^{-1}(t)$. Thus, variable change arguments similar to those used in the proof of Lemma 2.3 yield the identities:

$$\int_{\Pi^+} (f \circ \phi_z)(w) d\nu(w) = \int_{\Pi^+} \frac{4(\mathrm{Im} z)^2 f(t)}{|z - \overline{t}|^4} dA(t) = \int_{\Pi^+} f(t)|k_z(t)|^2 dA(t) = \mathcal{B}f(z).$$

□

The following corollary asserts that harmonic functions are fixed under the action of the Berezin transform:

Corollary 2.1. *Let $f \in \mathbb{L}^p(\Pi^+, dA), 1 \leq p \leq \infty$ be harmonic. Then $\widetilde{f}(z) = f(z)$.*

Proof. Combining Theorems 2.1 and 2.2, we obtain the desired result. □

The following deep property of harmonic functions is well-known in the case of harmonic functions in the disk $\mathbb{D}$. It has been proven in that framework independently by P. Ahern et al. [1] and by M. Engliš [11]. For a good exposition of their proofs, we refer to [15, 27].

Theorem 2.3. *Suppose that $f \in \mathbb{L}^p(\Pi^+, dA), 1 \leq p \leq \infty$. Then f is harmonic if and only if $\widetilde{f} = f$.*

To prove this assertion in our half-plane case, we shall need the following beautiful result due to M. Engliš (see [10] p.183-184), which asserts that the Berezin transform is invariant under biholomorphic equivalence:

Proposition 2.1. *Let Ω_1 and Ω_2 be domains from the complex plane $\mathbb{C}$, and let $A^2(\Omega_1)$ and $A^2(\Omega_2)$ be the corresponding Bergman spaces with reproducing kernels K_{Ω_1} and K_{Ω_2} respectively. Further denote by B_{Ω_1} and B_{Ω_2} the corresponding Berezin transforms, defined by $B_{\Omega_j} f(z) = \left\langle f(.)k_{\Omega_j}(z,.), k_{\Omega_j}(z,.)\right\rangle,\ j = 1, 2$. If $\varphi : \Omega_1 \longrightarrow \Omega_2$ is a biholomorphic map, then for any $f \in \mathbb{L}^1(\Omega_2, dA)$ we have:*

$$B_{\Omega_1}(f \circ \varphi)(\xi) = B_{\Omega_2} f(z), \quad \xi \in \Omega_1, \text{ and } z = \varphi(\xi).$$

In other words, we have $B_{\Omega_1}(f \circ \varphi) = (B_{\Omega_2} f) \circ \varphi$.

Proof of Theorem 2.3. The only if part is nothing but the assertion of Corollary 2.1. For the if part, let $f \in \mathbb{L}^p(\Pi^+, dA)$ be such that $\mathcal{B}_{\Pi^+} f = f$, and consider the Cayley transform $\varphi(z) = \frac{z-i}{z+i}$ whose inverse is $\varphi^{-1}(w) = i\frac{1+w}{1-w}$. By Lemmas 2.1 and 2.2 we have that $f \circ \varphi^{-1} \in \mathbb{L}^p(\mathbb{D}, dA)$. By Proposition 2.1, one has:

$$\mathcal{B}_{\mathbb{D}}(f \circ \varphi^{-1})(w) = (\mathcal{B}_{\Pi^+} f) \circ \varphi^{-1}(w), \quad \text{for any } \ w \in \mathbb{D}.$$

Hence, if $f \in \mathbb{L}^1(\Pi^+, dA)$ satisfies $\mathcal{B}_{\Pi^+} f = f$ as supposed, then we see that:

$$\mathcal{B}_{\mathbb{D}}(f \circ \varphi^{-1})(w) = (\mathcal{B}_{\Pi^+} f) \circ \varphi^{-1}(w) = f \circ \varphi^{-1}(w).$$

Now, the disk Ahern-Flores-Rudin and Engliš theorem mentioned above [1, 11, 15, 27] implies that $f \circ \varphi^{-1}$ is harmonic in $\mathbb{D}$, whence f is harmonic in Π^+. □

3. Commuting Toeplitz operators on $\mathcal{A}^2(\Pi^+)$

3.1. Case of two harmonic symbols. A very useful fact regarding bounded harmonic functions asserts that if $f \in \mathbb{L}^\infty(\Pi^+) \cap \mathbb{L}^2(\Pi^+, dA)$ is harmonic then it can be decomposed as $f = f_1 + \overline{f_2}$, with analytic components $f_1,\ f_2$. These components need not be bounded, but we know that they are Bergman functions, i.e. $f_1,\ f_2 \in \mathcal{A}^2(\Pi^+)$ due to the decomposition $\mathbb{L}^2_h(\Pi^+, dA) = \mathcal{A}^2(\Pi^+) \oplus \overline{\mathcal{A}^2(\Pi^+)}$. Accordingly, in the sequel, we may confine ourselves to the class of symbols that are in $\mathbb{L}^\infty(\Pi^+) \cap \mathbb{L}^2(\Pi^+, dA)$. Such an assumption may guarantee, on the other hand, the integrability of certain resulting function involved in the proof of the next theorem. Another fact we would like to emphasize here namely, the fact that the Berezin transform is a one to one transformation, see [2, 25].

Theorem 3.1. *Let $f,\ g \in \mathbb{L}^\infty(\Pi^+) \cap \mathbb{L}^2(\Pi^+, dA)$ be two bounded harmonic functions, then $T_fT_g = T_gT_f$ if and only if one of the following conditions is satisfied:*

1. *f and g are both analytic on Π^+,*
2. *f and g are both co-analytic on Π^+,*
3. *there exist constants $\alpha,\ \beta \in \mathbb{C}$ not both zero, such that $f = \alpha g + \beta$ on Π^+.*

Proof. Let us first calculate the Berezin transform of the commutator $[T_f, T_g]$. We know that every bounded harmonic function g in $\mathbb{L}^\infty(\Pi^+) \cap \mathbb{L}^2(\Pi^+, dA)$ admits the decomposition $g_1 + \overline{g_2}$ where $g_1, g_2 \in \mathcal{A}^2(\Pi^+)$, so we obtain

$$\widetilde{[T_f, T_g]}(z) = \Big(\langle(\overline{f_2}g_1 - f_1\overline{g_2})k_z, k_z\rangle\Big) - \Big(\overline{f_2}(z)g_1(z) - f_1(z)\overline{g_2}(z)\Big) = \widetilde{v}(z) - v(z),$$

with $v(z) = \overline{f_2}(z)g_1(z) - f_1(z)\overline{g_2}(z)$. Thus T_f and T_g commute if and only if $\widetilde{v} = v$. Now, as $v \in \mathbb{L}^1(\Pi^+, dA)$, we infer by Theorem 2.3 that v is a harmonic function on Π^+. Direct calculation leads to $\Delta v = 4(\overline{f_2'}g_1' - \overline{g_2'}f_1')$. Whence, v is harmonic if and only if $\overline{f_2'}g_1' = \overline{g_2'}f_1'$ on Π^+. Therefore, the desired conclusion is obtained as soon as we show by a standard discussion (see for instance [4, 25, 27]) that the latter equation is equivalent to the fulfillment of one of the three conditions listed in the statement of the theorem. □

The above commutativity theorem has an immediate important consequence on the characterization of normal Toeplitz operators. It asserts that the class of normal Toeplitz operators are not too far from the class of self-adjoint ones described above, namely that they are nothing but "rotations" of self-adjoint ones. Before stating this result, recall that an operator T is called normal if $TT^* = T^*T$, where T^* denotes the adjoint operator of T.

Corollary 3.1. *Let $f \in \mathbb{L}^\infty(\Pi^+) \cap \mathbb{L}^2(\Pi^+, dA)$ be harmonic. Then, T_f is normal if and only if $f(\Pi^+)$ is contained in a straight line in $\mathbb{C}$.*

Proof. Suppose that $f(\Pi^+)$ is contained in a straight line in $\mathbb{C}$, then there exist constants $\alpha, \beta, \gamma \in \mathbb{C}$ not all 0 such that $\alpha f(z) + \beta\overline{f(z)} = \gamma$, for any $z \in \Pi^+$. This implies by Theorem 2.3 that $T_fT_{\overline{f}} = T_{\overline{f}}T_f$.
Conversely, suppose that T_f is normal, then again by Theorem 2.3 we see that either f and $\overline{f}$ are both analytic or they are both co-analytic, which leads to the fact that f is constant over Π^+, or else there exist two complex constants α and β not both zero such that $f = \alpha\overline{f} + \beta$. In all of these cases we infer that $f(\Pi^+)$ is contained in a straight line in $\mathbb{C}$. □

3.2. Mixed case of a bi-analytic-type symbol with a harmonic one.
A function f is said to be of bi-analytic-type on Π^+ if there are two analytic functions $f_1,\ f_2$ defined on Π^+ such that $f(z) = f_1(z) + \overline{\varphi}(z) f_2(z)$ for all $z \in \Pi^+$, where φ is the Cayley transform. In other words, its transformation via the map $z = \varphi^{-1}(w)$ is a bi-analytic function in $\mathbb{D}$. Also, consider the a monomial function Θ_n is defined on Π^+ by $\Theta_n(z) = \left(\frac{z-i}{z+i}\right)^n$, for $n \in \mathbb{N}$. Recall that the family $\{\Theta_n k_i\}_{n\in\mathbb{N}}$ is total in the Bergman space $\mathcal{A}^2(\Pi^+)$, where k_i is the normalized reproducing kernel on Π^+ at the point $z = i$.

Lemma 3.1. *Let $n, m \in \mathbb{N}$. Then, we have*

(1) $\mathcal{P}\Big(\overline{\Theta_n(z)}\Theta_m(z)k_i(z)\Big)(w) = \frac{m-n+1}{m+1}\Theta_{m-n}(w)k_i(w)$, *if $m \geq n$,*

(2) $\mathcal{P}\Big(\overline{\Theta_n(z)}\Theta_m(z)k_i(z)\Big)(w) = 0$, *if $m < n$.*

Proof. Observe that, for all $n, m \in \mathbb{N}$, we have :

$$\begin{aligned}
\mathcal{P}\Big(\overline{\Theta_n(z)}\Theta_m(z)k_i(z)\Big)(w) &= \langle \overline{\Theta_n(z)}\Theta_m(z)k_i(z), K_w\rangle,\\
&= \int_{\Pi^+} \overline{\Theta_n(\xi)}\Theta_m(\xi)k_i(\xi)\overline{K_w(\xi)}dA(\xi),\\
&= \sum_{l=0}^{\infty}(l+1)\Theta_l(w)\frac{-2}{(w+i)^2}\int_{\Pi^+}\overline{\Theta_{n+l}(\xi)}\Theta_m(\xi)\frac{4dA(\xi)}{|\xi+i|^4},\\
&= \sum_{l=0}^{\infty}(l+1)\Theta_l(w)\frac{-2}{(w+i)^2}\int_{\mathbb{D}}\eta^m\overline{\eta}^{l+n}dA(\eta),\\
&= \begin{cases} \frac{m-n+1}{m+1}\Theta_{m-n}(w)k_i(w), & m \geq n,\\ 0 & m < n,\end{cases}
\end{aligned}$$

where polar coordinates are used in the latter line. Hence, the result follows. □

Example 3.1 (Example of non-commutativity)**.** Let $f(z) := \Theta_1(z) + \overline{\varphi(z)}\Theta_2(z)$, be a bounded bi-analytic-type function on Π^+, and $g(z) := \Theta_1(z) + \frac{1}{3}\overline{\Theta_3(z)}$ be a bounded harmonic function on Π^+. Let $h(z) = \Theta_l(z)k_i(z)$, where $l \geq 3$ is a positive integer. Invoking Lemma 3.1, we obtain:

$$(T_g h)(w) = \Theta_{l+1}(w)k_i(w) + \frac{l-2}{3(l+1)}\Theta_{l-3}(w)k_i(w).$$

Then, we obtain

$$T_f(T_g h)(w) = \frac{(l-2)(2l-1)}{3l(l+1)}\Theta_{l-2}(w)k_i(w) + \frac{2l+7}{l+4}\Theta_{l+2}(w)k_i(w).$$

Similarly, we get

$$T_g(T_f h)(w) = \frac{(2l+5)(l-1)}{3(l+3)(l+2)}\Theta_{l-2}(w)k_i(w) + \frac{2l+5}{l+3}\Theta_{l+2}(w)k_i(w).$$

This shows that $T_f T_g \neq T_g T_f$ in the general case where one of the symbols is of bi-analytic-type and the other one is harmonic. ■

Proposition 3.1. *Let $f(z) = \Theta_a(z) + \overline{\varphi(z)}\Theta_1(z)$, be a bounded bi-analytic-type function on Π^+, and let $g(z) = \Theta_s(z) + \overline{\Theta_t(z)}$ be a bounded harmonic function on Π^+ such that $a, s, t \in \mathbb{N}$. Then, $T_f T_g = T_g T_f$ holds if and only if g is constant.*

Proof. The if part is immediate. For the only if part, let $h(z) = \Theta_l(z)k_i(z)$, where $l \geq t$ is a positive integer. By Lemma 3.1, we obtain

$$T_g(h)(w) = \Theta_{s+l}(w)k_i(w) + \frac{l-t+1}{l+1}\Theta_{l-t}(w)k_i(w), \qquad l \geq t.$$

Thus, we get

$$\begin{aligned} T_f(T_g h)(w) &= \Theta_{a+s+l}(w)k_i(w)(w) + \frac{l-t+1}{l+1}\Theta_{a+l-t}(w)k_i(w) \\ &\quad + \frac{s+l+1}{s+l+2}\Theta_{s+l}(w)k_i(w) + \frac{l-t+1}{l+1} \cdot \frac{1+l-t}{l-t+2}\Theta_{l-t}(w)k_i(w). \end{aligned}$$

On the other hand, we have

$$(T_f h)(w) = \Theta_{a+l}(w)k_i(w) + \frac{l+1}{l+2}\Theta_l(w)k_i(w).$$

So, by Lemma 3.1 for all $l \geq t$, we see that

$$\begin{aligned} T_g(T_f h)(w) &= \Theta_{s+a+l}(w)k_i(w) + \frac{l+1}{l+2}\Theta_{l+s}(w)k_i(w) \\ &\quad + \frac{a+l-t+1}{a+l+1}\Theta_{a+l-t}(w)k_i(w) + \frac{l-t+1}{l+2}\Theta_{l-t}(w)k_i(w). \end{aligned}$$

Combining the above identities, we obtain

$$\begin{aligned} \Big(T_f T_g h - T_g T_f h\Big)(w) &= \Big(\frac{l-t+1}{l+1} - \frac{a+l-t+1}{a+l+1}\Big)\Theta_{a+l-t}(w)k_i(w) \\ &\quad + \Big(\frac{s+l+1}{s+l+2} - \frac{l+1}{l+2}\Big)\Theta_{s+l}(w)k_i(w) \\ &\quad + \Big(\frac{l-t+1}{l+1} \cdot \frac{l-t+1}{l-t+2} - \frac{l-t+1}{l+2}\Big)\Theta_{l-t}(w)k_i(w). \end{aligned}$$

By linear independence of the family $\{\Theta_n k_i\}_{n\in\mathbb{N}}$, the coefficients of this sum vanish. Thus, for $T_f T_g - T_g T_f$ to hold, we distinguish two cases:

(1) If $\boldsymbol{a - t \neq s}$, then we get

$$\frac{s+l+1}{s+l+2} = \frac{l+1}{l+2} \Longleftrightarrow s = 0, \tag{3.1}$$

and

$$\frac{l-t+1}{l+1} \cdot \frac{l-t+1}{l-t+2} = \frac{l-t+1}{l+2} \Longleftrightarrow t = 0, \tag{3.2}$$

and

$$\frac{l-t+1}{l+1} = \frac{a+l-t+1}{a+l+1} \Longleftrightarrow t = 0 \text{ or } a = 0. \tag{3.3}$$

The case $t = a = 0$ contradicts the fact that $a - t \neq s$.

(2) If $\boldsymbol{a - t = s}$: then the left-hand-side of (3.2), yields $t = 0$. Next, we consider the left-hand-side of (3.1), to get $s = 0$. Thus, we get $a = 0$.

□

Theorem 3.2. *Let $f(z) = \overline{\varphi(z)}\Theta_a(z)$ be a bounded bi-analytic-type function on Π^+, and let $g(z) = \Theta_s(z) + \overline{\Theta_t(z)}$ be a bounded harmonic function on Π^+ such that $a, s, t \in \mathbb{N}$. Then, $T_f T_g = T_g T_f$ holds if and only if one of the following holds*

(1) *g is constant Π^+,*
(2) *$a = 0$ and g is co-analytic on Π^+.*

Proof. The if part is immediate. For the only if part, let $f(z) = \overline{\varphi(z)}\Theta_a(z)$, be a bounded bi-analytic-type function on Π^+, and $g(z) = \Theta_s(z) + \overline{\Theta_t(z)}$ be a bounded harmonic function on Π^+ with $a, s, t \in \mathbb{N}$. Let $h(z) = \Theta_l(z)k_i(z)$, where $l \geq t + 1 - a$ is a positive integer. As above, by Lemma 3.1, we obtain

$$\Big(T_f T_g h - T_g T_f h\Big)(w) = \big(\frac{l-t+1}{l+1}\cdot\frac{a+l-t}{a+l-t+1} - \frac{a+l-t}{a+l+1}\big)\Theta_{a+l-1-t}(w)k_i(w) + \big(\frac{a+s+l}{a+s+l+1} - \frac{a+l}{a+l+1}\big)\Theta_{a+l+s-1}(w)k_i(w).$$

Thus, for $T_f T_g = T_g T_f$ to hold, it is necessary that the coefficients in the latter sum vanish, which yields the conditions:

- $\dfrac{a+s+l}{a+s+l+1} = \dfrac{a+l}{a+l+1} \implies s = 0;$
- $\dfrac{l-t+1}{l+1}\cdot\dfrac{a+l-t}{a+l-t+1} = \dfrac{a+l-t}{a+l+1} \implies t = 0$ or $a = 0$.

We conclude that

(1) $s = 0$ and $t = 0$: which implies that g is constant.
(2) $s = 0$, $t \neq 0$ and $a = 0$: which means that f_2 is constant and g is co-analytic.

□

3.3. Case of two bi-analytic-type symbols.

Theorem 3.3. *Let f and g be two bounded bi-analytic-type functions on Π^+, such that $f(z) = \overline{\varphi(z)}\Theta_a(z)$ and $g(z) = \overline{\varphi(z)}\Theta_b(z)$, where $a, b \in \mathbb{N}$. Then, $T_f T_g = T_g T_f$ if and only if $a = b$.*

Proof. The if part is immediate. For the only if part, let $h(z) = \Theta_l(z)k_i(z)$, where $l \geq 2 - a - b$ is a positive integer. Invoking Lemma 3.1, we get

$$T_f(T_g h)(w) = \frac{b+l}{b+l+1}\cdot\frac{a+b+l-1}{a+b+l}\Theta_{a+b+l-2}(w)k_i(w),$$

and

$$T_g(T_f h)(w) = \frac{a+l}{a+l+1}\cdot\frac{a+b+l-1}{a+b+l}\Theta_{a+b+l-2}(w)k_i(w).$$

Consequently, $T_f T_g = T_g T_f$ holds if and only if $\dfrac{a+l}{a+l+1} = \dfrac{b+l}{b+l+1}$, which reduces to $a = b$. □

Theorem 3.4. *Let $f(z) = f_1(z) + \overline{\varphi(z)}\Theta_a(z)$ and $g(z) = g_1(z) + \overline{\varphi(z)}\Theta_a(z)$ be two bounded bi-analytic-type functions on Π^+, such that $f_1, g_1 \in \mathcal{A}^2(\Pi^+) \cap \mathcal{A}^\infty(\Pi^+)$ and $a \in \mathbb{N}$. Then, $T_f T_g = T_g T_f$ holds if and only if $f_1 = \alpha$ and $g_2 = \beta$, where $\alpha, \beta \in \mathbb{C}$.*

Proof. The if part is immediate. For the only if part, let a be a positive integer and $f_1, g_1 \in \mathcal{A}^2(\Pi^+)\cap\mathcal{A}^\infty(\Pi^+)$, and let $h(z) = \Theta_l(z)k_i(z)$, where l is an arbitrary positive integer. As above, by direct computations, Lemma 3.1 implies that

$$\Big(T_fT_g - T_gT_f\Big)(h)(w) = P\Big(\big(g_1 - f_1\big)(z)\overline{\varphi(z)}\Theta_{l+a}(z)k_i(z)\Big)(w) - \big(g_1(w) - f_1(w)\big)\frac{a+l}{a+l+1}\Theta_{a+l-1}(w)k_i(w) = 0. \quad (3.4)$$

Since $g_1, f_1 \in \mathcal{A}^2(\Pi^+) \cap \mathcal{A}^\infty(\Pi^+)$, we have $\big(g_1 - f_1\big)(z) = \sum\limits_{n=0}^{\infty}(c_n - \lambda_n)\Theta_n(z)$, where $c_n, \lambda_n \in \mathbb{C}$ (and $c_n \neq \lambda_n$). Using Lemma 3.1 and the linearity of the projection, we get:

$$P\Big(\big(g_1 - f_1\big)\overline{\varphi(z)}\Theta_{l+a}k_i(z)\Big) = \sum_{n=0}^{\infty}\big(c_n - \lambda_n\big)\frac{a+l+n}{a+l+n+1}\Theta_{a+l+n-1}\ k_i.$$

On the other hand, we have

$$\big(g_1 - f_1\big)(w)\frac{a+l}{a+l+1}\Theta_{a+l-1}(w)k_i(w) = \sum_{n=0}^{\infty}\big(c_n - \lambda_n\big)\frac{a+l}{a+l+1}\Theta_{a+l+n-1}(w)k_i(w).$$

Thus, from (3.4), we get

$$\sum_{n=0}^{\infty}\big(c_n - \lambda_n\big)\frac{a+l+n}{a+l+n+1}\Theta_{a+l+n-1}\ k_i = \sum_{n=0}^{\infty}\big(c_n - \lambda_n\big)\frac{a+l}{a+l+1}\Theta_{a+l+n-1}\ k_i,$$

which holds if and only if:

$$\frac{a+l+n}{a+l+n+1} = \frac{a+l}{a+l+1}.$$

This is equivalent to $n = 0$. Consequently, $g_1 = c_0 = \alpha$ and $f_1 = \lambda_0 = \beta$. $\square$

Corollary 3.2. *Let $f(z) = \alpha+\beta\overline{\varphi(z)}\Theta_a(z)$ be a bounded bi-analytic-type function on Π^+, where $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$ and $a \in \mathbb{N}$. Then, T_f is normal if and only if $a = 1$.*

Proof. The if part is immediate. For the only if part, let $h(z) = \Theta_l(z)k_i(z)$, where l is an arbitrary positive integer. As above, by Lemma 3.1, we have :

$$\big(T_{\overline{f}}T_f - T_fT_{\overline{f}}\big)\big(\Theta_l(z)k_i(z)\big) = |\beta|^2\Big(\frac{a+l}{a+l+1}\cdot\frac{l+1}{a+l+1} - \frac{l+2-a}{l+2}\cdot\frac{l+1}{l+2}\Big)\Theta_l\ k_i.$$

Whence, T_f is normal if and only if :

$$\frac{a+l}{a+l+1}\cdot\frac{l+1}{a+l+1} = \frac{l+2-a}{l+2}\cdot\frac{l+1}{l+2},$$

In other words, if and only if $a = 1$. $\square$

Theorem 3.5. *Let $f(z) = \overline{\varphi(z)}f_2(z)$, $g(z) = \overline{\varphi(z)}g_2(z)$ be two bounded bi-analytic-type functions on Π^+, such that:*

$$f_2(z) = \sum_{m=1}^{\infty}\rho_m\Theta_m(z) \text{ and } g_2(z) = \sum_{t=1}^{\infty}\lambda_t\Theta_t(z),$$

for all $z \in \Pi^+$. Then, $T_fT_g = T_gT_f$ holds if and only if $f(z) = \alpha g(z)$, (or $g(z) = \alpha f(z)$), for some $\alpha \in \mathbb{C}$.

Proof. The if part is immediate. For the only if part, let $h(z) = \Theta_l(z)k_i(z)$, where l is an arbitrary positive integer. By lemma 3.1, we get :

$$T_f(T_g h)(w) = \sum_{s=2}^{\infty}\sum_{j=1}^{s} \rho_{s-j}\lambda_j \frac{l+j}{l+j+1} \cdot \frac{s+l-1}{s+l}\Theta_{s+l-2}(w)k_i(w),$$

and

$$T_g(T_f h)(w) = \sum_{s=2}^{\infty}\sum_{j=1}^{s-1} \lambda_{s-j}\rho_j \frac{l+j}{l+j+1} \cdot \frac{s+l-1}{s+l}\Theta_{s+l-2}(w)k_i(w).$$

Thus, $T_f T_g = T_g T_f$ holds if and only if :

$$\sum_{s=2}^{\infty}\sum_{j=1}^{s-1} \big(\rho_{s-j}\lambda_j - \lambda_{s-j}\rho_j\big)\frac{l+j}{l+j+1}\frac{s+l-1}{s+l}\Theta_{s+l-2}(w)k_i(w) = 0, \tag{3.5}$$

For any value of $s = N + j + 1$, where N is a positive integer and $j = 0, 1, 2, ...,$ we determine the corresponding coefficients as follows:

$$\Big(\big(\rho_{N+j}\lambda_1 - \lambda_{N+j}\rho_1\big)\frac{l+1}{l+2} + \big(\rho_{N+j-1}\lambda_2 - \lambda_{N+j-1}\rho_2\big)\frac{l+2}{l+3} + \ \ldots$$
$$+ \big(\rho_1\lambda_{N+j} - \lambda_{N+j}\rho_1\big)\frac{l+N+j}{l+N+j+1}\Big)\frac{l+N+j}{l+N+j+1}\Theta_{l+N+j-1}(w)k_i(w). \tag{3.6}$$

By the linear independence of the family $\{\Theta_n(w)k_i(w)\}_{n\in\mathbb{N}}$, the sum (3.5) vanishes if and only if the underlying coefficients are all zero. Making use of the established results, we can analyze all possible cases for the coefficients ρ_m, λ_t : The expression (3.6) yields a zero coefficient for $\lambda_1, \rho_1 \in \mathbb{C}$. In terms of the values of λ_1, ρ_1, and by iteration, four cases can be merged :

(1) Suppose that $\lambda_1 \neq 0$ and $\rho_1 \neq 0$: The expression (3.6) for $j = 0$ implies that $\rho_N\lambda_1 = \lambda_N\rho_1$, for all $N \in \mathbb{N}$, consequently $\rho_N = \dfrac{\rho_1}{\lambda_1}\lambda_N$, consequently $f_2(z) = \dfrac{\rho_1}{\lambda_1}g_2(z)$ for all $z \in \Pi^+$.

(2) Suppose that $\lambda_1 = 0$ and $\rho_1 \neq 0$: The expression (3.6) for $j = 0$ implies that $\rho_N\lambda_1 = \lambda_N\rho_1 = 0$, consequently $\lambda_N = 0$ and ρ_N is an arbitrary number in $\mathbb{C}$, consequently $f_2(z) = 0$ for all $z \in \Pi^+$, whence $f_2 = 0g_2$.

(3) Suppose that $\rho_1 = 0$ and $\lambda_1 \neq 0$: The expression (3.6) for $j = 0$ implies that $\rho_N\lambda_1 = \lambda_N\rho_1 = 0$, consequently $\rho_N = 0$ and λ_N is an arbitrary number in $\mathbb{C}$, consequently $g_2(z) = 0$ for all $z \in \Pi^+$, whence $g_2 = 0f_2$.

(4) Suppose that $\lambda_1 = 0$ and $\rho_1 = 0$: The expression (3.6) leads to a null coefficient for all $\lambda_2, \rho_2 \in \mathbb{C}$. In terms of λ_2, ρ_2, and by iteration, we arrive at the sub-cases :

   (a) Suppose that $\lambda_2 \neq 0$ and $\rho_2 \neq 0$: The expression (3.6) for $j = 1$ implies that $\rho_N\lambda_2 = \lambda_N\rho_2$, consequently $\rho_N = \dfrac{\rho_2}{\lambda_2}\lambda_N$, whence $f_2(z) = \dfrac{\rho_2}{\lambda_2}g_2(z)$ for all $z \in \Pi^+$.

   (b) Suppose that $\lambda_2 = 0$ and $\rho_2 \neq 0$: The expression (3.6) for $j = 1$ implies that $\rho_N\lambda_2 = \lambda_N\rho_2 = 0$, consequently $\lambda_N = 0$ and ρ_N is an arbitrary number in $\mathbb{C}$. Consequently $f_2(z) = 0$ for all $z \in \Pi^+$, whence $f_2 = 0g_2$.

(c) Suppose that $\rho_2 = 0$ and $\lambda_2 \neq 0$: The expression (3.6) for $j = 1$ implies that $\rho_N\lambda_2 = \lambda_N\rho_2 = 0$, consequently $\rho_N = 0$ and λ_N is an arbitrary number in $\mathbb{C}$, Consequently $g_2(z) = 0$ for all $z \in \Pi^+$, whence $g_2 = 0f_2$.

(d) Suppose that $\lambda_2 = 0$ and $\rho_2 = 0$: Then, arguing in a similar manner, we conclude that :

- $f_2(z) = \alpha g_2(z)$, such that $\alpha = \dfrac{\mathfrak{T}^{(\mathbf{n+1})}f_2(z)}{\mathfrak{T}^{(\mathbf{n+1})}g_2(z)}\Big|_{z=i}$ for some $n \in \mathbb{N}$, where $\mathfrak{T}^{(\mathbf{n+1})}$ is a differential operator defined recursively by :

$$\mathfrak{T}^{(\mathbf{n+1})}f_2(z) = \begin{cases} \dfrac{1}{\varphi'(z)}\dfrac{d}{dz}f_2(z) & \text{if } n = 0; \\ \dfrac{1}{\varphi'(z)}\dfrac{d}{dz}\left(\mathfrak{T}^{(\mathbf{n})}f_2(z)\right) & \text{if } n \geq 1. \end{cases}$$

- $f_2 = 0$ and g_2 is a bounded function on Π^+;
- $g_2 = 0$ and f_2 is a bounded function on Π^+.

□

Now, for $n \in \mathbb{N}$ and $\alpha_{s,j} \in \mathbb{C}$, a bi-variate-type polynomial of degree n on Π^+ is a function Q of the form :

$$Q(z) = \sum_{s=0}^{n}\sum_{j=0}^{s}\alpha_{s,j}\overline{\Theta_{s-j}(z)}\Theta_j(z).$$

Theorem 3.6. *Let $f(z) = \overline{\Theta_a(z)}\Theta_b(z)$ and $g(z) = \overline{\Theta_s(z)}\Theta_t(z)$ be two bounded bi-variate-type functions on Π^+, where $a, b, s, t \in \mathbb{N}$. Then, $T_fT_g = T_gT_f$ holds if and only if $f = g$.*

Proof. The if part is immediate. For the only if part, let $f(z) = \overline{\Theta_a(z)}\Theta_b(z)$ and $g(z) = \overline{\Theta_s(z)}\Theta_t(z)$ be two bounded bi-variate-type functions on Π^+ such that $a, b, s, t \in \mathbb{N}$. Let $h(z) = \Theta_l(z)k_i(z)$, where $l \geq a + s - t - b$ is a positive integer. Again by Lemma 3.1, we have :

$$T_f(T_gh)(w) = \frac{(t-s)+l+1}{t+l+1}\cdot\frac{(b-a)+(t-s)+l+1}{b+(t-s)+l+1}\Theta_{(b-a)+(t-s)+l}(w)k_i(w).$$

Similarly, we obtain

$$T_g(T_fh)(w) = \frac{(b-a)+l+1}{b+l+1}\cdot\frac{(b-a)+(t-s)+l+1}{(b-a)+t+l+1}\Theta_{(b-a)+(t-s)+l}(w)k_i(w).$$

Hence, $T_fT_g = T_gT_f$ holds if and only if :

$$\frac{(t-s)+l+1}{b+(t-s)+l+1} = \frac{(b-a)+l+1}{(b-a)+t+l+1}.$$

Comparing coefficients, we obtain $t = b$ and $s = a$. □

3.4. Case of two bi-harmonic-type symbols. Almansi type formula provides a decomposition of a bi-harmonic-type function f on Π^+ of the form :

$$f(z) = f_1(z) + |\varphi(z)|^2 f_2(z), \;\; \forall z \in \Pi^+,$$

where f_1 and f_2 are two harmonic functions on Π^+ and φ is the Cayley transform.

Example 3.2 (Example of non-commutativity)**.** Let $f(z) = \overline{\Theta_1(z)} + |\varphi(z)|^2\Theta_2(z)$ and $g(z) = \Theta_2(z) + |\varphi(z)|^2\overline{\Theta_1(z)}$ be two bounded bi-harmonic-type functions on Π^+, and let $h(z) = \Theta_l(z)k_i(z)$, where $l \geq 2$ is a positive integer. By Lemma 3.1, for all $l \geq 2$, we obtain :

$$\begin{aligned} T_f(T_g(h))(w) = {} & \frac{l+1}{l+2}\Theta_l(w)k_i(w) + \frac{l}{l+2}\cdot\frac{l-1}{l}\Theta_{l-2}(w)k_i(w) \\ & + \frac{l+4}{l+5}\Theta_{l+3}(w)k_i(w) + \frac{l}{l+2}\cdot\frac{l+2}{l+3}\Theta_{l+1}(w)k_i(w), \end{aligned}$$

and

$$\begin{aligned} T_g(T_f(h))(w) = {} & \frac{l}{l+1}\Theta_{l+1}(w)k_i(w) + \frac{l+3}{l+4}\Theta_{l+4}(w)k_i(w) \\ & + \frac{l}{l+1}\cdot\frac{l-1}{l+1}\Theta_{l-2}(w)k_i(w) + \frac{l+3}{l+4}\cdot\frac{l+2}{l+4}\Theta_{l+1}(w)k_i(w). \end{aligned}$$

This shows that $T_fT_g \neq T_gT_f$ in general. ■

Theorem 3.7. *Let $f(z) = \overline{\Theta_a(z)} + |\varphi(z)|^2\Theta_b(z)$ and $g(z) = \overline{\Theta_s(z)} + |\varphi(z)|^2\Theta_t(z)$ be two bounded bi-harmonic-type functions on Π^+, where $a, b, s, t \in \mathbb{N}$. Then, $T_fT_g = T_gT_f$ holds if an only if $b = t$ and $a = s = 0$.*

Proof. The if part is immediate. For the only if part, let $f(z) = \overline{\Theta_a(z)} + |\varphi(z)|^2\Theta_b(z)$ and $g(z) = \overline{\Theta_s(z)} + |\varphi(z)|^2\Theta_t(z)$ be two bounded biharmonic functions on Π^+ such that $a, b, s, t \in \mathbb{N}$, and let $h(z) = \Theta_l(z)k_i(z)$, where $l \geq s + a$ is a positive integer. By Lemma 3.1, we obtain :

$$\begin{aligned} \Big(T_fT_g - T_gT_f\Big)(h)(w) = {} & \Big(\frac{t+l+1-a}{t+l+2} - \frac{l-a+1}{l+1}\cdot\frac{t+l+1-a}{t+l-a+2}\Big)\Theta_{t+l-a}(w)k_i(w) \\ & + \Big(\frac{l-s+1}{l+1}\cdot\frac{l+b+1-s}{l+b+2-s} - \frac{b+l-s+1}{b+l+2}\Big)\Theta_{b+l-s}(w)k_i(w) \\ & + \frac{b+t+l+1}{b+t+l+2}\Big(\frac{t+l+1}{t+l+2} - \frac{b+l+1}{b+l+2}\Big)\Theta_{t+b+l}(w)k_i(w). \end{aligned}$$

Simplifying the resulting expression, we arrive to : $T_fT_g = T_gT_f$ if and only if

$$\frac{t+l+1-a}{t+l+2} = \frac{l-a+1}{l+1}\cdot\frac{t+l+1-a}{t+l-a+2} \quad \text{and} \quad \frac{l-s+1}{l+1}\frac{l+b+1-s}{l+b+2-s} = \frac{b+l-s+1}{b+l+2}.$$

This implies that $b = t$ and $a = 0$ and $s = 0$. □

Theorem 3.8. *Let $f(z) = \Theta_a(z) + |\varphi(z)|^2\overline{\Theta_b(z)}$ and $g(z) = \Theta_s(z) + |\varphi(z)|^2\overline{\Theta_t(z)}$ be two bounded bi-harmonic-type functions on Π^+, where $a, b, s, t \in \mathbb{N}$. Then, $T_fT_g = T_gT_f$ holds if and only if $b = t$ and $a = s$.*

Proof. The if part is immediate. For the only if part, consider the two bounded bi-harmonic-type functions on Π^+, namely $f(z) = \Theta_a(z) + |\varphi(z)|^2\overline{\Theta_b(z)}$ and

$g(z) = \Theta_s(z) + |\varphi(z)|^2\overline{\Theta_t(z)}$, with $a, b, s, t \in \mathbb{N}$. Let $h(z) = \Theta_l(z)k_i(z)$, where $l \geq b + t$ is a positive integer. By Lemma 3.1, we get :

$$\begin{aligned}\Big(T_fT_g - T_gT_f\Big)(h)(w) &= \left(\frac{l+1-t}{l+2} - \frac{l+a+1-t}{l+a+2}\right)\Theta_{l+a-t}(w)k_i(w)\\ &\quad+\left(\frac{l+s+1-b}{l+s+2} - \frac{l+1-b}{l+2}\right)\Theta_{l+s-b}(w)k_i(w)\\ &\quad+\frac{l+1-t-b}{l+2}\left(\frac{l+1-t}{l+2-t} - \frac{l+1-b}{l+2-b}\right)\Theta_{l-t-b}(w)k_i(w).\end{aligned}$$

Observe that :

$$\frac{l+1-t-b}{l+2}\left(\frac{l+1-t}{l+2-t} - \frac{l+1-b}{l+2-b}\right) = 0 \Longleftrightarrow t = b.$$

Thus, we distinguish two cases :

- **First case :** $T_fT_g = T_gT_f$ holds if and only if

$$\frac{l+1-t}{l+2} = \frac{l+a+1-t}{l+a+2} \quad \text{and} \quad \frac{l+s+1-b}{l+s+2} = \frac{l+1-b}{l+2}.$$

  These imply respectively that $a = 0$ and $s = 0$;
- **Second case :**

$$T_fT_g = T_gT_f \Longleftrightarrow a - t = s - b \Longleftrightarrow a = s.$$

□

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