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ON SCALAR CURVATURE OF GRADIENT r-ALMOST-NEWTON-EINSTEIN SOLITONS

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Abstract. In this work we establish the concept of r-almost-Newton-Einstein soliton immersed into a Riemannian manifold, which extends in a natural way the notion of Einstein solitons introduced by Catino et al. [8]. In this setting, under suitable hypothesis on the potential and soliton functions, we prove nonexistence and characterization results. Also, we proved some triviality results for the compact case and under some conditions we get a constant scalar curvature.

1. Introduction

A Ricci soliton is a $n(\geq 2)$-dimensional Riemannian manifold (M, g), endowed with a smooth vector field X satisfying

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where λ is a constant and $\mathcal{L}$ stands for the Lie derivative. Ricci solitons model the formation of singularities in the Ricci flow and they correspond to self-similar solutions, i.e., they are stationary points of this flow in the space of metrics modulo diffeomorphisms and scalings (see [10] for more details). Thus, classification of Ricci solitons or understanding their geometry is definitely an important issue.

If X is the gradient of a smooth function f on M, then such a Ricci soliton is called gradient Ricci soliton. In this case, (1.1) becomes

$$\mathrm{Ric} + \mathrm{Hess}(f) = \lambda g, \tag{1.2}$$

where $Hess(f)$ stands for the Hessian of f. Also, a Ricci soliton (M, g, X, λ) is called *expanding*, *steady* or *shrinking* according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Moreover, when the vector field X is *trivial* or f is constant, the Ricci soliton is called trivial. Therefore, Ricci solitons are natural extensions of Einstein manifolds.

Recently, based on the Ricci solitons theory, Catino et al. [7] introduced a notion of *ρ-Einstein solitons*, i.e., a Riemannian manifold satisfying the following equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g + \rho S g, \tag{1.3}$$

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where *Ric* is the Ricci curvature tensor, S is the scalar curvature, λ is a constant, and $\mathcal{L}_X g$ represents the Lie derivative of g in the direction of the vector field X and $\rho \in \mathbb{R}$, $\rho \neq 0$. This equation is associated to Ricci-Bourguignon flows developed by Catino et al. [7], which was first considered by Bourguignon [2].

If there exists a smooth function $f : M \to \mathbb{R}$ such that $X = \nabla f$ then the ρ-Einstein soliton is called a gradient ρ-Einstein soliton, denoted by (M, g, f, ρ) and in this case (1.3) takes the form

$$\mathrm{Ric} + \mathrm{Hess}(f) = \lambda g + \rho S g. \tag{1.4}$$

As usual, a ρ-Einstein soliton is called steady for $\lambda = 0$, shrinking for $\lambda > 0$ and expanding for $\lambda < 0$. The function f is called a ρ-Einstein potential of the gradient ρ-Einstein soliton.

For special values of the parameter ρ, a ρ-Einstein soliton is called [15]

i) gradient Einstein soliton if $\rho = \frac{1}{2}$,
ii) gradient traceless Ricci soliton if $\rho = \frac{1}{n}$,
iii) gradient Schouten soliton if $\rho = \frac{1}{2(n-1)}$.

In this paper we based on ideas of Cunha et. al. (see [4, 5]) to introduce a new geometric object which extends in a natural way a ρ-Einstein solitons. Let M^n be an oriented and connected hypersurface immersed into a $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$. We say that M^n is a *gradient r-almost-Newton-Einstein soliton*, for some $0 \leq r \leq n$, if there exists a smooth function $f : M^n \to \mathbb{R}$ such that the following equation holds:

$$\mathrm{Ric} + \mathrm{P_r} \circ \mathrm{Hess} f = \lambda g + \rho S g, \tag{1.5}$$

where g is the Riemannian metric induced by immersion, λ is a smooth function on M, $\rho \in \mathbb{R}\backslash\{0\}$, S is the scalar curvature of M with respect to g, and $\mathrm{P_r} \circ \mathrm{Hess} f$ stands for the tensor given by

$$\mathrm{P_r} \circ \mathrm{Hess} f(X, Y) = \langle P_r \nabla_X \nabla f, Y \rangle,$$

for tangent vector fields $X, Y \in \mathfrak{X}(M)$.

Moreover, Siddiqi et al. (see [9, 14]) also studied r-almost-Newton Ricci solitons which is merely closed to equation (1.5). The interest in the study of equation (1.5) comes, in particular, from the fact that when $r = 0$, a gradient r-almost-Newton-Einstein soliton reduces to a gradient ρ-Einstein soliton when λ is a constant.

If $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, then the gradient r-almost-Newton-Einstein soliton is called expanding, steady, or, shrinking respectively. When the potential f is constant, the gradient r-almost-Newton-Einstein soliton is called trivial. Otherwise, it is said nontrivial. Also, when λ is a constant, we say that the gradient r-almost-Newton-Einstein soliton is simply a *gradient r-Newton-Einstein soliton*.

This manuscript is organized in the following way: In Section 2 we recall some basic facts and notations that will appear throughout the paper. In Section 4 we approach the compact case and prove some triviality results . Furthermore we give a type Schur inequality. In Section 5 we approach the complete case and obtain constant scalar curvature under some conditions in the potential function. Finally in Section 6 we give some non existence of minimal r-almost Newton

Einstein solitons. Also, we characterize it to be totally geodesic and in some case we obtain that it is isometric to Euclidean sphere.

2. Preliminaries

Let M^n be an oriented and connected hypersurface immersed into a $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$. It is well known that the Gauss equation of the immersion is given by

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top + \langle AX, Z\rangle AY - \langle AY, Z\rangle AX$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(M)$, where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M})$ along M^n. Here, $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the second fundamental form (or shape operator) of M^n in $\overline{M}^{n+1}$ with respect to a fixed orientation, $\overline{R}$ and R denote the curvature tensors of $\overline{M}^{n+1}$ and M^n, respectively. In particular, the scalar curvature S of the hypersurface M^n satisfies

$$S = \sum_{i,j}^{n} \langle \overline{R}(e_i, e_j)e_j, e_i\rangle + n^2H^2 - |A|^2, \tag{2.1}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame on TM and $|\cdot|$ denotes the Hilbert-Schmidt norm. When $\overline{M}^{n+1}$ is a space form of constant sectional curvature c, we have the identity

$$S = n(n-1)c + n^2H^2 - |A|^2. \tag{2.2}$$

Associated to second fundamental form A of the hypersurface M^n there are n algebraic invariants, which are the elementary symmetric functions S_r of its principal curvatures $k_1, \ldots, k_n$, given by

$$S_0 = 1 \quad \text{and} \quad S_r = \sum_{i_1<\ldots<i_r} k_{i_1} \cdots k_{i_r}.$$

The r-th mean curvature H_r of the immersion is defined by

$$\binom{n}{r} H_r = S_r.$$

In the case $r = 1$, we have $H_1 = \frac{1}{n} tr(A) = H$ the mean curvature of M^n.

For each $0 \le r \le n$, one defines the r-th *Newton transformation* $P_r : \mathfrak{X}(M) \to \mathfrak{X}(M)$ of the hypersurface M^n by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$, via the recurrence relation

$$P_r = \sum_{j=0}^{r} (-1)^{r-j} \binom{n}{j} H_j A^{r-j}, \tag{2.3}$$

where $A^{(j)}$ denotes the composition of A with itself, j times ($A^{(0)} = I$). Let us remember that associated to each Newton transformation P_r one has the second order linear differential operator $L_r : C^\infty(M) \to C^\infty(M)$ defined by

$$L_r u = \mathrm{tr}(P_r \circ \text{Hess } u).$$

When $r = 0$, we note that L_0 is just the Laplacian operator. Moreover, it is not difficult to see that

$$\begin{aligned} \mathrm{div}_M(P_r\nabla u) &= \sum_{i=1}^{n}\langle(\nabla_{e_i}P_r)(\nabla u), e_i\rangle + \sum_{i=1}^{n}\langle P_r(\nabla_{e_i}\nabla u), e_i\rangle \\ &= \langle \mathrm{div}_M P_r, \nabla u\rangle + L_r u, \end{aligned} \tag{2.4}$$

where the divergence of P_r on M^n is given by

$$\mathrm{div}_M P_r = \mathrm{tr}(\nabla P_r) = \sum_{i=1}^{n}(\nabla_{e_i}P_r)(e_i).$$

In particular, when the ambient space has constant sectional curvature equation (2.4) reduces to $L_r u = \mathrm{div}_M(P_r\nabla u)$, because $\mathrm{div}_M P_r = 0$ (see [13] for more details). Useful consequences are given in the following lemma.

Lemma 2.1 ([13])**.** *If M is compact without boundary or if M is non compact and f has compact support then*

(i) $\displaystyle\int_M L_r(f)dM = 0$;

(ii) $\displaystyle\int_M fL_r(f)dM = -\int_M \langle P_r\nabla f, \nabla f\rangle.$

For our purpose, it also will be appropriate to deal with the so-called traceless second fundamental form of the hypersurface, which is given by $\Phi = A - HI$. Observe that $\mathrm{tr}\ \Phi = 0$ and $|\Phi|^2 = \mathrm{tr}(\Phi^2) = |A|^2 - nH^2 \geq 0$, with equality if and only if M^n is totally umbilical.

To conclude this section we recall the following lemma due to Yau, which corresponds to Theorem 3 of [17].

Lemma 2.2. *Let u be a non-negative smooth subharmonic function on a complete Riemannian manifold M^n. If $u \in L^p(M)$, for some $p > 1$, then u is constant.*

Here we use the notation $L^p(M) = \{u : M^n \to \mathbb{R}\ ;\ \int_M |u|^p dM < +\infty\}$ for each $p \geq 1$.

3. An example

Example 3.1. We consider the standard immersion of $\mathbb{S}^n$ into $\mathbb{S}^{n+1}$, which we know that is totally geodesic. In particular, $P_r \equiv 0$ for all $1 \leq r \leq n$, and choosing $\lambda = (n-1)/n - \rho$, we obtain that the immersion satisfies equation (1.5).

Also we can see that if M is constant scalar curvature, then the equation (1.5) becomes

$$\mathrm{Ric} + \mathrm{P_r} \circ \mathrm{Hess} f = \mu g,$$

where $\mu = \lambda + \rho S$. So, we can recall to Example 2 of [4] another example of gradient r-almost-Newton-Einstein soliton.

4. Triviality Results

We dedicate this section to present some triviality results and results when the gradient r-Newton-Einstein soliton is closed. More precisely:

Theorem 4.1. *Let (M, g, f, ρ) be a closed gradient r-Newton-Einstein soliton immersed into a Riemannian manifold $\bar{M}_c^{n+1}$ of constant sectional curvature c, such that P_r is bounded from above or from below (in the sense of quadratic forms). If holds any one of the following*

i) *either $\rho > \frac{1}{n}$ and the scalar curvature is $S \geq 0$ and $\lambda \geq 0$ or $S \leq 0$ and $\lambda \leq 0$ or;*
ii) *$\rho < \frac{1}{n}$ and the scalar curvature is $S \geq 0$ and $\lambda \leq 0$ or $S \leq 0$ and $\lambda \geq 0$ or,*
iii) *$\rho \neq \frac{1}{n}$ and the scalar curvature, either $S \geq \frac{-\lambda n}{\rho n - 1}$ or $S \leq \frac{-\lambda n}{\rho n - 1}$,*

then M must be a constant scalar curvature and trivial.

Proof. From Lemma 2.1 and the structural equation we obtain

$$0 = \int_M L_r f = \int_M (\lambda n + (\rho n - 1)S).$$

Hence, if holds (i), (ii) we obtain $S = \lambda = 0$ and from the structural equation we get $L_r f = 0$. Since P_r is bounded from above or from below (in the sense of quadratic forms), there is a positive constant $C > 0$ such that

$$0 = L_r f \leq C\Delta f, \text{ or } 0 = L_r f \geq -C\Delta f,$$

respectively. So, f is a subharmonic function. Since M is compact, we conclude from Hopf's theorem that f is a constant function. Therefore, M is trivial.

Finally, item (iii) follows identically to (i) and (ii). □

Remark 4.1. *Items (i) and (ii) in the theorem above imply that M is steady and $S = 0$. Being M trivial we get $Ric \equiv 0$. Finally, item (iii) implies $S = \frac{-\lambda n}{\rho n - 1}$. Since M is trivial, we obtain*

$$Ric = \left(\lambda - \frac{\rho n \lambda}{\rho n - 1}\right) g = -\frac{\lambda}{\rho n - 1} g = \frac{S}{n} g,$$

i.e., M is Einstein.

Theorem 4.2. *Let (M, g, f, ρ) be a closed gradient r-Newton-Einstein soliton immersed into a Riemannian manifold $\bar{M}_c^{n+1}$ of constant sectional curvature c, such that P_r is bounded from above or from below (in the sense of quadratic forms) and $\rho \neq \frac{1}{n}$. If M has constant scalar curvature, then M is Einstein and trivial.*

Proof. From Lemma 2.1 and the structural equation we have

$$\int_M |\lambda n + (\rho n - 1)S|^2 = \int_M (\lambda n + (\rho n - 1)S) L_r f = (\lambda n + (\rho n - 1)S) \int_M L_r f = 0.$$

Hence, we obtain $S = \frac{-\lambda n}{\rho n - 1}$ and $L_r f = 0$. Using that P_r is bounded from above or from below (in the sense of quadratic forms) we can proceed as in the proof of Theorem 4.1 to conclude that M is trivial. Finally, since M is trivial, we can proceed as in Remark 4.1 to conclude that M is Einstein. □

In the next result we prove an inequality type Schur. We prove the following.

Theorem 4.3. *Let (M, g, f, ρ) be a closed gradient r-Newton-Einstein soliton immersed into a Riemannian manifold $\bar{M}_c^{n+1}$ of constant sectional curvature c, such that P_r is bounded from below (in the sense of quadratic forms) and $\rho > \frac{1}{n}$. Then*

$$\int_M |S - \overline{S}|^2 \leq \frac{2nC}{(n-2)(\rho n - 1)} \, \|\overset{\circ}{\mathrm{Ric}}\|_{L^2} \, \|\nabla^2 f - \frac{\Delta f}{n} g\|_{L^2}. \tag{4.1}$$

Proof. We recall the contracted second Bianchi identity tells us that

$$\mathrm{divRic} + \frac{1}{2}\nabla S = 0,$$

and hence

$$\mathrm{div}\,\overset{\circ}{\mathrm{Ric}} = -\frac{n-2}{2n}\nabla S.$$

Since M is compact, we get using our assumption on P_r that

$$\begin{aligned}
\int_M |\lambda n + (\rho n - 1)S|^2 &= \int_M (\lambda n + (\rho n - 1)S) L_r f = \int_M (\lambda n + (\rho n - 1)S)\mathrm{div}(P_r \nabla f) \\
&= -(\rho n - 1)\int_M \langle \nabla S, P_r \nabla f\rangle \leq C(\rho n - 1)\int_M \langle \nabla S, \nabla f\rangle \\
&= -\frac{2nC(\rho n - 1)}{n-2}\int_M \langle \mathrm{div}\,\overset{\circ}{\mathrm{Ric}}, \nabla f\rangle \\
&= \frac{2nC(\rho n - 1)}{n-2}\int_M \langle \overset{\circ}{\mathrm{Ric}}, \nabla^2 f\rangle \\
&= \frac{2nC(\rho n - 1)}{n-2}\int_M \langle \overset{\circ}{\mathrm{Ric}}, \nabla^2 f - \frac{\Delta f}{n} g\rangle \\
&\leq \frac{2nC(\rho n - 1)}{n-2} \, \|\overset{\circ}{\mathrm{Ric}}\|_{L^2} \, \|\nabla^2 f - \frac{\Delta f}{n} g\|_{L^2},
\end{aligned}$$

where we used that $\langle \overset{\circ}{\mathrm{Ric}}, g\rangle = 0$. Since M is compact we have

$$\lambda n = -(\rho n - 1)\overline{S},$$

where $\overline{S}$ stands for the average of S. Therefore,

$$(\rho n - 1)^2 \int_M |S - \overline{S}|^2 = \int_M |\lambda n + (\rho n - 1)S|^2,$$

i.e.,

$$(\rho n - 1)^2 \int_M |S - \overline{S}|^2 \leq \frac{2nC(\rho n - 1)}{n-2} \, \|\overset{\circ}{\mathrm{Ric}}\|_{L^2} \, \|\nabla^2 f - \frac{\Delta f}{n} g\|_{L^2},$$

i.e.,

$$\int_M |S - \overline{S}|^2 \leq \frac{2nC}{(n-2)(\rho n - 1)} \, \|\overset{\circ}{\mathrm{Ric}}\|_{L^2} \, \|\nabla^2 f - \frac{\Delta f}{n} g\|_{L^2}. \tag{4.2}$$

This completes the proof. □

Remark 4.2. Observe that in the theorem above if M is Einstein, then both sides of (4.1) vanish, so the equality holds. It would be an interesting problem to prove the rigidity.

5. On complete r-Einstein-Newton solitons

In this section we consider the complete non-compact case. We start with the following theorem.

Theorem 5.1. *Let (M, g, f, ρ) be a complete r-Newton-Einstein soliton immersed into a Riemannian manifold $\bar{M}_c^{n+1}$ of constant sectional curvature c, such that the potential function is non-negative and belongs to $f \in L^p(M)$ for some $p > 1$. If holds any one of the following*

- i.) *either $\rho > \frac{1}{n}$, P_r is bounded from above (in the sense of quadratic forms) and $S \geq \frac{-\lambda n}{\rho n - 1}$, or*
- ii.) *$\rho > \frac{1}{n}$, P_r is bounded from below (in the sense of quadratic forms) and $S \leq \frac{-\lambda n}{\rho n - 1}$ or*
- iii.) *$\rho < \frac{1}{n}$, P_r is bounded from below (in the sense of quadratic forms) and $S \geq \frac{-\lambda n}{\rho n - 1}$, or*
- iv.) *$\rho < \frac{1}{n}$, P_r is bounded from above (in the sense of quadratic forms) and $S \leq \frac{-\lambda n}{\rho n - 1}$*

then $S = \frac{-\lambda n}{\rho n - 1}$ and M is Einstein trivial.

Proof. If P_r is bounded from above (in the sense of quadratic forms) there is a positive constant $\beta > 0$ such that

$$0 \leq \lambda n + (\rho n - 1)S = L_r f \leq \beta \Delta f.$$

Thus f is a subharmonic function, so from Lemma 2.2 we get f is a constant. Therefore, $S = -\frac{\lambda n}{\rho n - 1}$ and M is trivial. Since M is trivial, we have from the structural equation that

$$Ric = \left(\lambda - \frac{\rho n \lambda}{\rho n - 1}\right) g = -\frac{\lambda}{\rho n - 1} g = \frac{S}{n} g,$$

i.e., M is Einstein.

Finally, if (ii) holds, there is a positive constant $\beta > 0$ such that $L_r f \geq -\beta \Delta f$, so it's enough to proceed identically to (i). The cases (iii) and (iv) are analogous. □

The next two results follow the ideas of [6].

Theorem 5.2. *Let (M, g, f, ρ) be a complete gradient r-almost-Newton-Einstein soliton of dimension n immersed into a Riemannian manifold $\bar{M}_c^{n+1}$ of constant sectional curvature c, such that M has non-negative scalar curvature. Assume that the potential function f satisfies the condition*

$$\int_{M - B(q,r)} \frac{f}{d(x,q)^2} < \infty, \tag{5.1}$$

where $B(q, r)$ is a ball with radius $r > 0$ and center at q and $d(x, q)$ is the distance function from some fixed point $q \in M$. If any one of following

- i.) *either M is non-expanding, P_r is bounded from above (in the sense of quadratic forms) and $\rho > \frac{1}{n}$ or,*
- ii.) *M is expanding, P_r is bounded from below (in the sense of quadratic forms) and $\rho < \frac{1}{n}$,*

holds, then $S = 0$.

Proof. Let us see the item (i), the item (ii) is analogous. Taking trace at structural equation, we get

$$\lambda n + (\rho n - 1)S = L_r f. \tag{5.2}$$

Let us consider the cut-off function, introduced in [18], $\psi_r \in C_0^\infty(B(q, 2r))$ for $r > 0$ such that

$$\begin{cases} 0 \leq \psi_r \leq 1 & \text{in } B(q, 2r) \\ \psi_r = 1 & \text{in } B(q, r) \\ |\nabla \psi_r|^2 \leq \frac{C}{r^2} & \text{in } B(q, 2r) \\ \Delta \psi_r \leq \frac{C}{r^2} & \text{in } B(q, 2r), \end{cases} \tag{5.3}$$

where $C > 0$ is a constant. Now using (5.2), integration by parts and that P_r is bounded from above (in the sense of quadratic forms) we obtain

$$\begin{aligned} 0 \leq \int_{B(q,2r)} \psi_r S = \int_{B(q,2r)} \psi_r \left(\frac{1}{\rho n - 1} L_r f - \frac{\lambda n}{\rho n - 1}\right) \leq \frac{1}{\rho n - 1} \int_{B(q,2r)} \psi_r L_r f \\ \leq \frac{C_1}{\rho n - 1} \int_{B(q,2r)} \psi_r \Delta f \leq \frac{C_1}{\rho n - 1} \int_{B(q,2r) - B(q,r)} f \Delta \psi_r \\ \leq \frac{C_1}{\rho n - 1} \int_{B(q,2r) - B(q,r)} \frac{C_2}{r^2} f \to 0, \end{aligned}$$

as $r \to \infty$. Since $\psi_r = 1$ in $B(q, r)$, from the above inequality we have $S = 0$. This concludes the proof. □

Remark 5.1. *The above theorem still guarantees that a gradient r-almost-Newton-Einstein soliton is, in fact, a gradient r-almost-Newton-Ricci soliton studied in* [4]*. Therefore, any gradient r-almost-Newton-Einstein soliton satisfying the conditions of Theorem 5.2 is a gradient r-almost-Newton-Ricci soliton with scalar curvature* $S = 0$.

Theorem 5.3. *Let* $(M, g, f, \frac{1}{n})$ *be a non-expanding gradient traceless r-Newton-Einstein soliton, such that* P_r *is bounded from above (in the sense of quadratic forms) with non-negative potential function* f. *If* $f \in L^p(M)$ *for some* $p > 1$, *then* M *must be steady and Einstein trivial.*

Proof. Since for non-expanding solitons $\lambda \geq 0$, it follows from the structural equation that

$$L_r f = \lambda n \geq 0.$$

From hypothesis on P_r, there is a positive constant $C > 0$ such that

$$0 \leq \lambda n = L_r f \leq C \Delta f,$$

i.e., f is a non-negative subharmonic function. Hence from Lemma 2.2 we find that f is constant, and $0 = \Delta f \geq \lambda n \geq 0$. Therefore, $\lambda = 0$ and M is trivial. Finally, since M is trivial, we obtain $Ric = \frac{S}{n} g$, and M is Einstein.

□

6. Non existence results

In this section, we will use the following lemma due to Caminha et al. [3]

Lemma 6.1. *Let X be a smooth vector field on the n-dimensional, complete, non compact, oriented Riemannian manifold M^n such that $div_M X$ does not change sign on M^n. If $|X| \in L^1(M)$, then $div_M X = 0$.*

By following the idea of the authors of [4] we prove the following theorem.

Theorem 6.1. *Let M^n be complete r-almost Newton-Einstein soliton immersed into a Riemannian manifold M^{n+1} of constant sectional curvature c with bounded second fundamental form and potential function $f : M^n \to \mathbb{R}$ such that $|\nabla f| \in L^1(M)$. Then we have*

i) If $c \leq 0$, $\lambda > 0$ and $\rho < \frac{1}{n}$, then M^n can not be minimal,

ii) If $c < 0$, $\lambda \geq 0$ and $\rho < \frac{1}{n}$, then M^n can not be minimal.

iii) If $c = 0$, $\lambda \geq 0$, $\rho < \frac{1}{n}$, and M^n is minimal, then M^n is steady and isometric to the $\mathbb{R}^n$.

Proof. We know that the ambient space has constant sectional curvature, by equation (2.4) the operator L_r is a divergent type operator. On the other hand, since M^n has a bounded second fundamental form, it follows from (2.3) that the Newton transformation P_r has bounded norm. In particular,

$$|P_r \nabla f| \leq |P_r| \, |\nabla f| \in L^1(M), \tag{6.1}$$

Using (i) and (ii), let us consider by contradiction that M^n is minimal. Then, equation (2.2) jointly with considering $c \leq 0$ ($c < 0$) implies that the scalar curvature of M^n satisfies $S \leq 0 (S < 0)$. Hence, contracting (1.5) we have

$$L_r f = n\lambda + (n\rho - 1)S > 0$$

in both cases, which contradicts Lemma 6.1, since the fact is mentioned. This completes the proof of the first two assertions.

For the (iii) assertion, since the ambient space has constant sectional curvature $c = 0$ and M^n is minimal, the equation (2.2) becomes as

$$S = -|A|^2 \leq 0. \tag{6.2}$$

So, since $\lambda \geq 0$ and $\rho < \frac{1}{n}$, we have $L_r f = n\lambda + (n\rho - 1)S \geq 0$. Now, using the fact that L_r is a divergent type operator and $|P_r \nabla f| \in L^1(M)$, we have again from Lemma 6.1 that $L_r f = 0$ on M^n. Hence, we conclude that $0 \geq S = \frac{-n\lambda}{(n\rho - 1)} \geq 0$; that is, $S = \lambda = 0$. This implies that $|A|^2 = 0$. Therefore, the gradient r-almost-Newton-Einstein soliton M^n must be steady totally geodesic and flat. □

Further, we are in a condition to establish the following result, which holds when the ambient space is an arbitrary Riemannian manifold.

Theorem 6.2. *Let M^n be a complete r-almost-Newton-Einstein soliton immersed into a Riemannian manifold M^{n+1} of sectional curvature K, such that P_r is bounded from above (in the sense of quadratic forms) and its potential function $f : M^n \to \mathbb{R}$ is non-negative and $f \in L^p(M)$ for some $p > 1$. Then we have*

i) *If $K \leq 0$, $\lambda > 0$ and $\rho < \frac{1}{n}$, then M^n can not be minimal,*

ii) *If $K < 0$, $\lambda \geq 0$ and $\rho < \frac{1}{n}$, then M^n can not be minimal,*

iii) *If $K \leq 0$, $\lambda \geq 0$, $\rho < \frac{1}{n}$ and M^n is minimal, then M^n is steady flat and totally geodesic.*

Proof. For proving (i), we start with the contradiction that M^n is minimal, so our assumption on the sectional curvature of the ambient space and equation (2.1) imply that $S \leq 0$. Hence, contracting equation (1.5) we have

$$L_r f = n\lambda + (n\rho - 1)S > 0. \tag{6.3}$$

Thus, since we are considering that P_r is bounded from above, there is a positive constant $C > 0$ such that

$$C\Delta f \geq L_r f > 0. \tag{6.4}$$

In particular, from Lemma 2.2 we get that f must be constant, which gives a contradiction. Finally, reasoning as in the proof of Theorem 6.1 we can easily obtain (ii) and (iii). □

In our next result we obtain Theorem 1.5 of [1] for the case when $X = \nabla f$, giving conditions for a r-almost-Newton-Einstein soliton immersed be totally umbilical since it has bounded second fundamental form. Thus, we prove the following theorem:

Theorem 6.3. *Let M^n be a complete r-almost Newton-Einstein soliton immersed into a Riemannian manifold M^{n+1} of constant sectional curvature c, with bounded second fundamental form and potential function $f : M^n \to \mathbb{R}$ such that $|\nabla f| \in L^1(M)$. Then we have*

i) *If $\rho < \frac{1}{n}$ and $\lambda \geq n(\rho n - 1)H^2 - (n-1)(\rho n - 1)c$, then M^n is totally geodesic, with $\lambda = (n\rho - 1)(n-1)c$, and scalar curvature $S = n(n-1)c$,*

ii) *If M^n is compact, $\rho < \frac{1}{n}$ and $\lambda \geq n(\rho n - 1)H^2 - (n-1)(\rho n - 1)c$, then M^n is isometric to a Euclidean sphere,*

iii) *If $\rho < \frac{1}{n}$ and $\lambda \geq (n+1)(\rho n - 1)H^2 - (n-1)(\rho n - 1)c$, then M^n is totally umbilical. In particular, the scalar curvature $S = n(n-1)c - n(n+1)K_M$ is constant, where $K_M = \left[\frac{\lambda}{(n+1)(\rho n - 1)} + \frac{n-1}{n+1}c\right]$ is the sectional curvature of M^n.*

Proof. To prove (i), we can use the equation (2.2) and structural equation, to obtain

$$L_r f = n[\lambda + (n-1)(\rho n - 1)c - n(\rho n - 1)H^2] - (\rho n - 1)\,|A|^2. \tag{6.5}$$

Then, for our consideration on λ, we get that $L_r f$ is non-negative function on M^n. By Lemma 6.1 we find that $L_r f = 0$. Hence, from equation (6.5) we obtain that M^n is totally geodesic and $\lambda = -(\rho n - 1)(n-1)c$. Moreover, from structural equation we get that

$$S = \frac{-\lambda n}{\rho n - 1} = n(n-1)c,$$

which completes the proof of (i).

If M^n is compact, as it is totally geodesic, then the ambient space must be necessarily a sphere $\mathbb{S}^{n+1}$ and M^n is isometric to the Euclidean sphere $\mathbb{S}^n$, which proves (ii).

For the assertion (iii), we start with equation (6.5) that can be written in terms of the traceless second fundamental form Φ as

$$L_r f = n[\lambda + (n-1)(\rho n - 1)c - (n+1)(\rho n - 1)H^2] - (\rho n - 1)|\Phi|^2. \tag{6.6}$$

Therefore, our assumption on λ and ρ gives $L_r f \geq 0$. Then by applying Lemma 6.1 once again we have $L_r f = 0$. This implies that $|\Phi|^2 = 0$; that is, M^n is a totally umbilical hypersurface. In particular, the principal curvature κ of M^n is constant and M^n has constant sectional curvature given by $K_M = c + \kappa^2$. This combined with (6.6) yields

$$\begin{aligned} \lambda &= (n+1)(\rho n - 1)H^2 - (n-1)(\rho n - 1)c \\ &= (n+1)(\rho n - 1)(c + \kappa^2) - (n-1)(\rho n - 1)c \\ &= (n+1)(\rho n - 1)K_M - (n-1)(\rho n - 1)c. \end{aligned}$$

Since $L_r f = 0$, we get

$$S = \frac{-\lambda n}{\rho n - 1} = n(n-1)c - n(n+1)K_M,$$

as desired. □

Now, we have the following consequence of Theorem 6.3.

Corollary 6.1. *Let M^n be a compact r-almost-Newton-Einstein soliton immersed into $\mathbb{R}^{n+1}$ such that $\rho < \frac{1}{n}$. If $\lambda \geq (n+1)(\rho n - 1)H^2$, then M^n is isometric to $\mathbb{S}^n$.*

In [1, Theorem 1.6] it was proved that a nontrivial almost Ricci soliton M^n, minimally immersed in $\mathbb{S}^{n+1}$ with $S \geq n(n-2)$ and such that the norm of the second fundamental form attains its maximum, must be isometric to $\mathbb{S}^n$. Now, applying Theorem 6.3 we obtain a generalization of this result.

Corollary 6.2. *Let M^n be a complete r-almost-Newton-Einstein soliton minimally immersed in $\mathbb{S}^{n+1}$, such that $\rho < \frac{1}{n}$. Assume that $S \geq n(n-2)$, the norm of the second fundamental form attains its maximum and $\lambda \geq -(n-1)(\rho n - 1)$. Then, M^n is isometric to $\mathbb{S}^n$.*

Proof. Using the minimality of the immersion and that $S \geq n(n-2)$, we obtain from (2.2) that

$$|A|^2 = n(n-1) - S \leq n.$$

From Simons's formula [16], we obtain

$$\Delta |A|^2 = |\nabla A|^2 + (n - |A|^2)\,|A|^2 \geq 0. \tag{6.7}$$

Thus, we can apply Hopf's strong maximum principle to get that $\nabla A = 0$ on M^n. Therefore, Proposition 1 of [13] assures that M^n must be compact and, hence, the results of Theorem 6.3. □

As another application of Lemma 2.2, we can obtain the following theorem:

Theorem 6.4. *Let M^n be a complete r-almost Newton-Einstein soliton immersed into a Riemannian manifold M^{n+1} of constant sectional curvature c, such that P_r is bounded from above (in the sense of quadratic forms) with non-negative potential function $f \in L^p(M)$ for some $p > 1$. Then we have*

i) *If $\rho < \frac{1}{n}$ and $\lambda \geq n(\rho n - 1)H^2 - (n-1)(\rho n - 1)c$, then M^n is totally geodesic, with $\lambda = (n\rho - 1)(n-1)c$, and scalar curvature $S = n(n-1)c$,*

ii) *If $\rho < \frac{1}{n}$ and $\lambda \geq (n+1)(\rho n - 1)H^2 - (n-1)(\rho n - 1)c$, then M^n is totally umbilical. In particular, the scalar curvature $S = n(n-1)c - n(n+1)K_M$ is constant, where $K_M = \left[\frac{\lambda}{(n+1)(\rho n - 1)} + \frac{n-1}{n+1}c\right]$ is the sectional curvature of M^n.*

Proof. Observe that from equation (6.5) and assumption on λ we get

$$L_r f = n[\lambda + (n-1)(\rho n - 1)c - n(\rho n - 1)H^2] - (\rho n - 1)\,|A|^2 \geq 0. \tag{6.8}$$

Since we are assuming that P_r is bounded from above, there is a positive constant $C > 0$ such that

$$C\Delta f \geq L_r f \geq 0. \tag{6.9}$$

Using Lemma 2.2, we have that f must be constant. Therefore $L_r f = 0$, and by equation (6.8) we conclude that M^n is totally geodesic with

$$\lambda = -(n-1)(\rho n - 1)c \text{ and } S = n(n-1)c,$$

proving assertion (i). Finally, reasoning as in Theorem 6.3, it is easy to prove assertion (ii). □

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References

[1] A. Barros, J.N. Gomes and E. Ribeiro Jr., Immersion of almost Ricci solitons into a Riemannian manifold, *Mat. Contemp.* **40** (2011), 91–102.

[2] J. P., Bourguignon, Ricci curvature and Einstein metrics, *Glob. Diff. Geom. and Glob. Anal.,* **838** (1979), 42–63.

[3] A. Caminha, P. Sousa., F. Camargo, Complete foliations of space forms by hypersurfaces. *Bull. Braz. Math. Soc.* **41** (2010) 339–353.

[4] A. W. Cunha, H.F. de Lima and E.L. de Lima, r-Almost Newton-Ricci solitons immersed into a Riemannian manifold. *J. Math. Anal. Appl.,* **464** (2018), no. 1, 546–556.

[5] A. W Cunha, EL de Lima, M.D Siddiqi, On scalar curvature of r-almost Yamabe solitons, J. Geom. Phys., **175** (2022), 104472.

[6] A. W. Cunha; A. A. Shaikh and P. Mandal, Some characterizations of gradient ρ-Einstein solitons. *J. Geom. Phys.*, vol. **166** (2021), 104270.

[7] G. Catino, L. Cremaschi, C. Mantegazza, Z. Djadli and L. Mazzieri, The Ricci-Bourguignon flow, *Pac. J. Math.,* **287(2)** (2017), 337–370.

[8] G. Catino, C. Mantegazza and L. Mazzieri, On the global structure of conformal gradient solitons with nonnegative Ricci tensor, *Commun. Contemp. Math.* **14(06)** (2012), 12pp.

[9] U. C. De, M. D. Siddiqi, S. K. Chaubey, r-almost Newton-Ricci solitons on Legenderian submanifolds of Sasakian space forms, *Period Math. Hung.* **84** (2022), 76–88.

[10] R. S. Hamilton, The formation of singularities in the Ricci flow, *Surveys in Diff. Geom.,* **2** (1995), 7–136.

[11] R. S. Hamilton, Lectures on geometric flows, unpublished manuscript, 1989.

[12] S. Y. Hsu, A note on compact gradient Yamabe solitons, *J. Math. Anal. Appl.* **388** (2012), no. 2, 725–726.

[13] H. Rosenberg, Hypersurfaces of Constant Curvature in Space Forms, *Bull. Sc. Math.* **117** (1993), 217–239.

[14] M. D. Siddiqi, Newton-Ricci Bourguignon almost solitons on Lgarangian submanifolds of complex space form, *Acta Universit. Apulensis* **63** (2020), 81–96.

[15] M. D. Siddiqi, Ricci ρ-soliton and geometrical structure in a dust fluid and viscous fluid sapcetime, *Bulg. J. Phys.* **46** (2019), 163-173.

[16] J. Simons, Minimal varieties in Riemannian manifolds, *Ann. of Math.* **88** (1968), 62–105.

[17] S.T. Yau, Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, *Indiana Univ. Math. J.* **25** (1976), 659–670.

[18] J. Cheeger and T. H. Colding, Lower bounds on Ricci curvature and the almost rigidity of warped products, *Ann. of Math.* **144** (1996), no. 1, 189–237.

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