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IMPROVING BOUNDS FOR GENERALIZED BEREZIN NUMBER OF OPERATORS

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Abstract. This paper introduces an improvement of the Cauchy-Schwarz inequality in a semi-Hilbertian space with the inner product induced by an $\mathfrak{A}$-positive operator. This improved $\mathfrak{A}$-Cauchy-Schwarz inequality is then used to improve the bounds of the $\mathfrak{A}$-Berezin number of operators. First, we present upper bounds for the $\mathfrak{A}$-Berezin number of the product of two operators. Then, $\mathfrak{A}$-Berezin number bounds for the sum of two operators are developed. This work refines and extends some of the $\mathfrak{A}$-Berezin inequalities previously established in the literature.

1. Introduction

Operator theory is an advancement of functional analysis. The Berezin symbol serves as a bridge between operators and functions. It provides a way to approximate operators by scalar functions and offers a robust framework for studying operators on functional spaces, such as the reproducing kernel Hilbert space (RKHS). Likewise, it facilitates the study of spaces such as Fock, Bergman, and Hardy spaces. For example, some operators describe or relate to their boundedness, invertibility, compactness, and positivity. For more information about the Berezin symbol, readers are directed to ([4, 7, 9, 11, 14, 27, 28, 29, 30]).

To lay the foundation for our study, let $\mathcal{H}$ be a complex Hilbert space unless otherwise stated. A functional Hilbert space is a complex-valued function space defined on a domain Ω, equipped with an inner product $\langle\cdot,\cdot\rangle$ and satisfying the completeness property. It has a continuous point evaluation functional for each $\eta \in \Omega$, which is given as $f \longmapsto f(\eta)$. By the Riesz representation theorem, there is a unique element φ_η of $\mathcal{H}$ such that the functional can be represented as the inner product: $f(\eta) = \langle f, \varphi_\eta\rangle$ for all $f \in \mathcal{H}$. The reproducing kernel of $\mathcal{H}$ is the collection $\{\varphi_\eta : \eta \in \Omega\}$. According to problem 37 of [23], a set of vectors $\{e_n\}$ of a reproducing kernel Hilbert space is called an orthonormal basis if it satisfies: $\varphi_\eta(z) = \sum_n \overline{e_n(\eta)} e_n(z)$. Normalized reproducing kernel is defined as $\hat{\varphi}_\eta = \varphi_\eta / \|\varphi_\eta\|$, where $\eta \in \Omega$. It ensures the unit vector. Berezin initially introduced the Berezin symbol of a bounded linear operator M on $\mathcal{H}$ with the function $\tilde{\mathrm{M}}$ defined on Ω by $\tilde{\mathrm{M}}(\eta) = \langle \mathrm{M}\hat{\varphi}_\eta, \hat{\varphi}_\eta\rangle$, where $\hat{\varphi}_\eta$ represents Banach algebra that plays a crucial role in defining and studying Berezin number and

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Berezin norm. Let the adjoint operator and the range of operator M, belonging to Banach algebra of all bounded linear operators $\mathcal{B}(\mathcal{H})$, are denoted by M^* and $\mathcal{R}(\mathrm{M})$, respectively. Now, we are able to define the Berezin number and the Berezin norm of M as:

$$\operatorname{ber}(\mathrm{M}) = \sup_{\eta\in\Omega} \left|\langle \mathrm{M}\hat{\varphi}_\eta, \hat{\varphi}_\eta\rangle\right|, \text{ and } \|\mathrm{M}\|_{\mathrm{ber}} = \sup_{\eta\in\Omega} \|\mathrm{M}\hat{\varphi}_\eta\|,$$

respectively. The following properties of the Berezin number and the Berezin norm directly emerge from their definitions. Let for $\mathrm{M}, \mathrm{L} \in \mathcal{B}(\mathcal{H})$: (i) $\operatorname{ber}(\alpha\mathrm{M}) = |\alpha| \operatorname{ber}(\mathrm{M})$ for all $\alpha \in \mathbb{C}$; (ii) $\operatorname{ber}(\mathrm{M}+\mathrm{L}) \leq \operatorname{ber}(\mathrm{M}) + \operatorname{ber}(\mathrm{L})$; (iii) $\operatorname{ber}(\mathrm{M}) \leq \|\mathrm{M}\|_{\mathrm{ber}} \leq \|\mathrm{M}\|$; (iv) $\operatorname{ber}(\mathrm{M}^*) = \operatorname{ber}(\mathrm{M})$ and $\|\mathrm{M}^*\|_{\mathrm{ber}} = \|\mathrm{M}\|_{\mathrm{ber}}$.

Gürdal et al. [22] develop the following inequality for an operator as,

$$\operatorname{ber}^4(\mathrm{M}) \leq \frac{1}{2} \left\| |\mathrm{M}|^4 + |\mathrm{M}^*|^4 \right\|_{\mathrm{ber}}. \tag{1.1}$$

The same author develops the following inequality for the sum of two operators as

$$\operatorname{ber}^2(\mathrm{M}+\mathrm{L}) \leq \operatorname{ber}^2(\mathrm{M}) + \operatorname{ber}^2(\mathrm{L}) + \frac{1}{2} \left\| |\mathrm{M}|^2 + |\mathrm{L}^*|^2 \right\|_{\mathrm{ber}} + \operatorname{ber}(\mathrm{LM}). \tag{1.2}$$

Huban et al. [25] has presented the following estimate of Berezin radius for the product of two operators defined on a reproducing kernel Hilbert space as given below

$$\operatorname{ber}^{2r}(\mathrm{M}^*\mathrm{L}) \leq \frac{1}{2} \left\| |\mathrm{L}^*\mathrm{L}|^{2r} + |\mathrm{M}^*\mathrm{M}|^{2r} \right\|_{\mathrm{ber}}. \tag{1.3}$$

For more information, the reader is referred to ([5, 6, 13, 15, 16, 17, 18, 21, 24, 26, 36, 37, 20]).

We now concentrate on the semi-Hilbertian space operators to extend the concept of Berezin number. If $\mathfrak{A} \in \mathcal{B}(\mathcal{H})$ is the positive operator, $\mathfrak{A}$-inner product is defined as $\langle w_1, w_2\rangle_{\mathfrak{A}} = \langle \mathfrak{A}w_1, w_2\rangle$ for $w_1, w_2 \in \mathcal{H}$. Krein [32] and Zaanen [38] celebrated the structure of the semidefinite sesquilinear form: $\langle\cdot,\cdot\rangle_{\mathfrak{A}} : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$. This form generalize the inner product by ensuring non-negativity and it has closed tie to positive semidefinite operators. It yields a seminorm $\|\cdot\|_{\mathfrak{A}}$ as $\|w\|_{\mathfrak{A}} = \sqrt{\langle \mathfrak{A}w, w\rangle}$ for $w \in \mathcal{H}$. The Riesz representation theorem implies that if $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are inner products on $\mathcal{H}$ that induce the equivalent norms, then there is a unique invertible positive operator $\mathfrak{A}$ on $(\mathcal{H}, \langle,\rangle_1)$ such that $\langle w_1, w_2\rangle_2 = \langle \mathfrak{A}w_1, w_2\rangle_1$ for all $w_1, w_2 \in \mathcal{H}$. For a semi-Hilbertian space $\mathcal{H}$ equipped with the $\mathfrak{A}$-inner product, $\mathfrak{A}$-Cauchy-Schwarz inequality is expressed as:

$$|\langle w_1, w_2\rangle_{\mathfrak{A}}| \leq \|w_1\|_{\mathfrak{A}}\|w_2\|_{\mathfrak{A}}, \quad (w_1, w_2 \in \mathcal{H}).$$

An operator $\mathrm{L} \in \mathcal{B}(\mathcal{H})$ is said to be an $\mathfrak{A}$-adjoint operator of $\mathrm{M} \in \mathcal{B}(\mathcal{H})$ if it holds $\langle \mathrm{M}w_1, w_2\rangle_{\mathfrak{A}} = \langle w_1, \mathrm{L}w_2\rangle_{\mathfrak{A}}$ for every $w_1, w_2 \in \mathcal{H}$, which can be interpreted as $\mathfrak{A}\mathrm{L} = \mathrm{M}^*\mathfrak{A}$. $\mathfrak{A}$-adjoint operator examine the behavior of operators in semi-Hilbert space. It also plays an important role in generalizing the $\mathfrak{A}$-Berezin number to the classical Berezin number. R. G. Douglas explained in [12, Theorem 1] that an operator M on a semi-Hilbert space equipped with the $\mathfrak{A}$-inner product admits an $\mathfrak{A}$-adjoint operator if and only if the range of $\mathrm{M}^*\mathfrak{A}$ is contained in the range of $\mathfrak{A}$ as: $\mathcal{R}(\mathrm{M}^*\mathfrak{A}) \subseteq \mathcal{R}(\mathfrak{A})$. In that case, a unique bounded operator $Z \in \mathcal{B}(\mathcal{H})$ exists as $\mathfrak{A}Z = \mathrm{M}^*\mathfrak{A}$ and $\mathcal{R}(Z) \subseteq \overline{\mathcal{R}(\mathfrak{A})}$. This unique operator is usually denoted by $\mathrm{M}^{*\mathfrak{A}}$

and is called the distinguished $\mathfrak{A}$-adjoint operator of M. The set of all operators in $\mathcal{B}(\mathcal{H})$ admitting $\mathfrak{A}$-adjoint is denoted by $\mathcal{B}_{\mathfrak{A}}(\mathcal{H})$. It is easy to verify that for every $\mathrm{M}, \mathrm{L} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$, we have $\mathrm{ML} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$ such that $(\mathrm{ML})^{*_{\mathfrak{A}}} = \mathrm{L}^{*_{\mathfrak{A}}}\mathrm{M}^{*_{\mathfrak{A}}}$. Moreover, the set of all operators in $\mathcal{B}(\mathcal{H})$ accepting $\mathfrak{A}^{1/2}$-adjoint is conventionally represented by $\mathcal{B}_{\mathfrak{A}^{1/2}}(\mathcal{H})$. It can be demonstrated that $\mathcal{B}_{\mathfrak{A}}(\mathcal{H}) \subseteq \mathcal{B}_{\mathfrak{A}^{1/2}}(\mathcal{H})$. If $\mathrm{M} \in \mathcal{B}_{\mathfrak{A}^{1/2}}(\mathcal{H})$, then it holds that

$$\|\mathrm{M}\|_{\mathfrak{A}} = \sup_{0 \neq w \in \mathcal{R}(\mathfrak{A})} \frac{\|\mathrm{M}w\|_{\mathfrak{A}}}{\|w\|_{\mathfrak{A}}} = \sup\left\{\|\mathrm{M}w\|_{\mathfrak{A}} : w \in \mathcal{H}, \|w\|_{\mathfrak{A}} = 1\right\} < +\infty,$$

and $\|\mathrm{M}w\|_{\mathfrak{A}} \leq \|\mathrm{M}\|_{\mathfrak{A}}\|w\|_{\mathfrak{A}}$ for all $w \in \mathcal{H}$. For $\mathrm{M}, \mathrm{L} \in \mathcal{B}_{\mathfrak{A}^{1/2}}(\mathcal{H})$, it follows the inequality for $\mathfrak{A}$-norm: $\|\mathrm{ML}\|_{\mathfrak{A}} \leq \|\mathrm{M}\|_{\mathfrak{A}}\|\mathrm{L}\|_{\mathfrak{A}}$. Notice that it may happen that $\|\mathrm{M}\|_{\mathfrak{A}} = +\infty$ for some $\mathrm{M} \in \mathcal{B}(\mathcal{H}) \backslash \mathcal{B}_{\mathfrak{A}^{1/2}}(\mathcal{H})$. An operator $\mathrm{M} \in \mathcal{B}(\mathcal{H})$ is called $\mathfrak{A}$-positive if $\mathfrak{A}\mathrm{M}$ is positive, which ensures that

$$\|\mathrm{M}\|_{\mathfrak{A}} = \sup\left\{\langle \mathrm{M}w, w\rangle_{\mathfrak{A}} : w \in \mathcal{H}, \|w\|_{\mathfrak{A}} = 1\right\}.$$

Note that for $\mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H}), \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M}$ and $\mathrm{M}\mathrm{M}^{*_{\mathfrak{A}}}$ are both $\mathfrak{A}$-positive, and

$$\|\mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M}\|_{\mathfrak{A}} = \|\mathrm{M}\mathrm{M}^{*_{\mathfrak{A}}}\|_{\mathfrak{A}} = \|\mathrm{M}\|_{\mathfrak{A}}^2 = \|\mathrm{M}^{*_{\mathfrak{A}}}\|_{\mathfrak{A}}^2.$$

Further information about semi-Hilbertian space operators can be found in the references [3], [8], [33].

The $\mathfrak{A}$-Berezin number and the $\mathfrak{A}$-Berezin norm of $\mathrm{M} \in \mathcal{B}(\mathcal{H})$ were defined by Gürdal and Başaran [19], who extended the idea of Berezin number in semi-Hilbert space as follows:

$$\mathrm{ber}_{\mathfrak{A}}(\mathrm{M}) = \sup_{\eta \in \Omega}\left|\langle \mathrm{M}\hat{\varphi}_\eta, \hat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|, \text{ and } \|\mathrm{M}\|_{\mathfrak{A}-\mathrm{ber}} = \sup_{\eta \in \Omega}\|\mathrm{M}\hat{\varphi}_\eta\|_{\mathfrak{A}}.$$

Gürdal and Başaran [19] developed the inequalities of the $\mathfrak{A}$-Berezin number for the sum of operators as follows,

$$\mathrm{ber}_{\mathfrak{A}}(\mathrm{M}+\mathrm{L}) \leq \mathrm{ber}_{\mathfrak{A}}(\mathrm{M}) + \mathrm{ber}_{\mathfrak{A}}(\mathrm{L}) + \frac{1}{2}\|\mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L} + \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M}\|_{\mathfrak{A}-\mathrm{ber}} + \mathrm{ber}_{\mathfrak{A}}(\mathrm{LM}). \quad (1.4)$$

In particular, if we choose a positive operator $\mathfrak{A}$ as same as the identity operator in the definitions, then it gives $\mathrm{ber}_{\mathfrak{A}}(\mathrm{M}) = \mathrm{ber}(\mathrm{M})$ and $\|\mathrm{M}\|_{\mathfrak{A}-\mathrm{ber}} = \|\mathrm{M}\|_{\mathrm{ber}}$. For a comprehensive study of inequalities pertaining to the $\mathfrak{A}$-Berezin number of operators on reproducing kernel Hilbert space, see ([19]). The goal of this research is to expand $\mathfrak{A}$-Berezin number bounds, which are refinements of inequalities (1.1) and (1.3). The paper is organized as follows. In Section 2, foundational lemmas are presented, giving the context of the study. Next, we will provide a refinement of the Buzano inequality using these lemmas. The inequalities for the product of two operators in a semi-Hilbertian space are obtained in Section 3. Section 4 includes the bounds for the sum of two operators.

2. Basic Lemmas

This section introduces basic lemmas that can help provide primary results.

Lemma 2.1. ([10]) *Let $\mathfrak{A}$-positive operators* L *and* M *be in* $\mathrm{B}(\mathcal{H})$. *Then*

$$\left\|\frac{\mathrm{L}+\mathrm{M}}{2}\right\|_{\mathfrak{A}}^n \leq \left\|\frac{\mathrm{L}^n+\mathrm{M}^n}{2}\right\|_{\mathfrak{A}}, \quad \forall n \in \mathbb{N}^*.$$

Lemma 2.2. *([2]) Let* M *be an operator in* $\mathrm{B}(\mathcal{H})$ *such that* $\mathrm{M} \geq_{\mathfrak{A}} 0$. *Then, for any* $n \in \mathbb{N}^*$, *the inequality*

$$\langle \mathrm{M}\eta, \eta\rangle_{\mathfrak{A}}^n \leq \langle \mathrm{M}^n\eta, \eta\rangle_{\mathfrak{A}},$$

holds.

Lemma 2.3. *([31]) Let* $w_1, w_2 \in \mathcal{H}$. *Then*

$$|\langle w_1, w_2\rangle_{\mathfrak{A}}|^2 \leq \|w_1\|_{\mathfrak{A}}^2 \|w_2\|_{\mathfrak{A}}^2.$$

Lemma 2.4. *([35]) Let* $w_1, w_2, a \in \mathcal{H}$ *with* $\|a\|_{\mathfrak{A}} = 1$. *Then*

$$|\langle w_1, a\rangle_{\mathfrak{A}}\langle a, w_2\rangle_{\mathfrak{A}}| \leq \frac{1}{2}\left(\|w_1\|_{\mathfrak{A}}\|w_2\|_{\mathfrak{A}} + |\langle w_1, w_2\rangle_{\mathfrak{A}}|\right).$$

The next lemma is the polarization identity for real Hilbert spaces.

Lemma 2.5. *([34]) Let* $w_1, w_2 \in \mathcal{H}$, *where* $\mathcal{H}$ *is a real Hilbert space. Then*

$$\langle w_1, w_2\rangle_{\mathfrak{A}} = \frac{1}{4}\left(\|w_1 + w_2\|_{\mathfrak{A}}^2 - \|w_1 - w_2\|_{\mathfrak{A}}^2\right).$$

3. Bounds for the product of operators

In this section, we provide proofs for a few lemmas on the $\mathfrak{A}$-operator seminorm, inspired by the work of Najla Altwaijry et al. in [2]. In our first lemma, we employ classical Young's inequality [1] as a powerful tool to obtain the following new inequality.

Lemma 3.1. *Let* $w_1, w_2 \in \mathcal{H}$ *and* $\tau \in [0, 1]$. *Then*

$$\begin{aligned} |\langle w_1, w_2\rangle_{\mathfrak{A}}|^2 &\leq \frac{\tau}{1+\tau}\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2 + \frac{1}{1+\tau}\left(|\langle w_1, w_2\rangle_{\mathfrak{A}}|^{2\tau}\|w_1\|_{\mathfrak{A}}^{2-2\tau}\|w_2\|_{\mathfrak{A}}^{2-2\tau}\right) \\ &\leq \|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2. \end{aligned} \tag{3.1}$$

Proof. If $\tau = 0$ and $\tau = 1$ in (3.1), the result is satisfied. However, for $\tau \in (0, 1)$ the statement is

$$\frac{\tau}{1+\tau}\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2 + \frac{1}{1+\tau}\left(|\langle w_1, w_2\rangle_{\mathfrak{A}}|^{2\tau}\|w_1\|_{\mathfrak{A}}^{2-2\tau}\|w_2\|_{\mathfrak{A}}^{2-2\tau}\right) \geq 0,$$

which means that,

$$\begin{aligned} |\langle w_1, w_2\rangle_{\mathfrak{A}}|^{2\tau}|\langle w_1, w_2\rangle_{\mathfrak{A}}|^{2-2\tau} &\leq \frac{\tau}{1+\tau}\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2 \\ &\quad + \frac{1}{1+\tau}\left(|\langle w_1, w_2\rangle_{\mathfrak{A}}|^{2\tau}\|w_1\|_{\mathfrak{A}}^{2-2\tau}\|w_2\|_{\mathfrak{A}}^{2-2\tau}\right). \end{aligned}$$

This yields for all $w_1, w_2 \in \mathcal{H}$, and $\tau \in (0, 1)$,

$$|\langle w_1, w_2\rangle_{\mathfrak{A}}|^2 \leq \frac{\tau}{1+\tau}\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2 + \frac{1}{1+\tau}\left(|\langle w_1, w_2\rangle_{\mathfrak{A}}|^{2\tau}\|w_1\|_{\mathfrak{A}}^{2-2\tau}\|w_2\|_{\mathfrak{A}}^{2-2\tau}\right).$$

Using Young's inequality,

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b, \tag{3.2}$$

we obtain the following inequality,

$$|\langle w_1, w_2\rangle_{\mathfrak{A}}|^{2\tau}\|w_1\|_{\mathfrak{A}}^{2-2\tau}\|w_2\|_{\mathfrak{A}}^{2-2\tau} \leq \tau|\langle w_1, w_2\rangle_{\mathfrak{A}}|^2 + (1-\tau)\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2.$$

Therefore, the second inequality is obtained as,

$$\begin{aligned}
&\frac{\tau}{1+\tau}\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2+\frac{1}{1+\tau}\left(|\langle w_1,w_2\rangle_{\mathfrak{A}}|^{2\tau}\,\|w_1\|_{\mathfrak{A}}^{2-2\tau}\|w_2\|_{\mathfrak{A}}^{2-2\tau}\right)\\
&\le \frac{\tau}{1+\tau}\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2+\frac{1}{1+\tau}\left(\tau\,|\langle w_1,w_2\rangle_{\mathfrak{A}}|^{2}+(1-\tau)\|w_1\|_{\mathfrak{A}}^{2}\|w_2\|_{\mathfrak{A}}^{2}\right)\\
&\le \|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2.
\end{aligned}$$

□

Lemma 3.2. *Let $w_1, w_2 \in \mathcal{H}$ be such that $\|a\|_{\mathfrak{A}}=1$ and $\tau\in[0,1]$. Then*

$$\begin{aligned}
&|\langle w_1,a\rangle_{\mathfrak{A}}\langle a,w_2\rangle_{\mathfrak{A}}|^2\\
&\le \frac{2\tau+1}{2\tau+2}\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2+\frac{1}{2\tau+2}\left(|\langle w_1,w_2\rangle_{\mathfrak{A}}|^{2\tau}\,\|w_1\|_{\mathfrak{A}}^{2-2\tau}\|w_2\|_{\mathfrak{A}}^{2-2\tau}\right).
\end{aligned}$$

Proof. Lemma 2.4 and the convexity of the function $t\to t^2$ allow us to derive the following:

$$\begin{aligned}
|\langle w_1,a\rangle_{\mathfrak{A}}\langle a,w_2\rangle_{\mathfrak{A}}|^2 &\le \left(\frac{1}{2}\left(\|w_1\|_{\mathfrak{A}}\|w_2\|_{\mathfrak{A}}+|\langle w_1,w_2\rangle_{\mathfrak{A}}|\right)\right)^2\\
&\le \frac{1}{4}\left(\|w_1\|_{\mathfrak{A}}\|w_2\|_{\mathfrak{A}}+|\langle w_1,w_2\rangle_{\mathfrak{A}}|\right)^2\\
&\le \frac{1}{2}\left(\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2+|\langle w_1,w_2\rangle_{\mathfrak{A}}|^2\right).
\end{aligned}$$

Now, applying Lemma 3.1, we obtain

$$\begin{aligned}
|\langle w_1,a\rangle_{\mathfrak{A}}\langle a,w_2\rangle_{\mathfrak{A}}|^2 \le{}& \frac{1}{2}\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2+\frac{\tau}{2\tau+2}\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2\\
&+\frac{1}{2\tau+2}\left(|\langle w_1,w_2\rangle_{\mathfrak{A}}|^{2\tau}\,\|w_1\|_{\mathfrak{A}}^{2-2\tau}\|w_2\|_{\mathfrak{A}}^{2-2\tau}\right).
\end{aligned}$$

Therefore, we obtain the required result

$$\begin{aligned}
&|\langle w_1,a\rangle_{\mathfrak{A}}\langle a,w_2\rangle_{\mathfrak{A}}|^2\\
&\le \frac{2\tau+1}{2\tau+2}\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2+\frac{1}{2\tau+2}\left(|\langle w_1,w_2\rangle_{\mathfrak{A}}|^{2\tau}\,\|w_1\|_{\mathfrak{A}}^{2-2\tau}\|w_2\|_{\mathfrak{A}}^{2-2\tau}\right).
\end{aligned}$$

□

Remark 3.1. Let $w_1, w_2 \in \mathcal{H}$ be such that $\|a\|_{\mathfrak{A}}=1$. For $\tau=0,1$, Lemma 3.2 provides an improvement of the Cauchy-Schwarz inequality. Similarly, for $\tau=\frac{1}{2}$, we obtain that

$$\begin{aligned}
|\langle w_1,a\rangle_{\mathfrak{A}}\langle a,w_2\rangle_{\mathfrak{A}}|^2 &\le \frac{2}{3}\|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2+\frac{1}{3}\left(|\langle w_1,w_2\rangle_{\mathfrak{A}}|\,\|w_1\|_{\mathfrak{A}}\|w_2\|_{\mathfrak{A}}\right)\\
&\le \|w_1\|_{\mathfrak{A}}^2\|w_2\|_{\mathfrak{A}}^2.
\end{aligned}$$

Now we are ready to prove the first theorem of this research.

Theorem 3.1. *Let* $\mathrm{L},\mathrm{M}\in\mathcal{B}_{\mathfrak{A}}(\mathcal{H})$, $r\in N^*$, *and* $\tau\in[0,1]$. *Then*

$$\begin{aligned}
\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*_{\mathfrak{A}}}\mathrm{L}) \le{}& \frac{1}{2(1+\tau)}\,\mathrm{ber}_{\mathfrak{A}}^{2r\tau}(\mathrm{M}^{*_{\mathfrak{A}}}\mathrm{L})\left\|(\mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L})^{2r}+(\mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}^{1-\tau}\\
&+\frac{\tau}{2(1+\tau)}\left\|(\mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L})^{2r}+(\mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}. \qquad (3.3)
\end{aligned}$$

Proof. Let $\widehat{\varphi}_\eta$ be the normalized reproducing kernel of $\mathcal{H}$. Replacing w_1 with $\mathrm{L}\widehat{\varphi}_\eta$ and w_2 with $\mathrm{M}\widehat{\varphi}_\eta$ in Lemma 3.1, we obtain

$$\begin{aligned}
&|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r}\\
&= |\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r}\\
&\le \left(\frac{\tau}{1+\tau}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^2\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^2 + \frac{1}{1+\tau}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2\tau}\,\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2-2\tau}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2-2\tau}\right)\right)^{2r}.
\end{aligned}$$

Now, using the convexity of the function $h(t) \to t^r$, we have

$$\begin{aligned}
&|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r}\\
&\le \frac{\tau}{1+\tau}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r} + \frac{1}{1+\tau}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r\tau}\,\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{(2-2\tau)r}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{(2-2\tau)r}\right)\\
&\le \frac{\tau}{1+\tau}\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\\
&+ \frac{1}{1+\tau}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r\tau}\left(\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\right)^{1-\tau}\right)
\end{aligned}$$

Applying the Arithmetic-Geometric mean inequality, we conclude that

$$\begin{aligned}
&|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r}\\
&\le \frac{\tau}{2(1+\tau)}\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r} + \langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r}\\
&+ \frac{1}{2(1+\tau)}\left(|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r\tau}\left(\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r} + \langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r}\right)^{1-\tau}\right).
\end{aligned}$$

Applying Lemma 2.2 in the previous inequality yields

$$\begin{aligned}
|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r} &\le \frac{\tau}{2(1+\tau)}\left(\langle (\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right)\\
&+ \frac{1}{2(1+\tau)}\left(|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r\tau}\left(\langle (\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right)^{1-\tau}\right).
\end{aligned}$$

Finally, we take the supremum over all $\eta \in \Omega$ to complete the proof of our desired result. □

The corollaries described below are the direct outcomes of Theorem 3.1.

Corollary 3.1. *Let* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$, $r \in \mathbb{N}^*$. *Then*

$$\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \le \frac{1}{2}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}.$$

Corollary 3.2. *Let* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$, $r \in \mathbb{N}^*$. *Then*

$$\begin{aligned}
\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) &\le \frac{1}{4}\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) + \frac{1}{4}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}\\
&\le \frac{3}{8}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}.
\end{aligned}$$

Remark 3.2. If we put $\tau = 0$ and $\tau = 1$, then the bound in Theorem 3.1 yields Corollaries 3.1 and 3.2, respectively. Corollary 3.2 is an improvement of Corollary 3.1. Moreover, if we take $\mathfrak{A} = I$ in Corollary 3.1, then it gives inequality (1.3).

Our next theorem is a refinement of Corollary 3.1 and inequality (1.3).

Theorem 3.2. *Let* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$, $r \in N^*$, *and* $\tau \in [0,1]$. *Then*

$$\begin{aligned}\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*_{\mathfrak{A}}}\mathrm{L}) \leq{} & \frac{\tau^2}{1+\tau}\,\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*_{\mathfrak{A}}}\mathrm{L}) + \frac{\tau}{2(1+\tau)}\left\|(\mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L})^{2r} + (\mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\ & + \frac{\tau(1-\tau)}{(1+\tau)}\,\mathrm{ber}_{\mathfrak{A}}^{r}(\mathrm{M}^{*_{\mathfrak{A}}}\mathrm{L})\left\|(\mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L})^{r} + (\mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M})^{r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\ & + \frac{(1-\tau)^2}{(1+\tau)}\left\|(\mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L})^{r}\right\|_{\mathfrak{A}-\mathrm{Ber}}\left\|(\mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M})^{r}\right\|_{\mathfrak{A}-\mathrm{Ber}}. \qquad (3.4)\end{aligned}$$

Proof. Let $\widehat{\varphi}_\eta$ be the normalized reproducing kernel of $\mathcal{H}$. Replacing w_1 with $\mathrm{L}\widehat{\varphi}_\eta$ and w_2 with $\mathrm{M}\widehat{\varphi}_\eta$ in Lemma 3.1, we get

$$\begin{aligned}& \left|\langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{2r} \\ & = \left|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{2r} \\ & \leq \left(\frac{\tau}{1+\tau}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^2\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^2 + \frac{1}{1+\tau}\left(\left|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{2\tau}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2-2\tau}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2-2\tau}\right)\right)^{2r}.\end{aligned}$$

Now, using convexity of the function $h(t) \to t^r$, we obtain

$$\begin{aligned}& \left|\langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{2r} \\ & \leq \frac{\tau}{1+\tau}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r} + \frac{1}{1+\tau}\left(\left|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{2r\tau}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{(2-2\tau)r}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{(2-2\tau)r}\right) \\ & \leq \frac{\tau}{1+\tau}\langle \mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^r\langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^r \\ & + \frac{1}{1+\tau}\left(\left|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{2r\tau}\langle \mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r(1-\tau)}\langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r(1-\tau)}\right) \\ & \leq \frac{\tau}{1+\tau}\langle \mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^r\langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^r \\ & + \frac{1}{1+\tau}\left(\left|\langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{r\tau}\langle \mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r(1-\tau)}\right) \\ & \left(\left|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{r\tau}\langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r(1-\tau)}\right).\end{aligned}$$

Applying the Arithmetic-Geometric mean and Young's inequality (3.2), we conclude that

$$\begin{aligned}& \left|\langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{2r} \\ & \leq \frac{\tau}{2(1+\tau)}\left(\langle \mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r} + \langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r}\right) \\ & + \frac{1}{1+\tau}\left((\tau\left|\langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{r} + (1-\tau)\langle \mathrm{L}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\right) \\ & (\tau\left|\langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{r} + (1-\tau)\langle \mathrm{M}^{*_{\mathfrak{A}}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r})).\end{aligned}$$

Applying Lemma 2.2 in the previous inequality yields

$$
\begin{aligned}
&|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r} \\
&\leq \frac{\tau}{2(1+\tau)}\left(\langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right) + \frac{\tau^2}{1+\tau}|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r} \\
&+ \frac{\tau(1-\tau)}{1+\tau}|\langle(\mathrm{M}^{*\mathfrak{A}}\mathrm{L})^r\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|\left(\langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^r + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^r\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right) \\
&+ \frac{(1-\tau)^2}{1+\tau}\langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^r\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\langle(\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^r\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}.
\end{aligned}
$$

Finally, we take the supremum over all $\eta \in \Omega$ to get our desired result. □

Remark 3.3. If we put $\tau = 1$, then Theorem 3.2 is an improvement of Corollary 3.1 as,

$$
\begin{aligned}
&\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \leq \frac{1}{2}\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) + \frac{1}{4}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\
&\leq \frac{1}{4}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} + \frac{1}{4}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\
&\leq \frac{1}{2}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}.
\end{aligned}
$$

In fact, for $\mathfrak{A} = I$ the bound of Theorem 3.2 is an improvement of inequality (1.3).

We use Lemma 3.1 to prove our next theorem.

Theorem 3.3. *Let* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$, $r \in N^*$, *and* $\tau \in [0,1]$. *Then*

$$
\begin{aligned}
\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \leq \frac{\tau^2}{1+\tau}\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) &+ \frac{1+\tau^2-\tau}{2(1+\tau)}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\
&+ \frac{\tau(1-\tau)}{(1+\tau)}\mathrm{ber}_{\mathfrak{A}}^{r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L})\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{r}\right\|_{\mathfrak{A}-\mathrm{Ber}}.
\end{aligned}
\tag{3.5}
$$

Proof. Taking $w_1 = \mathrm{L}\widehat{\varphi}_\eta$ and $w_2 = \mathrm{M}\widehat{\varphi}_\eta$ in Lemma 3.1, we get

$$
\begin{aligned}
&|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r} \\
&= |\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r} \\
&\leq \left(\frac{\tau}{1+\tau}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^2\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^2 + \frac{1}{1+\tau}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2\tau}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2-2\tau}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2-2\tau}\right)\right)^{2r}.
\end{aligned}
$$

Now, using convexity of the function $h(t) \to t^r$, we obtain

$$
\begin{aligned}
&\left|\langle \mathrm{M}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^{2r} \\
&\leq \frac{\tau}{1+\tau} \|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r} \|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r} + \frac{1}{1+\tau} \left(\left|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^{2r\tau} \|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{(2-2\tau)r} \|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{(2-2\tau)r} \right) \\
&\leq \frac{\tau}{1+\tau} \langle \mathrm{L}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r \langle \mathrm{M}^{*\mathfrak{A}} \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r \\
&+ \frac{1}{1+\tau} \left(\left|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^{2r\tau} \langle \mathrm{L}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^{r(1-\tau)} \langle \mathrm{M}^{*\mathfrak{A}} \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^{r(1-\tau)} \right) \\
&\leq \frac{\tau}{1+\tau} \langle \mathrm{L}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r \langle \mathrm{M}^{*\mathfrak{A}} \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r \\
&+ \frac{1}{1+\tau} \left(\left|\langle \mathrm{M}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^{r\tau} \langle \mathrm{L}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^{r(1-\tau)} \right) \\
&\left(\left|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^{r\tau} \langle \mathrm{M}^{*\mathfrak{A}} \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^{r(1-\tau)} \right).
\end{aligned}
$$

Applying Young's inequality (3.2), we obtain

$$
\begin{aligned}
&\left|\langle \mathrm{M}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^{2r} \\
&\leq \frac{\tau}{(1+\tau)} \left(\langle \mathrm{L}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r \langle \mathrm{M}^{*\mathfrak{A}} \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r \right) \\
&+ \frac{1}{1+\tau} \left(\tau \left|\langle \mathrm{M}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^r + (1-\tau) \langle \mathrm{L}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r \right) \\
&\left(\tau \left|\langle \mathrm{M}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^r + (1-\tau) \langle \mathrm{M}^{*\mathfrak{A}} \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r \right)
\end{aligned}
$$

Now, we combine the terms to get

$$
\begin{aligned}
&\left|\langle \mathrm{M}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^{2r} \\
&\leq \frac{1+\tau^2-\tau}{(1+\tau)} \left(\langle \mathrm{L}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r \langle \mathrm{M}^{*\mathfrak{A}} \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r \right) + \frac{\tau^2}{1+\tau} \left|\langle \mathrm{M}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^{2r} \\
&+ \frac{\tau(1-\tau)}{1+\tau} \left|\langle \mathrm{M}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^r \left(\langle \mathrm{L}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r + \langle \mathrm{M}^{*\mathfrak{A}} \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}^r \right).
\end{aligned}
$$

Applying the Arithmetic-Geometric mean inequality and Lemma 2.2, we infer that

$$
\begin{aligned}
&\left|\langle \mathrm{M}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^{2r} \\
&\leq \frac{1+\tau^2-\tau}{(2(1+\tau)} \left(\langle (\mathrm{L}^{*\mathfrak{A}} \mathrm{L})^{2r} \widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}} + \langle (\mathrm{M}^{*\mathfrak{A}} \mathrm{M})^{2r} \widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}} \right) \\
&+ \frac{\tau^2}{1+\tau} \left|\langle \mathrm{M}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^{2r} \\
&+ \frac{\tau(1-\tau)}{1+\tau} \left|\langle \mathrm{M}^{*\mathfrak{A}} \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}}\right|^r \left(\langle (\mathrm{L}^{*\mathfrak{A}} \mathrm{L})^r + (\mathrm{M}^{*\mathfrak{A}} \mathrm{M})^r \widehat{\varphi}_\eta, \widehat{\varphi}_\eta \rangle_{\mathfrak{A}} \right).
\end{aligned}
$$

Finally, we take the supremum over all $\eta \in \Omega$ to get our desired result. $\square$

Remark 3.4. If we put $\tau = 1$, then Theorem 3.3 is an improvement of the Corollary 3.1 as,

$$\begin{aligned}\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) &\leq \frac{1}{2}\,\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) + \frac{1}{4}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2}\right\|_{\mathfrak{A}-\mathrm{Ber}}\\ &\leq \frac{1}{4}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} + \frac{1}{4}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}\\ &\leq \frac{1}{2}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}.\end{aligned}$$

In fact, for $\mathfrak{A} = I$ the bound of Theorem 3.3 is an improvement of inequality (1.3).

Next, we establish the following theorem using Lemma 3.2.

Theorem 3.4. *Let* $\mathrm{L},\mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$*,* $r \in N^*$*, and* $\tau \in [0,1]$*. Then*

$$\begin{aligned}\mathrm{ber}_{\mathfrak{A}}^{4r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \leq \frac{2\tau+1}{4\tau+4}\,&\mathrm{ber}_{\mathfrak{A}}^{2r\tau}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L})\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}^{1-\tau}\\ &+ \frac{1}{4\tau+4}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}.\end{aligned}\tag{3.6}$$

Proof. Let $\widehat{\varphi}_\eta$ be the normalized reproducing kernel of $\mathcal{H}$. Replacing w_1 with $\mathrm{L}\widehat{\varphi}_\eta$, w_2 with $\mathrm{M}\widehat{\varphi}_\eta$, a with $\widehat{\varphi}_\eta$ in Lemma 3.2 and using convexity of the function $h(t) \to t^r$, we get

$$\begin{aligned}&|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{4r}\\ &\leq \frac{2\tau+1}{2\tau+2}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r}\\ &+ \frac{1}{2\tau+2}\left(\left|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{2r\tau}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{r(2-2\tau)}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{r(2-2\tau)}\right)\\ &\leq \frac{2\tau+1}{2\tau+2}\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\\ &+ \frac{1}{2+2\tau}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r\tau}\left(\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\right)^{1-\tau}\right)\\ &\leq \frac{2\tau+1}{4\tau+4}\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r} + \langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r}\\ &+ \frac{1}{4\tau+4}\left(|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r\tau}\left(\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r} + \langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r}\right)^{1-\tau}\right).\end{aligned}$$

Using Lemma 2.2, we obtain

$$\begin{aligned}&|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{4r}\\ &\leq \frac{2\tau+1}{4\tau+4}\left(\langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right)\\ &+ \frac{1}{4\tau+4}\left(|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r\tau}\left(\langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right)^{1-\tau}\right).\end{aligned}$$

Finally, taking the supremum over all $\eta \in \Omega$ we obtain the desired result. ☐

The following corollary is an immediate consequence of Theorem 3.4.

Corollary 3.3. *Let* $\mathrm{L},\mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$*,* $r \in N^*$*. Then*

$$\mathrm{ber}_{\mathfrak{A}}^{4r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \leq \frac{1}{2}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}.$$

Remark 3.5. If we take $\tau = 0$, and $\tau = 1$, then the bound in Theorem 3.4 gives Corollary 3.3. Moreover, if we put $\mathfrak{A} = I$, $r = 1$ and $\mathrm{M} = \mathrm{L}$ in Corollary 3.3, then it gives inequality (1.1).

Next, we prove the following theorem, which is a refinement of Corollary 3.3.

Theorem 3.5. *Let* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$*,* $r \in N^*$*, and* $\tau \in [0,1]$*. Then*

$$\begin{aligned}\mathrm{ber}_{\mathfrak{A}}^{4r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \leq{}& \frac{\tau^2}{2\tau+2}\,\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) + \frac{2\tau+1}{(4\tau+4)}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\ &+ \frac{\tau(1-\tau)}{(2\tau+2)}\,\mathrm{ber}_{\mathfrak{A}}^{r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L})\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\ &+ \frac{(1-\tau)^2}{(2\tau+2)}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{r}\right\|_{\mathfrak{A}-\mathrm{Ber}}\left\|(\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{r}\right\|_{\mathfrak{A}-\mathrm{Ber}}. \qquad (3.7)\end{aligned}$$

Proof. Replacing w_1 with $\mathrm{L}\widehat{\varphi}_\eta$, w_2 with $\mathrm{M}\widehat{\varphi}_\eta$, a with $\widehat{\varphi}_\eta$ in Lemma 3.2 and using convexity of the function $h(t) \to t^r$, we get,

$$\begin{aligned}&|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{4r} \\ &\leq \frac{2\tau+1}{2\tau+2}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r} \\ &+ \frac{1}{2\tau+2}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r\tau}\,\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{r(2-2\tau)}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{r(2-2\tau)}\right) \\ &\leq \frac{2\tau+1}{2\tau+2}\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r} \\ &+ \frac{1}{2\tau+2}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r\tau}\,\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r(1-\tau)}\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r(1-\tau)}\right) \\ &\leq \frac{2\tau+1}{2\tau+2}\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r} \\ &+ \frac{1}{2\tau+2}\left(|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{r\tau}\,\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r(1-\tau)}\right) \\ &\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{r\tau}\,\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r(1-\tau)}\right).\end{aligned}$$

Applying the Arithmetic-Geometric mean inequality, Young's inequality (3.2), and then Lemma 2.2, we conclude that

$$\begin{aligned}&|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{4r} \\ &\leq \frac{2\tau+1}{(4\tau+4)}\left(\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r} + \langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{2r}\right) \\ &+ \frac{1}{2\tau+2}(\tau\,|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{r} + (1-\tau)\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}) \\ &(\tau\,|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{r} + (1-\tau)\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}) \\ &\leq \frac{2\tau+2}{(4\tau+4)}\left(\langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right) + \frac{\tau^2}{2\tau+2}\,|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r} \\ &+ \frac{\tau(1-\tau)}{2\tau+2}\,|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{r}\left(\langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{r}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right) \\ &+ \frac{(1-\tau)^2}{2\tau+2}\langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{r}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\langle(\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{r}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}.\end{aligned}$$

Finally, we take the supremum over all $\eta \in \Omega$ to get our desired result. $\square$

Remark 3.6. Let $\tau = 1$, then the bound in Theorem 3.5 is an improvement of Corollary 3.3,

$$\begin{aligned}
&\mathrm{ber}_{\mathfrak{A}}^{4r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \leq \frac{1}{4}\,\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) + \frac{3}{8}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\
&\leq \frac{1}{8}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} + \frac{3}{8}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\
&\leq \frac{1}{2}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}.
\end{aligned}$$

Remarks 3.6 shows that Theorem 3.5 is an improvement of inequality (1.1) if we put $\mathfrak{A} = I$ and $\mathrm{M} = \mathrm{L}$.

Next, we prove the following Theorem by using Lemma 3.2 and Young's inequality (3.2).

Theorem 3.6. *Let* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$ *and* $\tau \in [0,1]$. *Then*

$$\begin{aligned}
\mathrm{ber}_{\mathfrak{A}}^{4r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \leq{}& \frac{\tau^2}{2\tau+2}\,\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) + \frac{\tau^2+2}{(4\tau+4)}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\
&+ \frac{\tau(1-\tau)}{(2\tau+2)}\,\mathrm{ber}_{\mathfrak{A}}^{r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L})\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{r}\right\|_{\mathfrak{A}-\mathrm{Ber}}. \qquad (3.8)
\end{aligned}$$

Proof. Let $\widehat{\varphi}_\eta$ be the normalized reproducing kernel of $\mathcal{H}$. Replacing w_1 with $\mathrm{L}\widehat{\varphi}_\eta$, w_2 with $\mathrm{M}\widehat{\varphi}_\eta$, a with $\widehat{\varphi}_\eta$ in Lemma 3.2 and using convexity of the function $h(t) \to t^r$, we obtain

$$\begin{aligned}
&|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{4r} \\
&\leq \frac{2\tau+1}{2\tau+2}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{2r} + \frac{1}{2\tau+2}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r\tau}\,\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{r(2-2\tau)}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{r(2-2\tau)}\right) \\
&\leq \frac{2\tau+1}{2\tau+2}\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r} \\
&+ \frac{1}{2\tau+2}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2r\tau}\,\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r(1-\tau)}\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r(1-\tau)}\right).
\end{aligned}$$

Now, by applying Young's inequality (3.2), we obtain

$$\begin{aligned}
|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{4r} \leq{}& \frac{2\tau+1}{2\tau+2}\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r} \\
&+ \frac{1}{2\tau+2}(\tau\,|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{r} + (1-\tau)\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}) \\
&(\tau\,|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r}| + (1-\tau)\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^{r})
\end{aligned}$$

Now, we combine the terms and apply Arithmetic-Geometric mean inequality, and Lemma 2.2 to obtain the chain of inequalities

$$
\begin{aligned}
&|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{4r} \\
&\leq \frac{\tau^2+2}{2\tau+2}\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^r\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}^r + \frac{\tau^2}{2\tau+2}\left|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{2r} \\
&+ \frac{\tau(1-\tau)}{2\tau+2}\left|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^r\left(\langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^r + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^r\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right) \\
&\leq \frac{\tau^2+2}{4\tau+4}\langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}} + \frac{\tau^2}{2\tau+2}\left|\langle \mathrm{M}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|^{2r} \\
&+ \frac{\tau(1-\tau)}{2\tau+2}\left|\langle(\mathrm{M}^{*\mathfrak{A}}\mathrm{L})^r\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right|\left(\langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^r + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^r\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right).
\end{aligned}
$$

Finally, we take the supremum over all $\eta \in \Omega$ to get our desired result. $\square$

Remark 3.7. Let $\tau = 1$, then the bound in Theorem 3.6 gives Corollary 3.3,

$$
\begin{aligned}
&\mathrm{ber}_{\mathfrak{A}}^{4r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \leq \frac{1}{4}\,\mathrm{ber}_{\mathfrak{A}}^{2r}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) + \frac{3}{8}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\
&\leq \frac{1}{8}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} + \frac{3}{8}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}} \\
&\leq \frac{1}{2}\left\|(\mathrm{L}^{*\mathfrak{A}}\mathrm{L})^{2r} + (\mathrm{M}^{*\mathfrak{A}}\mathrm{M})^{2r}\right\|_{\mathfrak{A}-\mathrm{Ber}}.
\end{aligned}
$$

Remarks 3.7 shows that Theorem 3.6 is an improvement of inequality (1.1) if we put $\mathfrak{A} = I$ and $\mathrm{M} = \mathrm{L}$.

Now, we present an inequality in the context of semi-Hilbertian space, which is an improvement of the triangle inequality.

Lemma 3.3. *Let $w_1, w_2 \in \mathcal{H}$ and $\tau \geq 0$. Then*

$$
\begin{aligned}
&\|w_1 + w_2\|_{\mathfrak{A}}^2 \\
&\leq \frac{2}{3(1+\tau)}\left(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}\right)\|w_1 + w_2\|_{\mathfrak{A}} + \frac{1+3\tau}{3(1+\tau)}\left(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}\right)^2 \\
&\leq \left(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}\right)^2.
\end{aligned}
$$

Proof. Using triangle inequality of norm, we get

$$
\begin{aligned}
\|w_1 + w_2\|_{\mathfrak{A}}^2 &= \frac{1}{3}\|w_1 + w_2\|_{\mathfrak{A}}^2 + \frac{2}{3}\|w_1 + w_2\|_{\mathfrak{A}}^2 \\
&\leq \frac{1}{3}\|w_1 + w_2\|_{\mathfrak{A}}(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}) + \frac{2}{3}\|w_1 + w_2\|_{\mathfrak{A}}^2 \\
&\leq \frac{1}{3}\|w_1 + w_2\|_{\mathfrak{A}}(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}) + \frac{2}{3}\|w_1 + w_2\|_{\mathfrak{A}}^2 \\
&+ \tau\left((\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}})^2 - \|w_1 + w_2\|_{\mathfrak{A}}^2\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|w_1 + w_2\|_{\mathfrak{A}}^2 \\
&\leq \frac{2}{3(1+\tau)}\left(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}\right)\|w_1 + w_2\|_{\mathfrak{A}} + \frac{1+3\tau}{3(1+\tau)}\left(\|w_1\|_{\mathfrak{A}} + \|w_2.\|_{\mathfrak{A}}\right)^2
\end{aligned}
$$

Again by utilizing triangle inequality of $\|w_1 + w_2\| \le \|w_1\| + \|w_2\|$, we get the second inequality

$$\begin{aligned}\|w_1 + w_2\|_{\mathfrak{A}}^2 &\le \frac{2}{3(1+\tau)} \left(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}\right) \|w_1 + w_2\|_{\mathfrak{A}} + \frac{1+3\tau}{3(1+\tau)} \left(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}\right)^2 \\ &\le \frac{2}{3(1+\tau)} \left(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}\right)^2 + \frac{1+3\tau}{3(1+\tau)} \left(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}\right)^2. \\ &\le (\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}})^2.\end{aligned}$$

□

To obtain the following lemma, we combine Lemma 2.5 and Lemma 3.3.

Lemma 3.4. *Let $\mathcal{H}$ be a real Hilbert space such that $w_1, w_2 \in \mathcal{H}$. Then for $\tau \ge 0$*

$$|\langle w_1, w_2\rangle_{\mathfrak{A}}| \le \frac{2}{12(1+\tau)} \left(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}\right) \|w_1 + w_2\|_{\mathfrak{A}} + \frac{1+3\tau}{12(1+\tau)} \left(\|w_1\|_{\mathfrak{A}} + \|w_2\|_{\mathfrak{A}}\right)^2.$$

Next, we prove the following Theorem by using Lemma 3.4 which is constructed with the help of a refinement of the triangle inequality.

Theorem 3.7. *Let $\mathcal{H}$ be real Hilbert space such that $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$. Then for $\tau \ge 0$*

$$\begin{aligned}\mathrm{ber}_{\mathfrak{A}}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \le &\frac{2}{12(1+\tau)} \left(\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}} + \|\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\right) \|(\mathrm{L}+\mathrm{M})\|_{\mathfrak{A}-\mathrm{Ber}} \\ &+ \frac{1+3\tau}{12(1+\tau)} \|\mathrm{L}^{*\mathfrak{A}}\mathrm{L} + \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}. \end{aligned} \tag{3.9}$$

Proof. Let $\eta \in \Omega$ be arbitrary. Replacing w_1 with $\mathrm{L}\widehat{\varphi}_\eta$ and w_2 with $\mathrm{M}\widehat{\varphi}_\eta$ in Lemma 3.4, we get

$$\begin{aligned}|\langle(\mathrm{M}^{*\mathfrak{A}}\mathrm{L})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}| \le &\frac{2}{12(1+\tau)} \left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + \|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right) \|\mathrm{L}\widehat{\varphi}_\eta \\ &+ \mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + \frac{1+3\tau}{12(1+\tau)} \left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + \|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right)^2 \\ \le &\frac{2}{12(1+\tau)} \left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + \|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right) \|(\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta\|_{\mathfrak{A}} \\ &+ \frac{1+3\tau}{12(1+\tau)} \|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^2 + \|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^2 \\ \le &\frac{2}{12(1+\tau)} \left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + \|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right) \|(\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta\|_{\mathfrak{A}} \\ &+ \frac{1+3\tau}{12(1+\tau)} (\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}} + \langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}) \\ \le &\frac{2}{12(1+\tau)} \left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + \|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right) \|(\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta\|_{\mathfrak{A}} \\ &+ \frac{1+3\tau}{12(1+\tau)} \langle(\mathrm{L}^{*\mathfrak{A}}\mathrm{L} + \mathrm{M}^{*\mathfrak{A}}\mathrm{M})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}.\end{aligned}$$

In the second inequality, we apply the convexity of the function $t \to t^2$, and in the third and forth inequalities, we apply the norm property. Finally, we take the supremum over all $\eta \in \Omega$ to get our desired result. □

The following corollaries are direct outcomes of Theorem 3.1.

Corollary 3.4. *Let $\mathcal{H}$ be a real Hilbert space such that* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$. *Then*

$$\begin{aligned}\mathrm{ber}_{\mathfrak{A}}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \leq &\frac{1}{6}\left(\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}} + \|\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\right)\|(\mathrm{L}+\mathrm{M})\|_{\mathfrak{A}-\mathrm{Ber}} \\ &+ \frac{1}{6}\|\mathrm{L}^{*\mathfrak{A}}\mathrm{L} + \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}.\end{aligned}$$

Corollary 3.5. *Let $\mathcal{H}$ be a real Hilbert space such that* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$. *Then*

$$\begin{aligned}\mathrm{ber}_{\mathfrak{A}}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L}) \leq &\frac{1}{12}\left(\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}} + \|\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\right)\|(\mathrm{L}+\mathrm{M})\|_{\mathfrak{A}-\mathrm{Ber}} \\ &+ \frac{1}{12}\|\mathrm{L}^{*\mathfrak{A}}\mathrm{L} + \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}.\end{aligned}$$

4. Bounds for the sum of operators

In this section, bounds concerning the sum of two operators are developed. The results in this section provide some refinements of inequality (1.4).

Now, we provide our first theorem for the sum of two operators.

Theorem 4.1. *Let* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$ *and* $\tau \in [0, 1]$. *Then*

$$\begin{aligned}\mathrm{ber}^2_{\mathfrak{A}}(\mathrm{L}+\mathrm{M}) \leq \mathrm{ber}^2_{\mathfrak{A}}(\mathrm{L}) + \mathrm{ber}^2_{\mathfrak{A}}(\mathrm{M}) + &\sqrt{\frac{1+2\tau}{4(1+\tau)}}\|\mathrm{L}^{*\mathfrak{A}}\mathrm{L} + \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}} \\ + &\sqrt{\frac{1}{4(1+\tau)}}\,\mathrm{ber}^{\tau}_{\mathfrak{A}}(\mathrm{M}^{*\mathfrak{A}}\mathrm{L})\|\mathrm{L}^{*\mathfrak{A}}\mathrm{L} + \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\|^{1-\tau}_{\mathfrak{A}-\mathrm{Ber}}. \qquad (4.1)\end{aligned}$$

Proof. Let $\widehat{\varphi}_\eta$ be the normalized reproducing kernel of $\mathcal{H}$. Using Lemma 3.1, we get

$$\begin{aligned}&|\langle(\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 = \left(|\langle\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}| + |\langle\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|\right)^2 \\ &= |\langle\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + 2|\langle\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}||\langle\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}| \\ &= |\langle\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + 2|\langle\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\langle\widehat{\varphi}_\eta, \mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}| \\ &\leq |\langle\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + \|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + |\langle\mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|\end{aligned}$$

Using Lemma 3.1 in the previous inequality we obtain

$$\begin{aligned}&|\langle(\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 \\ &\leq |\langle\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + \|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + \|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|_{\mathfrak{A}} \\ &+ \Big(\tfrac{\tau}{1+\tau}\|\mathrm{L}\widehat{\varphi}_\eta\|^2_{\mathfrak{A}}\|\mathrm{M}\widehat{\varphi}_\eta\|^2_{\mathfrak{A}} + \\ &+ \tfrac{1}{1+\tau}\left(|\langle\mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2\tau}\,\|\mathrm{L}\widehat{\varphi}_\eta\|^{2-2\tau}_{\mathfrak{A}}\|\mathrm{M}\widehat{\varphi}_\eta\|^{2-2\tau}_{\mathfrak{A}}\right)\Big)^{1/2} \\ &\leq |\langle\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + \tfrac{1}{2}(\langle\mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}} + \langle\mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}) \\ &+ \sqrt{\tfrac{\tau}{1+\tau}}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}} \\ &+ \sqrt{\tfrac{1}{1+\tau}}\left(|\langle\mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{\tau}\,\|\mathrm{L}\widehat{\varphi}_\eta\|^{1-\tau}_{\mathfrak{A}}\|\mathrm{M}\widehat{\varphi}_\eta\|^{1-\tau}_{\mathfrak{A}}\right).\end{aligned}$$

Applying the Arithmetic-Geometric mean inequality in the previous inequality, we deduce that

$$\begin{aligned}
&|\langle (\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2\\
&\le |\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2\\
&+\sqrt{\frac{1+2\tau}{4(1+\tau)}}(\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}} + \langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}})\\
&+\sqrt{\frac{1}{4(1+\tau)}}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^\tau\left(\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}} + \langle \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right)^{1-\tau}\right)\\
&\le |\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + \sqrt{\frac{1+2\tau}{4(1+\tau)}}\langle (\mathrm{L}^{*\mathfrak{A}}\mathrm{L}+\mathrm{M}^{*\mathfrak{A}}\mathrm{M})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\\
&+\sqrt{\frac{1}{4(1+\tau)}}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^\tau\left(\langle (\mathrm{L}^{*\mathfrak{A}}\mathrm{L}+\mathrm{M}^{*\mathfrak{A}}\mathrm{M})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right)^{1-\tau}\right).
\end{aligned}$$

Finally, we take the supremum over all $\eta \in \Omega$ to get our desired result. □

Remark 4.1. Theorem 4.1 for $\tau = 1$ is a refinement of inequality (1.4). If we take $\mathfrak{A} = I$ in Remark 4.1, then it is a refinement of the inequality (1.2).

Now we derive the following theorem from Lemma 3.2.

Theorem 4.2. *Let* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$ *and* $\tau \in [0,1]$. *Then*

$$\begin{aligned}
\mathrm{ber}^2_{\mathfrak{A}}(\mathrm{L}+\mathrm{M}) \le \mathrm{ber}^2_{\mathfrak{A}}(\mathrm{L}) + \mathrm{ber}^2_{\mathfrak{A}}(\mathrm{M}) &+ \sqrt{\frac{2\tau+1}{2\tau+2}}\|\mathrm{L}^{*\mathfrak{A}}\mathrm{L}+\mathrm{M}\mathrm{M}^{*\mathfrak{A}}\|_{\mathfrak{A}-\mathrm{Ber}}\\
&+\sqrt{\frac{1}{2\tau+2}}\,\mathrm{ber}^\tau_{\mathfrak{A}}(\mathrm{ML})\,\|\mathrm{L}^{*\mathfrak{A}}\mathrm{L}+\mathrm{M}\mathrm{M}^{*\mathfrak{A}}\|^{1-\tau}_{\mathfrak{A}-\mathrm{Ber}}. \qquad (4.2)
\end{aligned}$$

Proof. Let $\widehat{\varphi}_\eta$ be the normalized reproducing kernel of $\mathcal{H}$. Using Lemma 3.2, we get

$$\begin{aligned}
&|\langle (\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2\\
&= (|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}| + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|)^2\\
&= |\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + 2|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}||\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|\\
&= |\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + 2|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\langle \widehat{\varphi}_\eta, \mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|\\
&\le |\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2\\
&+2\left(\frac{2\tau+1}{2\tau+2}\|\mathrm{L}\widehat{\varphi}_\eta\|^2_{\mathfrak{A}}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|^2_{\mathfrak{A}} +\right.\\
&\left.+\frac{1}{2\tau+2}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^{2\tau}\,\|\mathrm{L}\widehat{\varphi}_\eta\|^{2-2\tau}_{\mathfrak{A}}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|^{2-2\tau}_{\mathfrak{A}}\right)\right)^{1/2}.
\end{aligned}$$

Applying the Arithmetic-Geometric mean inequality in the previous inequality, we infer that

$$
\begin{aligned}
&|\langle (\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 \\
&\leq |\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 \\
&+ 2\sqrt{\frac{2\tau+1}{2\tau+2}}\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|_{\mathfrak{A}} \\
&+ 2\sqrt{\frac{1}{2\tau+2}}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^\tau \|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{1-\tau}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^{1-\tau}\right) \\
&\leq |\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 \\
&+ \sqrt{\frac{2\tau+1}{2\tau+2}}(\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}} + \langle \mathrm{M}\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}) \\
&+ \sqrt{\frac{1}{2\tau+2}}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^\tau \left(\langle \mathrm{L}^{*\mathfrak{A}}\mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}} + \langle \mathrm{M}\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right)^{1-\tau}\right) \\
&\leq |\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + \sqrt{\frac{2\tau+1}{2\tau+2}}\langle (\mathrm{L}^{*\mathfrak{A}}\mathrm{L} + \mathrm{M}\mathrm{M}^{*\mathfrak{A}})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}} \\
&+ \sqrt{\frac{1}{2\tau+2}}\left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^\tau \left(\langle (\mathrm{L}^{*\mathfrak{A}}\mathrm{L} + \mathrm{M}\mathrm{M}^{*\mathfrak{A}})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\right)^{1-\tau}\right).
\end{aligned}
$$

Finally, we take the supremum over all $\eta \in \Omega$ to get our desired result. □

Remark 4.2. Theorem 4.2 for $\tau = 1$ is a refinement of inequality (1.4). If we take $\mathfrak{A} = I$ in Remark 4.2, then it is a refinement of the inequality (1.2).

We prove the following theorem by using Lemma 3.4.

Theorem 4.3. *Let $\mathcal{H}$ be a real Hilbert space such that* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$. *Then for* $\tau \geq 0$

$$
\begin{aligned}
&\mathrm{ber}_{\mathfrak{A}}^2(\mathrm{L}+\mathrm{M}) \\
&\leq \mathrm{ber}_{\mathfrak{A}}^2(\mathrm{L}) + \mathrm{ber}_{\mathfrak{A}}^2(\mathrm{M}) + \frac{1+3\tau}{3(1+\tau)}\|\mathrm{L}^{*\mathfrak{A}}\mathrm{L} + \mathrm{M}^{*\mathfrak{A}}\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}} \\
&+ \|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}}\|\mathrm{M}^{*\mathfrak{A}}\|_{\mathfrak{A}-\mathrm{Ber}} + \frac{1}{3(1+\tau)}\left(\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}} + \|\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\right)\|(\mathrm{L}+\mathrm{M})\|_{\mathfrak{A}-\mathrm{Ber}}.
\end{aligned} \tag{4.3}
$$

Proof. Let $\eta \in \Omega$ be arbitrary. It is easy to check that

$$
\begin{aligned}
&|\langle (\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 \\
&= (|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}| + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|)^2 \\
&= |\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + 2|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}||\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}| \\
&= |\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + 2|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\langle \widehat{\varphi}_\eta, \mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}| \\
&\leq |\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + \|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + |\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|.
\end{aligned}
$$

Applying Lemma 3.4 and the Arithmetic-Geometric mean inequality, we obtain

$$
\begin{aligned}
&|\langle(\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2\\
&\le |\langle \mathrm{L}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2+|\langle \mathrm{M}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2+\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\\
&+\frac{4}{12(1+\tau)}\left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}+\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right)\|\mathrm{L}\widehat{\varphi}_\eta+\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\\
&+\frac{2(1+3\tau)}{12(1+\tau)}\left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}+\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right)^2\\
&\le |\langle \mathrm{L}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2+|\langle \mathrm{M}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2+\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\\
&+\frac{4}{12(1+\tau)}\left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}+\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right)\|(\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta\|_{\mathfrak{A}}\\
&+\frac{2(1+3\tau)}{12(1+\tau)}(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^2+\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}^2+2\|\mathrm{L}\widehat{\varphi}_\eta\|\|\mathrm{M}\widehat{\varphi}_\eta\|)\\
&\le |\langle \mathrm{L}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2+|\langle \mathrm{M}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2+\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\\
&+\frac{4}{12(1+\tau)}\left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}+\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right)\|(\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta\|_{\mathfrak{A}}\\
&+\frac{2(1+3\tau)}{12(1+\tau)}(\langle \mathrm{LL}^{*\mathfrak{A}}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}+\langle \mathrm{MM}^{*\mathfrak{A}}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\\
&+\langle \mathrm{LL}^{*\mathfrak{A}}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}+\langle \mathrm{MM}^{*\mathfrak{A}}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}})\\
&\le |\langle \mathrm{L}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2+|\langle \mathrm{M}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2\\
&+\frac{4}{12(1+\tau)}\left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}+\|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right)\|(\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta\|_{\mathfrak{A}}.\\
&+\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|_{\mathfrak{A}}+\frac{2(1+3\tau)}{12(1+\tau)}(2\langle \mathrm{LL}^{*\mathfrak{A}}+\mathrm{MM}^{*\mathfrak{A}}\widehat{\varphi}_\eta,\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}).
\end{aligned}
$$

Finally, we take the supremum over all $\eta\in\Omega$ to get our desired result. □

The following corollary can be derived from Theorem 4.3 by choosing $\tau=0$.

Corollary 4.1. *Let $\mathcal{H}$ be a real Hilbert space such that* $\mathrm{L},\mathrm{M}\in\mathcal{B}_{\mathfrak{A}}(\mathcal{H})$. *Then*

$$
\begin{aligned}
\mathrm{ber}_{\mathfrak{A}}^2(\mathrm{L}+\mathrm{M})&\le \mathrm{ber}_{\mathfrak{A}}^2(\mathrm{L})+\mathrm{ber}_{\mathfrak{A}}^2(\mathrm{M})+\frac{1}{3}\|\mathrm{L}^{*\mathfrak{A}}\mathrm{L}+\mathrm{M}^{*\mathfrak{A}}\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\\
&+\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}}\|\mathrm{M}^{*\mathfrak{A}}\|_{\mathfrak{A}-\mathrm{Ber}}+\frac{1}{3}\left(\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}}+\|\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\right)\|(\mathrm{L}+\mathrm{M})\|_{\mathfrak{A}-\mathrm{Ber}}.
\end{aligned}
$$

Next, we provide bounds of the $\mathfrak{A}$-Berezin norm by applying the supremum property in the following theorem.

Theorem 4.4. *Let $\mathcal{H}$ be a real Hilbert space such that* $\mathrm{L},\mathrm{M}\in\mathcal{B}_{\mathfrak{A}}(\mathcal{H})$. *Then for* $\tau\ge 0$

$$
\begin{aligned}
&\|(\mathrm{L}+\mathrm{M})\|_{\mathfrak{A}-\mathrm{Ber}}^2\\
&\le \mathrm{ber}_{\mathfrak{A}}^2(\mathrm{L})+\mathrm{ber}_{\mathfrak{A}}^2(\mathrm{L})+\frac{1}{3(1+\tau)}\left(\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}}+\|\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\right)^2 \qquad (4.4)\\
&+\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}}\|\mathrm{M}^{*\mathfrak{A}}\|_{\mathfrak{A}-\mathrm{Ber}}+\frac{1}{3(1+\tau)}\left(\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}}+\|\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\right)\|\mathrm{L}+\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}).
\end{aligned}
$$

Proof. Let $\eta \in \Omega$ be arbitrary. It is easy to check that

$$\begin{aligned}
&\|(\mathrm{L}+\mathrm{M})\|^2_{\mathfrak{A}-\mathrm{Ber}} = \mathrm{ber}^2_{\mathfrak{A}}(\mathrm{L}+\mathrm{M})\\
&= \sup_{\eta\in\Omega} \left(|\langle(\mathrm{L}+\mathrm{M})\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|\right)^2\\
&\le \sup_{\eta\in\Omega} \left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + 2|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}||\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|\right)\\
&\le \sup_{\eta\in\Omega} \left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + 2|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}\langle \widehat{\varphi}_\eta, \mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|\right)\\
&\le \sup_{\eta\in\Omega} \left(|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + \|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + |\langle \mathrm{L}\widehat{\varphi}_\eta, \mathrm{M}\widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|\right)\\
&\le \sup_{\eta\in\Omega}(|\langle \mathrm{L}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + |\langle \mathrm{M}\widehat{\varphi}_\eta, \widehat{\varphi}_\eta\rangle_{\mathfrak{A}}|^2 + \|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\|\mathrm{M}^{*\mathfrak{A}}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\\
&+ \frac{4}{12(1+\tau)}\left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + \|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right)\|\mathrm{L}\widehat{\varphi}_\eta + \mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\\
&+ \frac{2(1+3\tau)}{12(1+\tau)}\left(\|\mathrm{L}\widehat{\varphi}_\eta\|_{\mathfrak{A}} + \|\mathrm{M}\widehat{\varphi}_\eta\|_{\mathfrak{A}}\right)^2)\\
&\le \mathrm{ber}^2_{\mathfrak{A}}(\mathrm{L}) + \mathrm{ber}^2_{\mathfrak{A}}(\mathrm{L}) + \|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}}\|\mathrm{M}^{*\mathfrak{A}}\|_{\mathfrak{A}-\mathrm{Ber}}\\
&+ \frac{1}{3(1+\tau)}\left(\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}} + \|\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\right)\|\mathrm{L}+\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}})\\
&+ \frac{1}{3(1+\tau)}\left(\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}} + \|\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\right)^2.
\end{aligned}$$

□

The following corollary can be derived from Theorem 4.4 by choosing $\tau = 0$.

Corollary 4.2. *Let $\mathcal{H}$ be a real Hilbert space such that* $\mathrm{L}, \mathrm{M} \in \mathcal{B}_{\mathfrak{A}}(\mathcal{H})$. *Then*

$$\begin{aligned}
\|(\mathrm{L}+\mathrm{M})\|^2_{\mathfrak{A}-\mathrm{Ber}} &\le \mathrm{ber}^2_{\mathfrak{A}}(\mathrm{L}) + \mathrm{ber}^2_{\mathfrak{A}}(\mathrm{L}) + \frac{1}{3}\left(\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}} + \|\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\right)^2\\
&+ \|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}}\|\mathrm{M}^{*\mathfrak{A}}\|_{\mathfrak{A}-\mathrm{Ber}} + \frac{1}{3}\left(\|\mathrm{L}\|_{\mathfrak{A}-\mathrm{Ber}} + \|\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}\right)\|\mathrm{L}+\mathrm{M}\|_{\mathfrak{A}-\mathrm{Ber}}).
\end{aligned}$$

5. Conclusion

This study provides several refinements of $\mathfrak{A}$-Berezin number inequalities for semi-Hilbert space operators. The refinements of Cauchy-Schwarz and $\mathfrak{A}$-Buzano inequalities contribute to the development of our bounds. Examples are provided to illustrate the improvements achieved in our results. Several bounds obtained in this article refine inequalities (1.3) and (1.1). Moreover, our results offer refinements not only for the $\mathfrak{A}$-Berezin number but also for the Berezin number. In addition, an enhancement of the triangle inequality is developed within the framework of semi-Hilbert spaces, contributing to bounds for the sum of operators. We hope that these results will encourage further research in operator inequalities and related areas and serve as a useful foundation for future studies.

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