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VISCOELASTIC WAVE EQUATION WITH VARIABLE-EXPONENT AND LOGARITHMIC NONLINEARITIES: BLOW-UP OF SOLUTIONS

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Abstract. In this paper, we investigate a viscoelastic wave equation characterized by variable-exponent nonlinearity and a logarithmic source term, described by the following model:

$$u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(x,s)ds+au_t|u_t|^{m(\cdot)-2}=bu|u|^{p(\cdot)-2}\ln|u|,\ a,b>0,$$

where the exponents of nonlinearity $m(\cdot)$ and $p(\cdot)$ are given measurable functions that satisfy specific conditions. Using the energy method and various inequality techniques, a finite-time blow-up result is established for certain solutions with positive initial energy, assuming a decreasing positive function g.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $a,b>0$ are constants. We consider the following initial–boundary value problem posed in $\Omega\times(0,T)$:

$$\begin{cases} u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s+au_t|u_t|^{m(\cdot)-2}=bu|u|^{p(\cdot)-2}\ln|u|, \\ u(x,t)=0,\ \text{on } \partial\Omega\times(0,T) \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \text{in } \Omega, \end{cases} \tag{1.1}$$

where the kernel $g:\mathbb{R}^+\to\mathbb{R}^+$ is a C^1 nonincreasing function such that

$$\int_0^\infty g(s)ds=1-l<\frac{\frac{p_1}{2}-1}{(\frac{p_1}{2})-1+(\frac{1}{2p_1})},$$

and the exponents $m(\cdot)$ and $p(\cdot)$ are given measurable functions on Ω satisfying

$$\begin{aligned} 2\le\varphi_1\le\varphi(x)\le\varphi_2<\tfrac{2n}{n-2}, &\quad \text{for } n\ge 3,\\ 2\le\varphi_1\le\varphi(x)\le\varphi_2<+\infty, &\quad \text{for } n\le 2. \end{aligned} \tag{1.2}$$

with

$$\varphi_1:=\operatorname{ess\,inf}_{x\in\Omega}\varphi(x),\quad \varphi_2:=\operatorname{ess\,sup}_{x\in\Omega}\varphi(x),$$

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and the log-Hölder continuity condition:

$$|\varphi(x)-\varphi(y)| \le -\frac{A}{\log|x-y|}, \text{ for a.e. } x,y\in\Omega, \text{ with } |x-y|<\lambda<1,\ A>0. \tag{1.3}$$

In recent years, considerable attention has been devoted to the study of nonlinear mathematical models involving hyperbolic, parabolic, and elliptic equations with variable exponents of nonlinearity. Such models arise naturally in the description of various physical phenomena, including the flow of electro-rheological fluids, nonlinear viscoelasticity, filtration processes in porous media, and image processing. Further details and developments on these models can be found in [1, 4, 3, 2, 15, 7]. However, the literature concerning equations with variable exponents of nonlinearity remains relatively limited. Messaoudi et al. [17] established results concerning the well-posedness and finite-time blow-up of solutions for the nonlinear damped wave equations with variable exponents:

$$\begin{cases} u_{tt}-\Delta u + a u_t|u_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2}, & \text{in } \Omega\times(0,T)\\ u(x,t)=0, & \text{on } \partial\Omega\times(0,T)\\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), & \text{in } \Omega, \end{cases}$$

where $a,b\ge 0$ are constants and Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Using the Faedo–Galerkin method, the authors proved the existence of a unique weak solution under suitable conditions on the variable exponents m and p. Furthermore, they established the finite-time blow-up of solutions and provided a two-dimensional numerical example to illustrate this blow-up behavior.

Messaoudi and Talahmeh [16] investigated the blow-up behavior of solutions to a quasilinear wave equation with nonlinearities involving variable exponents:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{r(\cdot)-2}\nabla u\right) + a\,|u_t|^{m(\cdot)-2}\,u_t = b|u|^{p(\cdot)-2}u \quad \text{in} \quad \Omega\times(0,T).$$

They established blow-up results for solutions with negative initial energy, as well as for certain solutions with positive initial energy. For a nonincreasing positive function g, Park and Kang [18] established the blow-up of solutions with both positive and nonpositive initial energy for the following viscoelastic wave equation with variable exponents:

$$u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(x,s)ds + a u_t|u_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2}, \text{ in } \Omega\times(0,T)$$

where the exponents of nonlinearity $m(\cdot)$ and $p(\cdot)$ are given measurable functions, and $a,b>0$ are constants.

In the presence of the logarithmic nonlinear source, Kafini [14] considered a nonlinear wave equation with variable exponents:

$$u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(x,s)ds + u_t|u_t|^{m(\cdot)-2} = u|u|^{p(\cdot)-2}\ln|u|^k, \text{in } \Omega\times(0,T)$$

where $k>0$. Without imposing the Sobolev Logarithmic Inequality, Kafini established a global nonexistence result with negative initial data. The blow-up time is established with upper bound and lower bound. Additionally, under certain conditions on the initial data and for a specific class of relaxation functions, an infinite-time blow-up result was derived. Problems of this type, featuring a logarithmic source term, arise in various branches of physics, including nuclear physics, optics, geophysics, and quantum field theory. Further details on these problems can be found in [5, 13, 12, 6, 10].

Many researchers have increasingly focused on studying the blow-up phenomenon in problems involving variable exponents. In this context, Choucha et al.[9] studied a nonlinear viscoelastic Kirchhoff-type wave equation with distributed delay and variable-exponent nonlinearities. They proved the occurrence of blow-up under suitable assumptions on the initial data and model parameters. Moreover, in the absence of the source, they obtained a general energy decay estimate using Komornik's integral inequality. For further details on these studies, we refer the reader to the following works [8, 20, 21].

The aim of this work is to investigate the blow-up phenomenon and identify sufficient conditions on the functions m, p, g, and the initial data under which blow-up occurs.

2. Preliminaries

In this section, we present some preliminary information and fundamental results about Lebesgue and Sobolev spaces with variable exponents. Let $p : \Omega \subset \mathbb{R}^n \to [1, \infty]$ and $u : \Omega \to \mathbb{R}$ be measurable functions.

Definition 2.1. *We denote by $L^{p(\cdot)}(\Omega)$ all the real measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega |\lambda u(x)|^{p(x)}\, dx < \infty$, for some $\lambda > 0$.*

The variable-exponent Lebesque space $L^{p(\cdot)}(\Omega)$ equipped with the following Luxemburg-type norm

$$\|u\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

is a Banach space.

Definition 2.2. *The Banach variable-exponent Sobolev space $W^{1,p(.)}(\Omega)$ is defined as follows:*

$$W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

with respect to the norm

$$\|u\|_{W^{1,p(.)}(\Omega)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)} .$$

Moreover, we define $W_0^{1,p(.)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. As in the classical Sobolev spaces, the space $W^{-1,p'(\cdot)}(\Omega)$ is introduced as the dual of $W_0^{1,p(\cdot)}(\Omega)$, where the relation $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$ holds.

Lemma 2.1. *(Young's inequality [11]). Let $h, i, j \geq 1$ be measurable functions defined on Ω such that*

$$\frac{1}{h(y)} = \frac{1}{i(y)} + \frac{1}{j(y)}, \text{ for a.e. } y \in \Omega.$$

Then for all $a, b \geq 0$,

$$\frac{(ab)^{h(.)}}{h(.)} \leq \frac{(a)^{i(.)}}{i(.)} + \frac{(b)^{j(.)}}{j(.)}.$$

Lemma 2.2. *(Hölder's Inequality* [11]*). Let* $h, i, j \geq 1$ *be measurable functions defined on* Ω *such that*

$$\frac{1}{h(y)} = \frac{1}{i(y)} + \frac{1}{j(y)}, \text{ for a.e. } y \in \Omega.$$

If $f \in L^{i(\cdot)}(\Omega)$ *and* $g \in L^{j(\cdot)}(\Omega)$*, then* $fg \in L^{h(\cdot)}(\Omega)$ *and*

$$\|fg\|_h \leq 2\|f\|_i\|g\|_j.$$

Lemma 2.3. *(Poincaré's inequality* [11]*). Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain and* $p(\cdot)$ *satisfy* (1.3)*. Then*

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}, \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where the positive constant depends on Ω, p_1, *and* p_2*. Consequently, the space* $W_0^{1,p(\cdot)}(\Omega)$ *has an equivalent norm given by* $\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot)}$.

Lemma 2.4. *(*[11]*). If* $p : \Omega \to [1, \infty)$ *is a measurable function and*

$$\begin{array}{l} 2 \leq p_1 \leq p(x) \leq p_2 \leq \frac{2n}{n-2}, \quad n > 2, \\ 2 \leq p_1 \leq p(x) \leq p_2 < +\infty, \quad n \leq 2, \end{array}$$

then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ *is continuous and compact.*

Definition 2.3. *(Weak solution).* $u = u(x,t)$ *is called a weak solution of problem* (1.1) *on* $\Omega \times [0, T)$, *if* $u \in L^\infty\left(0, T; H_0^1(\Omega)\right)$ *with* $u_t \in L^2\left(0, T; L^2(\Omega)\right)$ *and satisfies the problem* (1.1) *in the weak sense, i.e.*

$$\begin{array}{c} (u_{tt}, \psi)_2 + (\nabla u, \nabla \psi)_2 + \int_0^t g(t-s)(\nabla u(s), \nabla \psi)_2 ds + a(u_t|u_t|^{m(\cdot)-2}, \psi)_2 \\ = b(u|u|^{p(\cdot)-2} \ln |u|, \psi)_2, \end{array}$$

for any $\psi \in H_0^1(\Omega)$, $t \in (0, T)$, *where* $u(0, x) = u_0(x) \in H_0^1(\Omega)$, $u_t(x, 0) = u_1(x) \in L^2(\Omega)$, *and* $(\cdot, \cdot)_2$ *means the inner product* $(\cdot, \cdot)_{L^2(\Omega)}$.

The following theorem presents the well-posedness result, which can be inferred from [14] :

Theorem 2.1. *Let* $m(\cdot)$ *and* $p(\cdot)$ *satisfy* (1.2) *and* (1.3)*. Moreover,* $p(\cdot)$ *satisfies*

$$\begin{array}{l} 2 < p_1 \leq p(x) \leq p_2 < 2\frac{n-1}{n-2}, \quad \text{for } n \geq 3, \\ 2 < p_1 \leq p(x) \leq p_2 < +\infty, \quad \text{for } n \leq 2. \end{array}$$

Then, for any given $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$*, there exists* $T > 0$ *and a unique solution* u *of the problem* (1.1) *on* $(0, T)$ *such that*

$$u \in C\left((0, T), H_0^1(\Omega)\right) \cap C^1\left((0, T), L^2(\Omega)\right) \cap C^2\left((0, T), H^{-1}(\Omega)\right). \tag{2.1}$$

Furthermore, the energy functional $E(t)$ *associated with the problem* (1.1)*, defined as*

$$\begin{aligned} E(t) = & \frac{1}{2}\|u_t(t)\|_2^2 + \frac{l}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u) \\ & + \int_\Omega \frac{b|u|^{p(x)}}{p^2(x)} \mathrm{d}x - \int_\Omega \frac{b|u|^{p(x)} \ln |u|}{p(x)} \mathrm{d}x, \end{aligned} \tag{2.2}$$

is absolutely continuous and satisfies

$$E'(t) = \frac{1}{2}\left(g' \circ \nabla u\right) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - a\int_{\Omega} |u_t|^{m(x)}\mathrm{d}x \leq 0, \quad a.e\ t \in [0,T). \quad (2.3)$$

where $(g \circ v)(t) = \int_0^t g(t-s)\|v(t)-v(s)\|_2^2 \mathrm{d}s$.

3. Blowup result

In this section, we study the blow-up phenomenon for certain solutions with positive initial energy. To formally state and prove our result, let $B = B_1 l^{1/2}$ be the optimal constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\sigma}(\Omega)$ with

$$p_2 < \sigma \leq \sigma_{max},\ \sigma_{max} := \begin{cases} \frac{2n}{n-2} & \text{if } n \geq 3 \\ +\infty & \text{if } n \leq 2 \end{cases},$$

and determine

$$\delta_1 = \left(\frac{ep_1\left(\sigma - p_2\right)}{b\sigma}\right)^{\frac{2}{\sigma-2}} (B_1)^{\frac{-2\sigma}{\sigma-2}}, E_1 = \left(\frac{1}{2} - \frac{1+\frac{1-l}{4\tau l}}{p_1}\right)\delta_1,$$

$$H(t) = E_1 - E(t), \quad (3.1)$$

where $0 < \tau < \frac{p_1}{2}$. We now present some useful lemmas, where a generic positive constant is denoted by C.

Lemma 3.1. *Let* $q : \Omega \to \mathbb{R}$ *be a measurable function satisfying* (1.2)*, and we set* $\|u\|_{q_1,\Omega_+}^{q_1} := \int_{\Omega_+} |u|^{q_1}\,\mathrm{d}x$, $\|u\|_{q_2,\Omega_+}^{q_2} := \int_{\Omega_+} |u|^{q_2}\,\mathrm{d}x$ *and* $\varrho_{q(x)}(u) := \int_{\Omega} |u|^{q(x)}\,\mathrm{d}x$, *then*

$$\|u\|_{q_1,\Omega_+}^{q_1} \leqslant \varrho_{q(x)}(u) \leqslant C\|u\|_{q_2,\Omega_+}^{q_2},$$

where $\Omega_- := \{x \in \Omega; |u| \leqslant 1\}$, $\Omega_+ := \{x \in \Omega; |u| > 1\}$ *and* C, $|\Omega_+| > 0$.

Proof. From the definition, we have

$$\varrho_{q(x)}(u) := \int_{\Omega} |u|^{q(x)}\,\mathrm{d}x = \int_{\Omega_-} |u|^{q(x)}\,\mathrm{d}x + \int_{\Omega_+} |u|^{q(x)}\,\mathrm{d}x,$$

hence

$$\begin{aligned} \|u\|_{q_1,\Omega_+}^{q_1} \leqslant \int_{\Omega_+} |u|^{q(x)}\,\mathrm{d}x &\leqslant \int_{\Omega_-} |u|^{q(x)}\,\mathrm{d}x + \int_{\Omega_+} |u|^{q(x)}\,\mathrm{d}x \\ &\leqslant C\int_{\Omega_+} |u|^{q(x)}\,\mathrm{d}x \leqslant C\|u\|_{q_2,\Omega_+}^{q_2}. \end{aligned}$$

□

Lemma 3.2. *(*[19]*). Let* u *be the solution of* (1.1)*. If*

$$E(0) < E_1 \text{ and } \|\nabla u_0\|_2^2 \geq \delta_1 > 0,$$

then there exists $\delta_2 > \delta_1 > 0$ *such that*

$$\|\nabla u(t)\|_2^2 \geq \delta_2, t \geq 0. \quad (3.2)$$

Lemma 3.3. *Let the assumptions in Lemma 3.2 be satisfied, then we have*

$$0 < H(0) \leq H(t) \leq \frac{b}{p_1}\int_{\Omega} |u|^{p(x)} \ln|u|\,\mathrm{d}x.$$

Proof. Using (2.2), (2.3) and (3.1), we obtain

$$
\begin{aligned}
0 &< H(0) \le H(t) \\
&\le E_1 - \left[\frac{1}{2}\int_\Omega u_t^2 \, \mathrm{d}x + \frac{l}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \frac{1}{2}(g \circ \nabla u) + \int_\Omega \frac{b}{p^2(x)}|u|^{p(x)} \, \mathrm{d}x\right] \\
&+ \int_\Omega \frac{b}{p(x)}|u|^{p(x)} \ln|u| \, \mathrm{d}x,
\end{aligned}
$$

and, from (3.2), we get

$$
\begin{aligned}
E_1 - &\left[\frac{1}{2}\int_\Omega u_t^2 \, \mathrm{d}x + \frac{l}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \frac{1}{2}(g \circ \nabla u) + \int_\Omega \frac{b}{p^2(x)}|u|^{p(x)} \, \mathrm{d}x\right] \\
&\le E_1 - \frac{l}{2}\|\nabla u\|_2^2 \le E_1 - \frac{\delta_1}{2} = -\left(1 + \frac{1-l}{4\tau l}\right)\frac{\delta_1}{p_1} < 0.
\end{aligned}
$$

Hence

$$
0 < H(0) \le H(t) \le \frac{b}{p_1}\int_\Omega |u|^{p(x)} \ln|u| \, \mathrm{d}x, \ \forall t \ge 0.
$$

□

The following theorem presents the main result of this article:

Theorem 3.1. *Let the conditions of Lemma 3.2 be satisfied and assume that*

$$
\begin{aligned}
&2 \le m_1 \le m(x) \le m_2 < p_1 \le p(x) \le p_2 < \tfrac{2n}{n-2}, \quad \text{for } n \ge 3, \\
&2 \le m_1 \le m(x) \le m_2 < p_1 \le p(x) \le p_2 < +\infty, \quad \text{for } n \le 2.
\end{aligned}
\tag{3.3}
$$

Then, the solution of problem (1.1) *given by Theorem 2.1 blows up in finite time.*

Proof. We introduce the following auxiliary function:

$$
L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u \, u_t(x,t) \, \mathrm{d}x, \tag{3.4}
$$

where $\varepsilon > 0$ is a sufficiently small parameter to be chosen later. The parameter α is assumed to satisfy the condition

$$
0 < \alpha \le \min\left\{\frac{p_1 - 2}{2p_1}, \frac{p_1 - m_2}{p_1(m_2 - 1)}\right\}. \tag{3.5}
$$

By utilizing equation (1.1) and differentiating the auxiliary function defined in (3.4), we obtain

$$
\begin{aligned}
L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega \left[u_t^2 - |\nabla u|^2\right] \mathrm{d}x + \varepsilon b \int_\Omega |u|^{p(x)} \ln|u| \mathrm{d}x \\
&+ \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t) \cdot \nabla u(s) \mathrm{d}x \mathrm{d}s - a\varepsilon \int_\Omega u u_t |u_t|^{m(x)-2} \mathrm{d}x.
\end{aligned}
\tag{3.6}
$$

We proceed by estimating the fourth term on the right-hand side:

$$
\begin{aligned}
&\int_0^t g(t-s) \int_\Omega \nabla u(t) \cdot \nabla u(s) \mathrm{d}x \mathrm{d}s \\
&= \left(\int_0^t g(s) \mathrm{d}s\right) \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_0^t g(t-s) \int_\Omega \nabla u(t) \cdot (\nabla u(s) - \nabla u(t)) \mathrm{d}x \mathrm{d}s.
\end{aligned}
$$

For any $\tau > 0$, the second term above can be estimated using Young's inequality:

$$\begin{aligned}&\int_0^t g(t-s)\int_\Omega \nabla u(t)\cdot(\nabla u(s)-\nabla u(t))\mathrm{d}x\mathrm{d}s\\&\le \tau(g\circ\nabla u)(t)+\frac{1}{4\tau}\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega|\nabla u(t)|^2\mathrm{d}x.\end{aligned}$$

Using the above estimate and by adding and subtracting the term $\varepsilon p_1 H(t)$ on the right-hand side of equation (3.6), we obtain the following inequality:

$$\begin{aligned}L'(t)\ge&(1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon p_1H(t)+\varepsilon\frac{p_1+2}{2}\|u_t\|_2^2\\&+\varepsilon\left[\left(\frac{p_1}{2}-1\right)-\left(\frac{p_1}{2}-1+\frac{1}{4\tau}\right)(1-l)\right]\|\nabla u\|_2^2+\varepsilon\frac{bp_1}{p_2^2}\|u\|_{p_1,\Omega_+}^{p_1}\\&+\varepsilon\left(\frac{p_1}{2}-\tau\right)(g\circ\nabla u)-\varepsilon p_1E_1-a\varepsilon\int_\Omega uu_t|u_t|^{m(x)-2}\mathrm{d}x,\end{aligned}$$

by virtue of Lemma 3.1.
It follows that, for $0<\tau<\frac{p_1}{2}$, we have

$$\begin{aligned}L'(t)\ge&(1-\alpha)H^{1-\alpha}(t)H'(t)+\varepsilon\beta\left[H(t)+\|u\|_{p_1,\Omega_+}^{p_1}+\|u_t\|_2^2+\|\nabla u\|_2^2+g\circ\nabla u\right]\\&-a\varepsilon\int_\Omega uu_t|u_t|^{m(x)-2}\,\mathrm{d}x,\end{aligned}\tag{3.7}$$

where $\beta>0$ is a positive constant.
We estimate the last term in (3.7) using Young's inequality, as follows:

$$\int_\Omega|u_t|^{m(x)-1}|u|\,\mathrm{d}x\le\frac{1}{m_1}\int_\Omega\theta^{m(x)}|u|^{m(x)}\,\mathrm{d}x+\frac{m_2-1}{m_2}\int_\Omega\theta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}\,\mathrm{d}x,\tag{3.8}$$

for $\theta>0$. We now choose θ such that

$$\theta^{-\frac{m(x)}{m(x)-1}}=k\cdot H^{-\alpha}(t),\quad k>0,$$

which yields the following estimate:

$$\begin{aligned}\int_\Omega|u_t|^{m(x)-1}|u|\,\mathrm{d}x\le&\frac{1}{m_1}\max_{x\in\Omega}\left\{k^{1-m(x)}H^{\alpha(m(x)-1)}(t)\right\}\int_\Omega|u|^{m(x)}\,\mathrm{d}x\\&+\frac{(m_2-1)\,k}{am_2}H^{-\alpha}(t)H'(t).\end{aligned}\tag{3.9}$$

Combining inequality (3.7) with estimate (3.9), we obtain

$$\begin{aligned}L'(t)\ge&\left[(1-\alpha)-\varepsilon\left(\frac{m_2-1}{m_2}\right)k\right]H^{-\alpha}(t)H'(t)\\&-\varepsilon\frac{k^{1-m_1}a}{m_1}\max_{x\in\Omega}\left\{H^{\alpha(m(x)-1)}(t)\right\}\int_\Omega|u|^{m(x)}\,\mathrm{d}x\\&+\varepsilon\beta\left[H(t)+\|u_t\|_2^2+\|u\|_{p_1,\Omega_+}^{p_1}+\|\nabla u\|_2^2+g\circ\nabla u\right].\end{aligned}\tag{3.10}$$

Applying Lemma 3.1, we obtain the following estimate:

$$\max_{x\in\Omega}\left\{H^{\alpha(m(x)-1)}(t)\right\}\int_\Omega |u|^{m(x)}\,\mathrm{d}x \le C\max_{x\in\Omega}\left\{H^{\alpha(m(x)-1)}(t)\right\}\|u\|_{p_1,\Omega_+}^{m_2}$$
$$\le C\left[(H(t))^{\frac{p_1\alpha(m_1-1)}{p_1-m_2}}+(H(t))^{\frac{p_1\alpha(m_2-1)}{p_1-m_2}}+\|u\|_{p_1,\Omega_+}^{p_1}\right] \tag{3.11}$$

thanks to Young's inequality.
We now apply the well-known algebraic inequality, which states:

$$z^\tau \le z+1 \le \left(1+\frac{1}{d}\right)(z+d)\,,\forall z\ge 0, 0<\tau\le 1, d\ge 0. \tag{3.12}$$

Substituting $z = H(t)$, $d = H(0)$ and $\tau_{i(i=1,2)} = \frac{p_1\alpha(m_i-1)}{(p_1-m_2)}$, we arrive at the following inequality:

$$\max_{x\in\Omega}\left\{H^{\alpha(m(x)-1)}(t)\right\}\int_\Omega |u|^{m(x)}\ \mathrm{d}x \le C\left(H(t)+\|u\|_{p_1,\Omega_+}^{p_1}\right). \tag{3.13}$$

Combining inequalities (3.10) and (3.13), we get the following result:

$$L'(t)\ge\left[(1-\alpha)-\varepsilon\left(\frac{m_2-1}{m_2}\right)k\right]H^{-\alpha}(t)\,H'(t)$$
$$+\varepsilon\left(\beta-\frac{k^{1-m_1}a}{m_1}C\right)\left[H(t)+\|u_t\|_2^2+\|u\|_{p_1,\Omega_+}^{p_1}+\|\nabla u\|_2^2+g\circ\nabla u\right]. \tag{3.14}$$

Here, we choose k to be big enough that

$$\gamma=\beta-\frac{a\cdot k^{1-m_1}}{m_1}C>0.$$

Once k is fixed, we find ε small enough to guarantee

$$(1-\alpha)-\varepsilon\left(\frac{m_2-1}{m_2}\right)k\ge 0 \text{ and } L(0)=H^{1-\alpha}(0)+\varepsilon\int_\Omega u_0(x)u_1(x)\ \mathrm{d}x>0.$$

Thus, (3.14) takes the form

$$L'(t)\ge\gamma\varepsilon\left[H(t)+\|u_t\|_2^2+\|u\|_{p_1,\Omega_+}^{p_1}+\|\nabla u\|_2^2+g\circ\nabla u\right]. \tag{3.15}$$

Consequently, we have

$$L(t)\ge L(0)>0,\quad \text{for all } t\ge 0.$$

However, from (3.4), we derive:

$$L^{\frac{1}{1-\alpha}}(t)=\left[H^{1-\alpha}(t)+\varepsilon\int_\Omega uu_t(x,t)\ \mathrm{d}x\right]^{\frac{1}{1-\alpha}}$$
$$\le 2^{1/1-\alpha}\left[H(t)+\varepsilon\left(\int_\Omega uu_t(x,t)\ \mathrm{d}x\right)^{\frac{1}{1-\alpha}}\right]. \tag{3.16}$$

We observe that

$$\left|\int_\Omega uu_t(x,t)\ \mathrm{d}x\right|\le\|u\|_2\|u_t\|_2\le C\|u\|_{p_1}\|u_t\|_2,$$

which implies

$$\left|\int_\Omega uu_t(x,t)\ \mathrm{d}x\right|^{\frac{1}{1-\alpha}} \le C\|u\|_{p_1}^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}}.$$

Once again, Young's inequality provides the following estimate:

$$\left|\int_\Omega uu_t(x,t)\ \mathrm{d}x\right|^{\frac{1}{1-\alpha}} \le C\left[\|u\|_{p_1}^{\frac{2}{1-2\alpha}} + \|u_t\|_2^2\right], \text{ for } \frac{1-2\alpha}{2(1-\alpha)} + \frac{1}{2(1-\alpha)} = 1.$$

Recalling condition (3.5) and applying inequality (3.12), we find

$$\left|\int_\Omega uu_t(x,t)\ \mathrm{d}x\right|^{\frac{1}{1-\alpha}} \le C\left[H(t) + \|u_t\|_2^2 + \|u\|_{p_1,\Omega_+}^{p_1}\right]. \tag{3.17}$$

By inserting (3.17) into (3.16), we obtain

$$L^{\frac{1}{1-\alpha}}(t) \le \varepsilon C\left[H(t) + \|u_t\|_2^2 + \|u\|_{p_1,\Omega_+}^{p_1}\right]. \tag{3.18}$$

From (3.15) and (3.18), we conclude that

$$L'(t) \ge \Lambda L^{\frac{1}{1-\alpha}}(t), \quad \text{for all } t \ge 0. \tag{3.19}$$

where Λ is a positive constant depending on $\Omega, u_{0,1}, m_{1,2}$ and $p_{1,2}$ only.
When inequality (3.19) is integrated over the interval $(0,t)$, it produces

$$L^{\frac{\alpha}{1-\alpha}}(t) \ge \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha}{1-\alpha}\Lambda t}$$

Therefore, $L(t)$ blows up in finite time

$$T^* \le \frac{1-\alpha}{\Lambda\alpha\left[L(0)\right]^{\frac{\alpha}{1-\alpha}}}.$$

This completes the proof. □

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