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AVERAGING OPERATORS ON HERZ SEQUENCE SPACES WITH VARIABLE EXPONENTS

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Abstract. The main result of this paper establishes the boundedness of the averaging operators on the Herz sequence spaces with variable exponents. The Herz sequence spaces with variable exponents are extensions of the discrete Lebesgue spaces with variable exponent and the Herz spaces. We obtain our main result by extending the boundedness of the averaging operator on the discrete Lebesgue spaces with variable exponent to Herz sequence spaces with variable exponents.

1. Introduction

This paper gives the boundedness of the averaging operator on the Herz sequence spaces with variable exponents.

Recently, a number of function spaces on the Euclidean spaces had been discretized. For instance, we have the Morrey sequence spaces (the discrete Morrey spaces) introduced and studied in [5, 6, 7, 8]. We also have the discrete Lebesgue spaces with variable exponents [4, 22].

The Herz sequence spaces with variable exponents introduced in this paper are extensions of the discrete Lebesgue spaces with variable exponents. The discrete Lebesgue spaces with variable exponents are introduced in [22]. It is one of the important members of the modular space [21]. The embedding result for the discrete Lebesgue spaces with variable exponents was obtained in [23] and the boundedness of the averaging operator on the discrete Lebesgue spaces with variable exponents was established in [4]. The results obtained for the discrete Lebesgue spaces with variable exponents have applications on the studies of the Lebesgue spaces with variable exponents on Euclidean spaces [9, 24].

Moreover, the Herz sequence spaces with variable exponents are also the discrete analogue of the Herz space with variable exponents. The Herz spaces with variable exponents are extensions of the Lebesgue spaces with variable exponents [2, 3] and the Herz spaces [1, 10, 20, 25, 26, 27, 29].

This paper is organized as follows. Section 2 recalls the definitions of the discrete Lebesgue spaces with variable exponents and the boundedness of the averaging operators on the discrete Lebesgue spaces with variable exponents. The definition of the Herz sequence space with variable exponent is also given in

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this section. The main result on the boundedness of the averaging operators on the Herz sequence spaces with variable exponents is established in Section 3.

2. Preliminaries and Definitions

For any $a = \{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ and $b = \{b_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$, we write $a \le b$ if $a_n \le b_n$, $\forall n \in \mathbb{Z}$. Define $|a| = \{|a_n|\}_{n\in\mathbb{Z}}$ and $\operatorname{supp} a = \{n \in \mathbb{Z} : a_n \neq 0\}$.

Let $k \in \mathbb{N}\backslash\{0,1\}$. For any $a = \{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$, the averaging operator $T_k a$ is defined as

$$(T_k a)_n = \frac{a_n + \cdots + a_{n+k-1}}{k}, \quad n \in \mathbb{Z}.$$

We now recall the definition of the discrete Lebesgue space with variable exponent from [4].

Definition 2.1. Let $\{p_n\}_{n\in\mathbb{Z}} \subset (1,\infty)$. The discrete Lebesgue space with variable exponent $l^{\{p_n\}}$ consists of all $a = \{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ satisfying

$$\|a\|_{l^{\{p_n\}}} = \inf\left\{\lambda > 0 : \sum_{n\in\mathbb{Z}} \left|\frac{a_n}{\lambda}\right|^{p_n} \le 1\right\} < \infty.$$

Whenever $p_n = p \in (1,\infty)$ for all $n \in \mathbb{Z}$, we see that $l^{\{p_n\}}$ becomes ℓ^p.

For the studies and applications of the discrete Lebesgue spaces with variable exponents, see [4, 9, 23]. The following is the condition that guarantees the boundedness of the averaging operator on $l^{\{p_n\}}$ from [4, Definition 3.5].

Definition 2.2. Let $\{p_n\}_{n\in\mathbb{Z}} \subset (1,\infty)$. We write $\{p_n\}_{n\in\mathbb{Z}} \in \mathcal{P}$ if there exist $\{\epsilon_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ such that $p_n = p_0 + \epsilon_n$, $n \in \mathbb{Z}$ and $\sum_{n\in\mathbb{Z}} |\epsilon_n| < \infty$.

The following result is the boundedness of the averaging operator on $l^{\{p_n\}}$.

Proposition 2.1. *Let $k \in \mathbb{N}$ and $\{p_n\}_{n\in\mathbb{Z}} \subset (1,\infty)$. If $\{p_n\}_{n\in\mathbb{Z}} \in \mathcal{P}$, then $T_k : l^{\{p_n\}} \to l^{\{p_n\}}$ is bounded.*

For the proof of the above result, see [4, Theorem 3.6].

We now turn to the definition of the Herz sequence spaces with variable exponents. For any $i \in \mathbb{N}\backslash\{0\}$, write $D_i = \{n \in \mathbb{Z} : 2^i < n \le 2^{i+1}\}$, $D_{-i} = \{n \in \mathbb{Z} : -2^{i+1} < n \le -2^i\}$ and $D_0 = \{n \in \mathbb{Z} : |n| \le 2\}$. For any $i \in \mathbb{Z}\backslash\{0\}$ and $a = \{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$, define $\chi_i a = \{(\chi_i a)_n\}_{n\in\mathbb{Z}}$ by

$$(\chi_i a)_n = \begin{cases} a_n, & n \in D_i, \\ 0, & n \notin D_i, \end{cases} \quad n \in \mathbb{Z}$$

and $\chi_0 a$ by

$$(\chi_0 a)_n = \begin{cases} a_n, & n \in D_0, \\ 0, & n \notin D_0, \end{cases} \quad n \in \mathbb{Z}.$$

It is easy to see that $\operatorname{supp}(\chi_i a) = \operatorname{supp} a \cap D_i$.

Definition 2.3. Let $\alpha \in \mathbb{R}$, $\theta \in (0,\infty)$ and $\{p_n\}_{n\in\mathbb{Z}} \subset (1,\infty)$. The Herz sequence space with variable exponent $h_\theta^{\alpha,\{p_n\}}$ consists of all $a = \{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ satisfying

$$\|a\|_{h_\theta^{\alpha,\{p_n\}}} = \left(\sum_{j\in\mathbb{Z}} (2^{j\alpha} \|\chi_j a\|_{l^{\{p_n\}}})^\theta\right)^{\frac{1}{\theta}} < \infty.$$

Let $\alpha \in \mathbb{R}$, $\theta \in (0,\infty)$ and $p \in (1,\infty)$. The Herz sequence space $h_\theta^{\alpha,p}$ consists of all $a = \{a_n\}_{n\in\mathbb{Z}}$ satisfying

$$\|a\|_{h_\theta^{\alpha,p}} = \left(\sum_{j\in\mathbb{Z}}(2^{j\alpha}\|\chi_j a\|_{\ell^p})^\theta\right)^{\frac{1}{\theta}} < \infty.$$

Let $p \in (1,\infty)$. Whenever $p_n = p$ for all $n \in \mathbb{Z}$, the Herz sequence space with variable exponent $h_\theta^{\alpha,\{p_n\}}$ becomes the Herz sequence space $h_\theta^{\alpha,p}$.

The Herz sequence spaces with variable exponents are the extensions of the Herz spaces with variable exponents on Euclidean spaces in the discrete setting. For the studies of the Herz spaces with variable exponents on Euclidean spaces, see [11, 12, 13, 14, 15, 17, 18, 19, 28, 30].

3. Main result

This section establishes the main result of this paper. We extend the boundedness of the averaging operator to the Herz sequence spaces with variable exponents.

Theorem 3.1. *Let $k \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\theta \in (0,\infty)$ and $\{p_n\}_{n\in\mathbb{Z}} \subset (1,\infty)$. If $\{p_n\}_{n\in\mathbb{Z}} \in \mathcal{P}$, then $T_k : h_\theta^{\alpha,\{p_n\}} \to h_\theta^{\alpha,\{p_n\}}$ is bounded.*

Proof. Let h be the smallest positive integer such that $k \leq 2^h$. Let $a = \{a_n\}_{n\in\mathbb{Z}} \in h_\theta^{\alpha,\{p_n\}}$.

For any $i \in \mathbb{N}$ with $2^{i-1} > k$, when $n \in D_i$, we have

$$n + k - 1 < 2^{i+1} + 2^{i-1} - 1 < 2^{i+2}.$$

Therefore,

$$\{n, \cdots, n + k - 1\} \subset D_i \cup D_{i+1}.$$

Consequently,

$$|(\chi_i T_k a)_n \leq \frac{|a_n| + \cdots |a_{n+k-1}|}{k} \leq T_k(|\chi_i a| + |\chi_{i+1} a|).$$

By applying the norm $\|\cdot\|_{l^{\{p_n\}}}$ and, then multiplying $2^{i\alpha}$ on both sides of the above inequality, we obtain

$$2^{i\alpha}\|\chi_i T_k a\|_{l^{\{p_n\}}} \leq 2^{i\alpha}\|T_k(\chi_i a)\|_{l^{\{p_n\}}} + 2^{i\alpha}\|T_k(\chi_{i+1} a)\|_{l^{\{p_n\}}}.$$

Proposition 2.1 yields

$$2^{i\alpha}\|\chi_i T_k a\|_{l^{\{p_n\}}} \leq C 2^{i\alpha}(\|\chi_i a\|_{l^{\{p_n\}}} + \|\chi_{i+1} a\|_{l^{\{p_n\}}}). \tag{3.1}$$

When $n \in D_{-i}$, we have

$$n + k - 1 < -2^i + 2^{i-1} - 1 \leq -2^{i-1} - 1.$$

That is,

$$\{n, \cdots, n + k - 1\} \subset D_i \cup D_{i-1}.$$

We find that

$$|(|\chi_{-i} T_k a|)_n \leq \frac{|a_n| + \cdots |a_{n+k-1}|}{k} \leq \big(T_k(|\chi_{-i} a| + |\chi_{-i+1} a|)\big)_n.$$

Therefore,

$$|\chi_i T_k a| \le T_k(|\chi_{-i}a| + |\chi_{-i+1}a|).$$

Similar to (3.1), Proposition 2.1 gives

$$2^{i\alpha}\|\chi_i T_k a\|_{l\{p_n\}} \le C2^{i\alpha}(\|\chi_{-i}a\|_{l\{p_n\}} + \|\chi_{-i+1}a\|_{l\{p_n\}}). \tag{3.2}$$

When i satisfies $k \ge 2^{i-1}$, we have $i \le \frac{\ln k}{\ln 2} + 1$. For any $n \in D_{-i} \cup D_i$, we have

$$n + k - 1 \le 2^{i+1} + 2^h - 1 \le 2^{i+h+1}$$

because for any $b, c > 1$, we have $bc - c - b + 1 = (c-1)(b-1) > 0$. Consequently,

$$\{n, \cdots, n+k-1\} \subset \bigcup_{j=-i}^{i+h} D_i$$

and

$$|(\chi_i T_k a)_n| \le \frac{|a_n| + \cdots |a_{n+k-1}|}{k} \le T_k\left(\sum_{j=-i}^{i+h} |\chi_j a|\right)$$

Therefore, Proposition 2.1 guarantees that

$$2^{i\alpha}\|\chi_i T_k a\|_{l\{p_n\}} \le C\left(\sum_{j=-i}^{i+h} 2^{i\alpha}\|\chi_j a\|_{l\{p_n\}}\right). \tag{3.3}$$

Write $m = [\frac{\ln k}{\ln 2} + 1]$ where $[a]$ is the integral part of $a \in (0, \infty)$. As $\|\cdot\|_{\ell^\theta}$ is a quasi-norm, we have a constant $C > 0$ such that

$$\begin{aligned}\|\{2^{i\alpha}\|\chi_i T_k a\|_{l\{p_n\}}\}_{i\in\mathbb{Z}}\|_{\ell^\theta} \le C\Bigg(&\left(\sum_{i=m}^{\infty}(2^{i\alpha}\|\chi_i T_k a\|_{l\{p_n\}})^\theta\right)^{\frac{1}{\theta}} \\ &+ \left(\sum_{i=-m+1}^{m-1}(2^{i\alpha}\|\chi_i T_k a\|_{l\{p_n\}})^\theta\right)^{\frac{1}{\theta}} \\ &+ \left(\sum_{i=-\infty}^{-m}(2^{i\alpha}\|\chi_i T_k a\|_{l\{p_n\}})^\theta\right)^{\frac{1}{\theta}}\Bigg) \\ &= I + II + III.\end{aligned}$$

According to (3.1), we have

$$I \le C\left(\sum_{i=m}^{\infty}(2^{i\alpha}(\|\chi_i a\|_{l\{p_n\}} + \|\chi_{i+1}a\|_{l\{p_n\}}))^\theta\right)^{\frac{1}{\theta}} \le C\|a\|_{h_\theta^{\alpha,\{p_n\}}} \tag{3.4}$$

for some $C > 0$ independent of a.

Similarly, (3.2) gives

$$\begin{aligned}III &\le C\left(\sum_{i=-\infty}^{-m}(2^{i\alpha}(\|\chi_i a\|_{l\{p_n\}} + \|\chi_{i-1}a\|_{l\{p_n\}}))^\theta\right)^{\frac{1}{\theta}} \\ &\le C\|a\|_{h_\theta^{\alpha,\{p_n\}}} \end{aligned} \tag{3.5}$$

for some $C > 0$ independent of a.

For II, for any $j \in \{-i, \cdots, i+h\}$, since $0 \le i \le m$, we have $2^{(i-j)\alpha} \le C$ because $-m-h \le i-j \le 2m$. Consequently, (3.3) yields

$$\begin{aligned}(2^{i\alpha}\|\chi_i T_k a\|_{l\{p_n\}})^\theta &\le C\Big(\sum_{j=-i}^{i+h} 2^{(i-j)\alpha}2^{j\alpha}\|\chi_j a\|_{l\{p_n\}}\Big)^\theta \\ &\le C\left(\sum_{j=-i}^{i+h}(2^{j\alpha}\|\chi_j a\|_{l\{p_n\}})^\theta\right) \\ &\le C\|a\|^\theta_{l\{p_n\}}\end{aligned}$$

for some $C > 0$ independent of i because there are at most $2i+h \le m+h$ terms in the above summations. Therefore,

$$II \le C\left(\sum_{i=-m+1}^{m-1}\sum_{j=-i}^{i+h}(2^{j\alpha}\|\chi_j a\|_{l\{p_n\}})^\theta\right)^{\frac{1}{\theta}} \le C\|a\|_{h_\theta^{\alpha,\{p_n\}}} \tag{3.6}$$

for some $C > 0$ independent of a. Consequently, (3.4), (3.5) and (3.6) show that $T_k a \in h_\theta^{\alpha,\{p_n\}}$ and there is a constant $C > 0$ such that for any $a \in h_\theta^{\alpha,\{p_n\}}$, $\|T_k a\|_{h_\theta^{\alpha,\{p_n\}}} \le C\|a\|_{h_\theta^{\alpha,\{p_n\}}}$. □

Theorem 3.1 gives the following application on the discrete Herz spaces.

Corollary 3.1. *Let $k \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\theta \in (0,\infty)$ and $p \in (1,\infty)$. The averaging operator T_k is bounded on $h_\theta^{\alpha,p}$.*

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