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ELLIPTIC EQUATIONS WITH VARIABLE EXPONENT, CONVECTION TERM AND INTRINSIC OPERATOR

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Abstract. In this paper, we explore a nonlinear elliptic Dirichlet problem driven by the negative $p(\cdot)$-Laplace operator. The reaction term of the problem is the composition of a Carathéodory function, which exhibits dependence on the solution and its gradient, with a continuous map on the Sobolev space, called intrinsic operator. Under very general structure conditions, we first establish the existence of at least one solution for the problem under consideration. Then, we produce an upper bound for solutions to the problem.

1. Introduction

Let $D \subseteq \mathbb{R}^N$ with $N \geq 2$ be a bounded domain whose boundary is Lipschitz. Also, let $p \in C(\overline{D})$ be such that $1 < p(x) < N$ for all $x \in \overline{D}$. In the present paper, we focus on the following Dirichlet problem

$$\begin{aligned} -\Delta_{p(\cdot)}u &= f(x, G(u), \nabla G(u)) \quad \text{in } D,\\ u = 0 & \qquad\qquad\qquad\qquad\quad\ \text{on } \partial D \end{aligned} \tag{1.1}$$

driven by the negative $p(\cdot)$-Laplace operator $-\Delta_{p(\cdot)}$. An interesting feature of problem (1.1) is that the reaction term (right-hand side of the equation) is expressed as the composition of a Carathéodory function $f : D \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, which exhibits dependence on the solution and its gradient (convection term), with a continuous map $G : W_0^{1,p(\cdot)}(D) \to W_0^{1,p(\cdot)}(D)$, which we call intrinsic operator (see Section 2 for the precise definition of the function space $W_0^{1,p(\cdot)}(D)$). Now, the presence of G in the convection term makes the problem more general. This for way of fact that the operator G cannot be incorporated in the function f, as it acts on the whole space $W_0^{1,p(\cdot)}(D)$ and not pointwise.

We stress that in the case in which G is the identity map, problem (1.1) reduces to a $p(\cdot)$-Laplacian equation with convection term. Equations driven by the negative $p(\cdot)$-Laplace operator were explored widely in the last decades, as one can see for example by [1, 3, 4, 10] and their references. On the contrary, elliptic equations involving an intrinsic operator were considered only recently. Precisely, the study of such type of equations started in the work of Motreanu

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et al. [7]. There, the authors explored (p, q)-Laplacian equations with reaction term in convection form, providing the existence of at least one solution. A similar problem to one under consideration here but with constant exponent was analyzed by Motreanu in [8]. Also, we mention the work of Medeiros et al. [5] for problem involving competing and intrinsic operators.

To the best of our knowledge this is the first work which combines variable exponent, convection term and intrinsic operator. The main objective of this paper is to provide the existence of at least one solution for problem (1.1) (see Theorem 4.1), under very general structure conditions (see hypotheses (H_G) and (H_f)). Further, we here give an upper bound for the norm of solutions to problem (1.1) (see Theorem 4.1 again). We emphasize that such an upper bound allows to answer the question if there can exist a solution to (1.1) belonging to a prescribed ball in the function space $W_0^{1,p(\cdot)}(D)$. Finally, as the presence of the gradient-dependence in the reaction term inhibits the use of variational methods, we follow a topological approach. Thus, a key tool in order to produce our main result is given by a well-known surjectivity theorem for pseudomonotone operators, which we recall in Section 2 (see Theorem 2.1).

2. Preliminaries

The function spaces framework with which we will work are variable exponent Lebesgue and Sobolev spaces. In this section, we then recall some basic notions from the theory of such spaces. The reader can find more details in the books of Diening et al. [2] and Rădulescu et al. [9]. Also, we give some remark about pseudomonotone operators. For these topics, we refer to the book of Motreanu et al. [6]

Let $D \subseteq \mathbb{R}^N$ with $N \geq 2$ be a bounded domain whose boundary is Lipschitz. Given $m \in C(\overline{D})$ with $m(x) > 1$ for all $x \in \overline{D}$, we write m' in order to denote the conjugate variable exponent of m, namely,

$$\frac{1}{m(x)} + \frac{1}{m'(x)} = 1 \quad \text{for all } x \in \overline{D}.$$

Also, we put

$$m^- := \min_{x \in \overline{D}} m(x) \quad \text{and} \quad m^+ := \max_{x \in \overline{D}} m(x).$$

Then, the variable exponent Lebesgue space $L^{m(\cdot)}(D)$ is given by

$$L^{m(\cdot)}(D) = \left\{ u \in M(D) \,:\, \rho_{m(\cdot)}(u) := \int_D |u|^{m(x)}\, dx < +\infty \right\}$$

where $M(D)$ stands for the set of all measurable functions $u\colon D \to \mathbb{R}$. We here equip $L^{m(\cdot)}(D)$ with the usual Luxemburg norm, that is,

$$\|u\|_{m(\cdot)} := \inf \left\{ \lambda > 0 \,:\, \rho_{m(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}$$

for all $u \in L^{m(\cdot)}(D)$. Such norm makes $L^{m(\cdot)}(D)$ a separable, uniformly convex and hence reflexive Banach space whose dual space is given by $L^{m'(\cdot)}(D)$. Also, we stress that the Hölder-type inequality

$$\int_D |u\,h|\,\mathrm{d}x \le \left[\frac{1}{m^-} + \frac{1}{(m')^-}\right] \|u\|_{m(\cdot)}\, \|h\|_{m'(\cdot)}$$

holds for all $u \in L^{m(\cdot)}(D)$ and all $h \in L^{m'(\cdot)}(D)$. Further, we recall the following useful relations.

Proposition 2.1. *Let $m \in C(\overline{D})$ be such that $m(x) > 1$ for all $x \in \overline{D}$. Then the following hold:*

(i) $\|u\|_{m(\cdot)} < 1$ *(resp.* $> 1, = 1$*) if and only if* $\rho_{m(\cdot)}(u) < 1$ *(resp.* $> 1, = 1$*);*

(ii) *if* $\|u\|_{m(\cdot)} < 1$ *then* $\|u\|_{m(\cdot)}^{m^+} \le \rho_{m(\cdot)}(u) \le \|u\|_{m(\cdot)}^{m^-}$*;*

(iii) *if* $\|u\|_{m(\cdot)} > 1$ *then* $\|u\|_{m(\cdot)}^{m^-} \le \rho_{m(\cdot)}(u) \le \|u\|_{m(\cdot)}^{m^+}$.

We point out that, according to Proposition 2.1, we have that

$$\|u\|_{m(\cdot)}^{m^-} - 1 \le \rho_{m(\cdot)}(u) \le \|u\|_{m(\cdot)}^{m^+} + 1 \tag{2.1}$$

for all $u \in L^{m(\cdot)}(D)$.

The variable exponent Sobolev space corresponding to $L^{m(\cdot)}(D)$ is

$$W^{1,m(\cdot)}(D) = \{u \in L^{m(\cdot)}(D) \,:\, |\nabla u| \in L^{m(\cdot)}(D)\}$$

endowed with the norm

$$\|u\|_{1,m(\cdot)} = \|u\|_{m(\cdot)} + \|\nabla u\|_{m(\cdot)},$$

where $\|\nabla u\|_{m(\cdot)} = \|\,|\nabla u|\,\|_{m(\cdot)}$ for all $u \in W^{1,m(\cdot)}(D)$. We here use $W_0^{1,m(\cdot)}(D)$ by denoting the closure of $C_0^\infty(D)$ with respect to the norm $\|\cdot\|_{1,m(\cdot)}$. We note that $W^{1,m(\cdot)}(D)$ and $W_0^{1,m(\cdot)}(D)$ are uniformly convex, separable and reflexive Banach spaces. Also, the Poincaré inequality is valid for $W_0^{1,m(\cdot)}(D)$. This means that there exists a constant $k_m > 0$ such that the following inequality

$$\|u\|_{m(\cdot)} \le k_m \|\nabla u\|_{m(\cdot)}$$

is satisfied for all $u \in W_0^{1,m(\cdot)}(D)$. Taking this into account, we will consider on $W_0^{1,p(\cdot)}(D)$ the equivalent norm given by

$$\|u\| := \|\nabla u\|_{p(\cdot)} \quad \text{for all } u \in W_0^{1,p(\cdot)}(D).$$

Next, the following Sobolev embedding result holds.

Proposition 2.2. *Let $m \in C(\overline{D})$ be such that*

$$1 < m(x) \le p^*(x) := \frac{Np(x)}{N - p(x)}$$

for all $x \in \overline{D}$. Then, we have the continuous embedding

$$W_0^{1,p(\cdot)}(D) \hookrightarrow L^{m(\cdot)}(D).$$

If $1 < m(x) < p^*(x)$ for all $x \in \overline{D}$, then the above embedding is compact.

In the sequel, according to Proposition 2.2, we will denote by $e_{m(\cdot)}$ the best positive constant such that

$$\|u\|_{m(\cdot)} \leq e_{m(\cdot)} \|u\| \quad \text{for all } u \in W_0^{1,p(\cdot)}(D). \tag{2.2}$$

Also, as we have the continuous embedding $L^{p(\cdot)}(D) \hookrightarrow L^{p^-}(D)$ we will denote by e the best positive constant such that

$$\|u\|_{p^-}^{p^-} \leq e \|u\|_{p(\cdot)}^{p^-} \quad \text{for all } \ u \in L^{p(\cdot)}(D). \tag{2.3}$$

Finally, the negative $p(\cdot)$-Laplace operator

$$-\Delta_{p(\cdot)} : W_0^{1,p(\cdot)}(D) \to (W_0^{1,p(\cdot)}(D))^*,$$

with $(W_0^{1,p(\cdot)}(D))^*$ being the dual space of $W_0^{1,p(\cdot)}(D)$, is defined by

$$\langle -\Delta_{p(\cdot)} u, w\rangle = \int_D |\nabla u|^{p(x)-2} \nabla u \cdot \nabla w \, dx$$

for all $u, w \in W_0^{1,p(\cdot)}(D)$. Such operator have several properties. In particular, we point out that it is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone and further satisfies the $(S)_+$-properties, which means that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(\cdot)}(D) \quad \text{and} \quad \limsup_{n\to+\infty} \langle -\Delta_{p(\cdot)} u_n, u_n - u\rangle \leq 0$$

imply

$$u_n \to u \text{ in } W_0^{1,p(\cdot)}(D).$$

Also, we recall that an operator $O : W_0^{1,p(\cdot)}(D) \to (W_0^{1,p(\cdot)}(D))^*$ is called:

- pseudomonotone if

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p(\cdot)}(D) \quad \text{and} \quad \limsup_{n\to+\infty} \langle O(u_n), u_n - u\rangle \leq 0$$

imply

$$\liminf_{n\to+\infty} \langle O(u_n), u_n - w\rangle \geq \langle O(u), u - w\rangle$$

for all $w \in W_0^{1,p(\cdot)}(D)$;

- coercive if

$$\lim_{\|u\|\to+\infty} \frac{\langle O(u), u\rangle}{\|u\|} = +\infty.$$

Lastly, we point out that the following surjectivity result holds for pseudomonotone operators.

Theorem 2.1. *Let X be a real, reflexive Banach space with dual space X^*. Also, let $O : X \to X^*$ be a pseudomonotone, bounded, and coercive operator and $h \in X^*$. Then, the equation $Ou = h$ admits at least one solution.*

3. Hypotheses on the reaction term

In this section, we are going to formulate our assumptions on the reaction term. We start by the hypothesis on intrinsic operator G.

(H_G) $G : W_0^{1,p(\cdot)}(D) \to W_0^{1,p(\cdot)}(D)$ is a continuous operator satisfying the following condition:
there exist constants a_1, $a_2 \geq 0$ such that

$$\|G(u)\|_{p(\cdot)} + \|\nabla G(u)\|_{p(\cdot)} \leq a_1 \|u\| + a_2$$

for all $u \in W_0^{1,p(\cdot)}(D)$.

Remark 3.1. We stress that the hypothesis (H_G) guarantees that the operator G is bounded.

Now, we can make our hypotheses on the nonlinearity f.

(H_f) $f : D \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions:

(i) there exist $\vartheta_1 \in L^{\ell'(\cdot)}(D)$, with $1 \leq \ell(x) < p^*(x)$ for all $x \in \overline{D}$, $\tau_1 \in \left[0, (p^*)^- - 1\right[$, $\tau_2 \in \left[0, \frac{p^-}{((p^*)')^-}\right[$ and constants $b_1, b_2 \geq 0$ such that

$$|f(x,s,\xi)| \leq \vartheta_1(x) + b_1|s|^{\tau_1} + b_2|\xi|^{\tau_2}$$

for a.e. $x \in D$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$;

(ii) there exist $\vartheta_2 \in L^1(D)$ and constants $c_1, c_2 \geq 0$ with

$$2^{p^- -1}\, e\, a_1^{p^-}\, (c_1 + c_2) < 1,$$

being e from (2.3), such that

$$f(x,s,\xi)s \leq \vartheta_2(x) + c_1|s|^{p^-} + c_2|\xi|^{p^-}$$

for a.e. $x \in D$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$.

In the sequel, with the purpose of streamlining the notation, we will put

$$\delta := 2^{p^- -1}\, e\, a_1^{p^-}\, (c_1 + c_2) \quad \text{and} \quad \tilde{\delta} := 2^{p^- -1}\, e\, a_2^{p^-}\, (c_1 + c_2). \tag{3.1}$$

Remark 3.2. We point out that as $\tau_1 < (p^*)^- - 1$ we have that

$$p^*(x) - \tau_1 > p^*(x) - [(p^*)^- - 1] \geq 1$$

for all $x \in \overline{D}$, which implies

$$\frac{p^*(x)}{p^*(x) - \tau_1} < p^*(x)$$

for all $x \in \overline{D}$. Also, from $\tau_2 < \frac{p^-}{((p^*)^-)'}$ we deduce that

$$\begin{aligned} p(x) - \tau_2 > p(x) - \frac{p^-}{((p^*)^-)'} &= p(x) - p^- \frac{(p^*)^- - 1}{(p^*)^-} \\ &= p(x) - p^- + p^- \frac{(N - p^-)}{Np^-} \end{aligned}$$

$$= \frac{N(p(x) - p^-) + N - p^-}{N}$$

for all $x \in \overline{D}$. This leads to

$$\frac{p(x)}{p(x) - \tau_2} < \frac{Np(x)}{N(p(x) - p^-) + N - p^-}$$

for all $x \in \overline{D}$. Taking into account that $N - p^- \geq N - p(x)$ and $N(p(x) - p^-) \geq 0$ for all $x \in \overline{D}$, we have that

$$\frac{Np(x)}{N(p(x) - p^-) + (N - p^-)} \leq \frac{Np(x)}{N - p(x)} := p^*(x)$$

and consequently

$$\frac{p(x)}{p(x) - \tau_2} < p^*(x)$$

for all $x \in \overline{D}$. Then, in accordance of the previous facts, we can affirm that the conditions imposed in hypothesis (H_f)(i) on ℓ, τ_1 and τ_2 guarantee that $W_0^{1,p(\cdot)}(D)$ is compactly embedded in the spaces

$$L^{\ell(\cdot)}(D), \quad L^{\frac{p^*(\cdot)}{p^*(\cdot)-\tau_1}}(D) \quad \text{and} \quad L^{\frac{p(\cdot)}{p(\cdot)-\tau_2}}(D).$$

4. Main Result

In this section, we present and prove our main result. In detail, we establish the existence of at least one solution for problem (1.1) in $W_0^{1,p(\cdot)}(D)$, supposed that hypotheses (H_G) and (H_f) are verified. Further, we produce an upper bound for the norm of solutions to (1.1). We point out that when we speak about solution, we mean weak solution. Thus, we call $u \in W_0^{1,p(\cdot)}(D)$ a solution of problem (1.1) if

$$\int_D |\nabla u|^{p(\cdot)-2} \nabla u \cdot \nabla w \, dx = \int_D f(x, G(u), \nabla G(u)) \, w \, dx$$

is satisfied for all $w \in W_0^{1,p(\cdot)}(D)$. Now, we introduce the operator

$$\mathcal{A} : W_0^{1,p(\cdot)}(D) \to (W_0^{1,p(\cdot)}(D))^*$$

defined by

$$\langle \mathcal{A} u, w \rangle := \int_D f(x, G(u), \nabla G(u)) w \, dx \tag{4.1}$$

for all $u, w \in W_0^{1,p(\cdot)}(D)$. Using $\mathcal{A}$, we can express problem (1.1) as the operator equation

$$-\Delta_{p(\cdot)} u - \mathcal{A} u = 0. \tag{4.2}$$

Consequently, in order to establish our existence result, we have to show that equation (4.2) has at least one solution in $W_0^{1,p(\cdot)}(D)$. Thus, our idea is to apply Theorem 2.1 to the operator

$$-\Delta_{p(\cdot)} - \mathcal{A} : W_0^{1,p(\cdot)}(D) \to (W_0^{1,p(\cdot)}(D))^*,$$

which drives equation (4.2) and is defined by

$$\langle(-\Delta_{p(\cdot)} - \mathcal{A})\, u, w\rangle := \langle -\Delta_{p(\cdot)} u, w\rangle + \langle -\mathcal{A}\, u, w\rangle$$

for all $u, w \in W_0^{1,p(\cdot)}(D)$. With this in mind, we start by exploring some properties which characterize the operator $\mathcal{A}$.

Proposition 4.1. *Let hypotheses* (H_G) *and* (H_f)(i) *be satisfied. Also, let* $\mathcal{A}$ *be as introduced in* (4.1)*. Then*

$$\|\mathcal{A}\, u\|_{(W_0^{1,p(\cdot)}(D))^*} \le e_{\ell(\cdot)}\, \|\vartheta_1\|_{\ell'(\cdot)} + b_1\, e_{\frac{p^*(\cdot)}{p^*(\cdot)-\tau_1}}\, \|G(u)\|_{p^*(\cdot)} + b_2\, e_{\frac{p(\cdot)}{p(\cdot)-\tau_2}}\, \|\nabla G(u)\|_{p(\cdot)} \tag{4.3}$$

for all $u \in W_0^{1,p(\cdot)}(D)$*, where* $e_{\ell(\cdot)}$, $e_{\frac{p^*(\cdot)}{p^*(\cdot)-\tau_1}}$, $e_{\frac{p(\cdot)}{p(\cdot)-\tau_2}}$ *are positive constants as given in* (2.2)*.*

Proof. First, we recall that as hypothesis (H_G) holds the operator G is bounded. Then, according to hypothesis (H_f)(i) we have that

$$\left|\int_D f(x, G(u), \nabla G(u))\, w\, dx\right| \le \int_D \left(|\vartheta_1(x)| + b_1 |G(u)|^{\tau_1} + b_2 |\nabla G(u)|^{\tau_2}\right) |w| d\, x$$

for all $w \in W_0^{1,p(\cdot)}(D)$. Now, using Remark 3.2 along with the Hölder's inequality, we derive that

$$\begin{aligned}\left|\int_D f(x, G(u), \nabla G(u))\, w\, dx\right| &\le \|\vartheta_1\|_{\ell'(\cdot)}\, \|w\|_{\ell(\cdot)} + b_1\, \|G(u)^{\tau_1}\|_{\frac{p^*(\cdot)}{\tau_1}}\, \|w\|_{\frac{p^*(\cdot)}{p^*(\cdot)-\tau_1}} \\ &\quad +\ b_2 \|(\nabla G(u))^{\tau_2}\|_{\frac{p(\cdot)}{\tau_2}}\, \|w\|_{\frac{p(\cdot)}{p(\cdot)-\tau_2}} \\ &\le \Big[e_{\ell(\cdot)}\, \|\vartheta_1\|_{\ell'(\cdot)} + b_1\, e_{\frac{p^*(\cdot)}{p^*(\cdot)-\tau_1}}\, \|G(u)\|_{p^*(\cdot)} \\ &\quad +\ b_2\ e_{\frac{p(\cdot)}{p(\cdot)-\tau_2}}\, \|\nabla G(u)\|_{p(\cdot)} \Big] \|w\|,\end{aligned}$$

where $e_{\ell(\cdot)}$, $e_{\frac{p^*(\cdot)}{p^*(\cdot)-\tau_1}}$, $e_{\frac{p(\cdot)}{p(\cdot)-\tau_2}}$ are positive constants as given in (2.2). Hence, taking (4.1) into account, we conclude that bound (4.3) holds. ☐

Proposition 4.2. *Let hypotheses* (H_G) *and* (H_f)(i) *be satisfied. Also, let* $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p(\cdot)}(D)$ *be a sequence such that*

$$u_n \rightharpoonup u \quad in \quad W_0^{1,p(\cdot)}(D).$$

Then

$$\lim_{n\to+\infty} \langle \mathcal{A}\, u_n,\, u_n - u\rangle = 0. \tag{4.4}$$

Proof. In order to achive the claim, we first stress that hypothesis (H_f)(i) assures that

$$\left| \int_D f(x, G(u_n), \nabla G(u_n))\,(u_n - u)\,dx \right| \le \int_D \left(|\vartheta_1(x)| + b_1|G(u_n)|^{\tau_1} + b_2|\nabla G(u_n)|^{\tau_2}\right)|u_n - u|\,dx$$

for all $n \in \mathbb{N}$. Now, we recall that G is bounded due to hypothesis (H_G). This guarantees that the sequence $\{G(u_n)\}_{n\in\mathbb{N}}$ is bounded both in $W_0^{1,p(\cdot)}(D)$ and $L^{p^*(\cdot)}(D)$. Then, using Proposition 2.2 along with the Hölder's inequality, we are able to derive the following estimates

$$\int_D |\vartheta_1(x)|\,|u_n - u|\,dx \le \|\vartheta_1\|_{\ell'(\cdot)}\,\|u_n - u\|_{\ell(\cdot)} \to 0 \quad n \to +\infty,$$

$$\int_D |G(u_n)|^{\tau_1}|u_n - u|\,dx \le \|G(u)\|_{p^*(\cdot)}\,\|u_n - u\|_{\frac{p^*(\cdot)}{p^*(\cdot)-\tau_1}} \to 0 \quad n \to +\infty \quad \text{(see Remark 3.2)},$$

$$\int_D |\nabla G(u_n)|^{\tau_2}\,|u_n - u|\,dx \le \|\nabla G(u)\|_{p(\cdot)}\,\|u_n - u\|_{\frac{p(\cdot)}{p(\cdot)-\tau_2}} \to 0 \quad n \to +\infty \quad \text{(see Remark 3.2)}.$$

This according to the definition of $\mathcal{A}$ produces the claim. □

Now, we are in the position to show that the driven operator in equation (4.2) is bounded, pseudomonotone and coercive.

Proposition 4.3. *Let hypotheses (H_G) and (H_f) be satisfied. Then, the driven operator in* (4.2), *namely $-\Delta_{p(\cdot)} - \mathcal{A}$, is bounded, pseudomonotone and coercive.*

Proof. We point out that the operator $-\Delta_{p(\cdot)} - \mathcal{A}$ is sum of two bounded operators, namely, $-\Delta_{p(\cdot)}$ and $-\mathcal{A}$ (see Section 2 and Proposition 4.1, respectively). Consequently, it is bounded as well. Thus, in order to achive the claim, we have only to show that $-\Delta_{p(\cdot)} - \mathcal{A}$ is a pseudomonotone and coercive operator.

We start by proving that $-\Delta_{p(\cdot)} - \mathcal{A}$ is pseudomonotone. To this end, we consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p(\cdot)}(D)$ satisfying the following conditions:

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p(\cdot)}(D) \quad \text{and} \quad \limsup_{n\to+\infty}\langle -\Delta_{p(\cdot)}u_n - \mathcal{A}\,u_n,\, u_n - u\rangle \le 0. \tag{4.5}$$

Our aim is then prove that the following inequality

$$\liminf_{n\to+\infty}\langle -\Delta_{p(\cdot)}u_n - \mathcal{A}\,u_n,\, u_n - w\rangle \ge \langle -\Delta_{p(\cdot)}u - \mathcal{A}\,u,\, u - w\rangle \tag{4.6}$$

is verified for all $w \in W_0^{1,p(\cdot)}(D)$. Now, according to Proposition 4.2 we have that (4.4) holds. Then, from (4.5) along with (4.4), we deduce that

$$\lim_{n\to+\infty}\langle -\Delta_{p(\cdot)}u_n, u_n - u\rangle \le 0.$$

Taking into account that the negative $p(\cdot)$-Laplace operator satisfies the S_+-property (see Section 2), we have that

$$u_n \to u \quad \text{in } W_0^{1,p(\cdot)}(D) \quad \text{as } n \to +\infty,$$

and this leads to (4.6). Thus, $-\Delta_{p(\cdot)} - \mathcal{A}$ is a pseudomonotone operator. We in addition note that according to the fact that (4.5) yields $u_n \to u$ in $W_0^{1,p(\cdot)}(D)$, we have that $-\Delta_{p(\cdot)} - \mathcal{A}$ satisfies the S_+-property.

Finally, we show that $-\Delta_{p(\cdot)} - \mathcal{A}$ is a coercive operator. Using hypotheses (H_G), (H_f)(ii) and the continuous embedding $L^{p(\cdot)}(D) \hookrightarrow L^{p^-}(D)$, we see that for all $u \in W_0^{1,p(\cdot)}(D)$ it results

$$\begin{aligned}\int_D f(x, G(u), \nabla G(u))\, u\, dx &\le \int_D \left(\vartheta_2(x) + c_1|G(u)|^{p^-} + c_2|\nabla G(u)|^{p^-}\right) dx \\ &\le \|\vartheta_2\|_1 + c_1\|G(u)\|_{p^-}^{p^-} + c_2\|\nabla G(u)\|_{p^-}^{p^-} \\ &\le \|\vartheta_2\|_1 + c_1\, e\, \|G(u)\|_{p(\cdot)}^{p^-} + c_2\, e\, \|\nabla G(u)\|_{p(\cdot)}^{p^-} \\ &\le \|\vartheta_2\|_1 + e\,(c_1 + c_2)(a_1\|u\| + a_2)^{p^-} \\ &\le \|\vartheta_2\|_1 + 2^{p^- -1}\, e\,(c_1 + c_2)\left(a_1^{p^-}\|u\|^{p^-} + a_2^{p^-}\right)\end{aligned}$$

being $e > 0$ as given in (2.3). Then, according to Proposition 2.1, for $u \in W_0^{1,p(\cdot)}(D)$ with $\|u\| > 1$, we have that

$$\begin{aligned}&\langle -\Delta_{p(\cdot)}u - \mathcal{A}\,u, u\rangle \\ &= \langle -\Delta_{p(\cdot)}u, u\rangle + \langle -\mathcal{A}\,u, u\rangle \\ &\ge \rho_{p(\cdot)}(\nabla u) - \left[\|\vartheta_2\|_1 + 2^{p^- -1}\, e\,(c_1 + c_2)(a_1^{p^-}\|u\|^{p^-} + a_2^{p^-})\right] \\ &\ge (1-\delta)\,\|u\|^{p^-} - \|\vartheta_2\|_1 - \tilde{\delta}\end{aligned}$$

where δ and $\tilde{\delta}$ are as given in (3.1). Now, as $\delta < 1$ due to hypothesis (H_f)(ii), we are able to affirm that

$$\lim_{\|u\| \to +\infty} \frac{\langle -\Delta_{p(\cdot)}u - \mathcal{A}\,u, u\rangle}{\|u\|} = +\infty.$$

Therefore, $-\Delta_{p(\cdot)} - \mathcal{A}$ is coercive. This completes the proof of proposition. □

Now, our main result reads as follow.

Theorem 4.1. *Let hypotheses* (H_G) *and* (H_f) *be satisfied. Then, problem* (1.1) *admits at least one solution in* $W_0^{1,p(\cdot)}(D)$. *Further, if* $u \in W_0^{1,p(\cdot)}(D)$ *is a solution of problem* (1.1), *then*

$$\|u\| \le \left(\frac{\|\vartheta_2\|_1 + \tilde{\delta} + 1}{1-\delta}\right)^{\frac{1}{p^-}}$$

with δ *and* $\tilde{\delta}$ *being from* (3.1).

Proof. We point out that problem (1.1) is equivalent to the operator equation (4.2). Taking this into account, in order to produce the claim we have to show that equation (4.2) admits at least one solution in $W_0^{1,p(\cdot)}(D)$. Now, from Proposition

4.3 we know that the driven operator in (4.2), namely $-\Delta_{p(\cdot)} - \mathcal{A}$, is a bounded, pseudomonotone and coercive operator. According of this, we are in the position to apply Theorem 2.1 which guarantees the existence of at least one solution for equation (4.2) in $W_0^{1,p(\cdot)}(D)$.

Thus, we suppose that $u \in W_0^{1,p(\cdot)}(D)$ is a solution of problem (1.1). Then, we have that

$$\rho_{p(\cdot)}(\nabla u) := \int_D |\nabla u|^{p(\cdot)}\,dx = \int_D f(x, G(u), \nabla G(u))\,u\,dx.$$

Now, using hypotheses (H_G), (H_f)(ii) along with (2.1) and arguing similar to proof of Proposition 4.3, we are able to derive that

$$\|u\|^{p^-} - 1 \le \|\vartheta_2\|_1 + 2^{p^- -1}\,e\,(c_1 + c_2)(a_1^{p^-}\|u\|^{p^-} + a_2^{p^-}),$$

which yields

$$\|u\| \le \left(\frac{\|\vartheta_2\|_1 + \tilde{\delta} + 1}{1 - \delta}\right)^{\frac{1}{p^-}},$$

with δ and $\tilde{\delta}$ being as defined in (3.1) and e being from (2.3). So, the claim is proved. □

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