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TITCHMARSH AND BOAS TYPE THEOREMS FOR THE MULTIDIMENSIONAL FOURIER-BESSEL TRANSFORM

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Abstract. Using a generalized spherical mean operator, we introduce the Lipschitz class associated with the multidimensional Fourier-Bessel operator Δ_α. We establish two versions of Titchmarsh's theorem for the multidimensional Fourier-Bessel transform $\mathscr{F}_\alpha$. Furthermore, we introduce the generalized Lipschitz classes $Lip_k(\beta)$ and $lip_k(\beta)$, proving two versions of Boas's theorem for the transform $\mathscr{F}_\alpha$, and conclude with an application to multidimensional Fourier-Bessel multipliers. The harmonic analysis associated with the operator Δ_α plays an important role in establishing the results of this paper.

1. Introduction

Titchmarsh ([27], Theorem 85) characterizes the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform. Similarly, Younis ([31], Theorem 5.2) characterizes the set of functions in $L^2(\mathbb{R})$ satisfying the Dini-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform.

Theorem 1.1. *Let $\beta \in (0,1)$ and suppose that $f \in L^2(\mathbb{R})$. Then the following assertions are equivalent.*

(i) $\|f(\cdot+h)-f(\cdot)\|_{L^2(\mathbb{R})} = O\left(h^\beta\right)$ *as* $h \to 0$.
(ii) $\int_{|\lambda|\geq s} |\mathscr{F}(f)(\lambda)|^2 d\lambda = O\left(s^{-2\beta}\right)$ *as* $s \to \infty$, *where $\mathscr{F}$ is the standard Fourier transform.*

Building on Titchmarsh's results, Boas established the necessary and sufficient conditions for the Fourier coefficients of a function to belong to a generalized Lipschitz class. In 1967, Boas provided the first such characterization (see [3]). Later, in [12], Móricz studied the continuity and regularity properties of a function f with an absolutely convergent Fourier series. Continuing these findings, the author in [13] extended the results as follows.

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Theorem 1.2. *If* $f \in L^1(\mathbb{R})$*, and for some* $0 < \beta \leq k$*,* $k \in \mathbb{N}$*, we have*

$$\int_{|\lambda|<s} |\lambda|^k |\mathscr{F}(f)(\lambda)| d\lambda = O(s^{k-\beta}) \quad \textit{for all} \quad s > 0,$$

then $\mathscr{F}(f) \in L^1(\mathbb{R})$ *and* f *satisfies the smooth Lipschitz condition of order* k*.*

Recently, many analogues of Titchmarsh and Boas type theorems have been established in harmonic analysis [10, 11, 16, 25, 26, 28]. Another fundamental tool in harmonic analysis is the multidimensional Fourier-Bessel transform, which is the focus of this paper.

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in [-\frac{1}{2}, \infty)^d$, we consider the multidimensional Fourier-Bessel operator Δ_α (see [1, 30, 9]) defined for $x = (x_1, \dots, x_d) \in \mathbb{R}^d_+$ by

$$\Delta_\alpha := \sum_{k=1}^{d} \left[\frac{\partial^2}{\partial x_k^2} + \frac{2\alpha_k + 1}{x_k} \frac{\partial}{\partial x_k} \right].$$

This operator has important applications in both pure and applied mathematics and leads to generalizations of multivariate analytic structures such as the Fourier-Bessel transform and the Fourier-Bessel convolution [2, 17, 18, 19, 20, 22, 23, 24].

For any $\lambda \in \mathbb{R}^d_+$, the system

$$\Delta_\alpha u(x) = -|\lambda|^2 u(x), \quad u(0) = 1, \quad \frac{\partial}{\partial x_k} u(x)\Big|_{x_k=0} = 0, \quad k = 1, \dots, d,$$

admits a unique solution $j_\alpha(\lambda, x)$, given by

$$j_\alpha(\lambda, x) := \prod_{k=1}^{d} j_{\alpha_k}(\lambda_k x_k), \tag{1.1}$$

where j_{α_k} is the normalized Bessel function of the first kind and order α_k (see [29]) given by

$$j_{\alpha_k}(x_k) := \Gamma(\alpha_k + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \alpha_k + 1)} \left(\frac{x_k}{2}\right)^{2n} = 2^{\alpha_k}\Gamma(\alpha_k + 1) \frac{J_{\alpha_k}(x_k)}{x_k^{\alpha_k}}.$$

Here J_{α_k} is the Bessel function of first kind and order α_k, see [8].

We denote by μ_α the measure defined by

$$\mathrm{d}\mu_\alpha(x) := w_\alpha(x)\mathrm{d}x,$$

where

$$w_\alpha(x) := c_\alpha \prod_{k=1}^{d} x_k^{2\alpha_k+1}, \quad c_\alpha := \prod_{k=1}^{d} \frac{1}{2^{\alpha_k}\Gamma(\alpha_k + 1)}.$$

Let $L^p_\alpha(\mathbb{R}^d_+)$, $p \in [1, \infty]$, be the space of measurable functions f on $\mathbb{R}^d_+$, for which

$$\|f\|_{L^p_\alpha(\mathbb{R}^d_+)} := \left[\int_{\mathbb{R}^d_+} |f(x)|^p \mathrm{d}\mu_\alpha(x) \right]^{1/p} < \infty,$$

$$\|f\|_{L^\infty_\alpha(\mathbb{R}^d_+)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^d_+} |f(x)| < \infty.$$

In this paper, we consider the multidimensional Fourier-Bessel transform defined for $f \in L^1_\alpha(\mathbb{R}^d_+)$ by

$$\mathscr{F}_\alpha(f)(\lambda) := \int_{\mathbb{R}^d_+} j_\alpha(\lambda, x) f(x) \mathrm{d}\mu_\alpha(x), \quad \lambda \in \mathbb{R}^d_+. \tag{1.2}$$

The multidimensional Fourier-Bessel transform can be considered a generalization of the Fourier-Bessel transform [6, 7, 14]. Numerous results have already been established for the multidimensional Fourier-Bessel transform $\mathscr{F}_\alpha$ (see [2, 17, 19, 22]).

The main objective of this work is to extend Theorems 1.1 and 1.2 to the multidimensional Fourier-Bessel transform $\mathscr{F}_\alpha$ applied to functions belonging to the multidimensional Fourier-Bessel Lipschitz classes in the space $L^2_\alpha(\mathbb{R}^d_+)$. To achieve this, we employ the spherical mean operator $\mathscr{M}_{r,\alpha}$ defined by the relation

$$\mathscr{F}_\alpha(\mathscr{M}_{r,\alpha} f)(\lambda) = j_{\langle\alpha\rangle+d-1}(r|\lambda|)\mathscr{F}_\alpha(f)(\lambda),$$

where $\langle\alpha\rangle = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

This work is organized as follows. In Section 2, we recall some results on the multidimensional Fourier-Bessel transform $\mathscr{F}_\alpha$ and the multidimensional Fourier-Bessel translation operators τ_x, $x \in \mathbb{R}^d_+$. In Section 3, we define Lipschitz class and we prove a Titchmarsh-type theorem for the multidimensional Fourier-Bessel transform $\mathscr{F}_\alpha$. In the final section, we define the multidimensional Lipschitz classes and prove two versions of Boas-type theorem for the multidimensional Fourier-Bessel transform. We also provide an application to multidimensional Fourier-Bessel multipliers.

2. Generalized spherical mean operator

In this section we recall some basic results related to the multidimensional Fourier-Bessel harmonic analysis [2, 4, 5, 17, 18, 19, 20, 22].

Lemma 2.1. *(See* [15]*). For $x_k \in \mathbb{R}_+$ the following inequalities are satisfied.*

(i) $|j_{\alpha_k}(x_k)| \leq 1$.
(ii) $|1 - j_{\alpha_k}(x_k)| \leq x_k$.
(iii) $|1 - j_{\alpha_k}(x_k)| \geq c$ *with* $x_k \geq 1$, *where* $c > 0$ *is a certain constant which depends only on* α_k.

Proof. (i) For $\alpha_k > -\frac{1}{2}$, we have

$$j_{\alpha_k}(x_k) = \frac{2\Gamma(\alpha_k+1)}{\sqrt{\pi}\Gamma\left(\alpha_k+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\alpha_k-\frac{1}{2}} \cos(x_k t)\mathrm{d}t.$$

Then

$$|j_{\alpha_k}(x_k)| \leq \frac{2\Gamma(\alpha_k+1)}{\sqrt{\pi}\Gamma\left(\alpha_k+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\alpha_k-\frac{1}{2}} \mathrm{d}t \leq 1.$$

(ii) Let's start with the integral formula

$$1 - j_{\alpha_k}(x_k) = \frac{2\Gamma(\alpha_k+1)}{\sqrt{\pi}\Gamma\left(\alpha_k+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\alpha_k-\frac{1}{2}} [1-\cos(x_k t)]\mathrm{d}t.$$

Using the inequality $|1-\cos u| \leq |u|$, we obtain

$$|1-j_{\alpha_k}(x_k)| \leq \frac{2\Gamma(\alpha_k+1)}{\sqrt{\pi}\Gamma\left(\alpha_k+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\alpha_k-\frac{1}{2}} x_k t \mathrm{d}t$$
$$\leq \frac{2x_k\Gamma(\alpha_k+1)}{\sqrt{\pi}\Gamma\left(\alpha_k+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\alpha_k-\frac{1}{2}} \mathrm{d}t = x_k.$$

(iii) The asymptotic formula:

$$j_{\alpha_k}(x_k) \sim 2^{\alpha_k}\Gamma(\alpha_k+1)\frac{\cos\left(x_k-\frac{\pi}{2}\alpha_k-\frac{\pi}{4}\right)}{\sqrt{2\pi}x_k^{\alpha_k+\frac{1}{2}}} \quad \text{as} \quad x_k \to \infty,$$

imply that $\lim_{x_k\to\infty} j_{\alpha_k}(x_k) = 0$. Consequently, a number $A > 0$ exists such that with $x_k \geq A$ the inequality $|j_{\alpha_k}(x_k)| \leq \frac{1}{2}$ is true. Let $m = \min_{x_k\in[1,A]} |1-j_{\alpha_k}(x_k)|$. With $x_k \geq 1$ we get the inequality $|1-j_{\alpha_k}(x_k)| \geq c$, where $c = \min\left(\frac{1}{2}, m\right)$. □

Lemma 2.2. *The kernel $x \longrightarrow j_\alpha(\lambda, x)$ possesses the following properties.*

(i) $|j_\alpha(\lambda, x)| \leq 1$, *for* $\lambda, x \in \mathbb{R}_+^d$.

(ii) $j_\alpha(\lambda, rx) = j_\alpha(r\lambda, x)$, *for* $x, \lambda \in \mathbb{R}_+^d$, $r > 0$.

Proof. Part (i) follows from Lemma 2.1, while (ii) is obvious due to the definition (1.1). □

The kernel $j_\alpha(\lambda, y)$ gives rise to an integral transform, known as the multidimensional Fourier-Bessel transform on $\mathbb{R}_+^d$, for which many fundamental properties have been established [1]. The multidimensional Fourier-Bessel transform is given by (1.2). Some of the properties of multidimensional Fourier-Bessel transform $\mathscr{F}_\alpha$ are summarized below.

Theorem 2.1. *(See* [2, 17, 19]*).*

(i) $L^1 - L^\infty$*-boundedness for* $\mathscr{F}_\alpha$. *For all* $f \in L_\alpha^1(\mathbb{R}_+^d)$ *the function* $\lambda \longrightarrow \mathscr{F}_\alpha(f)(\lambda)$ *is continuous on* $\mathbb{R}_+^d$ *and satisfies*

$$\|\mathscr{F}_\alpha(f)\|_{L_\alpha^\infty(\mathbb{R}_+^d)} \leq \|f\|_{L_\alpha^1(\mathbb{R}_+^d)}.$$

(ii) *Plancherel formula for* $\mathscr{F}_\alpha$. *The transform* $\mathscr{F}_\alpha$ *extends uniquely to an isometric isomorphism on* $L_\alpha^2(\mathbb{R}_+^d)$, *onto itself. In particular,*

$$\|\mathscr{F}_\alpha(f)\|_{L_\alpha^2(\mathbb{R}_+^d)} = \|f\|_{L_\alpha^2(\mathbb{R}_+^d)}.$$

(iii) *Inversion formula for* $\mathscr{F}_\alpha$. *If* f *and* $\mathscr{F}_\alpha(f)$ *are both in* $L_\alpha^1(\mathbb{R}_+^d)$, *then*

$$f(x) = \int_{\mathbb{R}_+^d} j_\alpha(\lambda, x)\mathscr{F}_\alpha(f)(\lambda)\, d\mu_\alpha(\lambda), \quad a.e. \quad \lambda \in \mathbb{R}_+^d,$$

and

$$\mathscr{F}_\alpha^{-1}(f)(x) = \mathscr{F}_\alpha(f)(x).$$

We denote by $C_*(\mathbb{R}_+^d)$ the space of continuous functions on $\mathbb{R}_+^d$, that are even with respect to each variable. For $f \in C_*(\mathbb{R}_+^d)$ and $x, y \in \mathbb{R}_+^d$, we define the multidimensional Fourier-Bessel translation operator (see [2, 20, 30]) by

$$\tau_x f(y) := a_\alpha \int_{(0,\pi)^d} f\left([x_1, y_1]_{\theta_1}, \ldots, [x_d, y_d]_{\theta_d}\right) \times \prod_{k=1}^d (\sin\theta_k)^{2\alpha_k} \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_d,$$

where $[x_i, y_i]_{\theta_i} := \sqrt{x_i^2 + y_i^2 + 2x_i y_i \cos\theta_i}$, $i = 1, \dots, d$ and

$$a_\alpha := \prod_{k=1}^{d} \frac{\Gamma(\alpha_k + 1)}{\sqrt{\pi}\Gamma(\alpha_k + 1/2)}.$$

The following properties of the multidimensional Fourier-Bessel translation operator are established in [2].

(i) For suitable function f and for all $x, y \in \mathbb{R}_+^d$, we have

$$\tau_x f(y) = \tau_y f(x) \quad \text{and} \quad \tau_0 f(x) = f(x). \tag{2.1}$$

(ii) For all $\lambda, x, y \in \mathbb{R}_+^d$, we have the product formula

$$\tau_x(j_\alpha(\lambda, .))(y) = j_\alpha(\lambda, x) j_\alpha(\lambda, y).$$

(iii) For $f \in L_\alpha^p(\mathbb{R}_+^d)$, $p \in [1, \infty]$, and $x \in \mathbb{R}_+^d$, then $\tau_x f \in L_\alpha^p(\mathbb{R}_+^d)$ and

$$\|\tau_x f\|_{L_\alpha^p(\mathbb{R}_+^d)} \leq \|f\|_{L_\alpha^p(\mathbb{R}_+^d)}. \tag{2.2}$$

(iv) For $f \in L_\alpha^p(\mathbb{R}_+^d), p = 1, 2$ and $x \in \mathbb{R}_+^d$, we have

$$\mathscr{F}_\alpha(\tau_x f)(\lambda) = j_\alpha(\lambda, x)\mathscr{F}_\alpha(f)(\lambda), \quad \lambda \in \mathbb{R}_+^d. \tag{2.3}$$

We denote by $\mathbb{S}_+^{d-1} = \{x \in \mathbb{R}_+^d : |x| = 1\}$, and $\mathrm{d}\sigma_\alpha(x) = w_\alpha(x)\mathrm{d}\sigma(x)$, where $\mathrm{d}\sigma$ is the normalized surface measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, we have

$$d_\alpha^+ = \sigma_\alpha(\mathbb{S}_+^{d-1}) = \frac{1}{2^{\langle\alpha\rangle+d-1}\Gamma(\langle\alpha\rangle + d)},$$

where $\langle\alpha\rangle = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

We put $\lambda = ty$, with $t > 0$ and $y \in \mathbb{S}_+^{d-1}$, we obtain [21, page 165]

$$\int_{\mathbb{R}_+^d} \psi(\lambda)\mathrm{d}\mu_\alpha(\lambda) = \int_0^\infty \left[\int_{\mathbb{S}_+^{d-1}} \psi(ty)\mathrm{d}\sigma_\alpha(y)\right] t^{2\langle\alpha\rangle+2d-1}\mathrm{d}t. \tag{2.4}$$

The generalized spherical mean operator associated with the multidimensional Fourier-Bessel operator is defined for $f \in L_\alpha^2\left(\mathbb{R}_+^d\right)$ by

$$\mathscr{M}_{r,\alpha} f(x) := \frac{1}{d_\alpha^+} \int_{\mathbb{S}_+^{d-1}} \tau_x(f)(ry)\mathrm{d}\sigma_\alpha(y), \quad x \in \mathbb{R}_+^d, \quad r > 0. \tag{2.5}$$

Lemma 2.3. *For $\lambda \in \mathbb{R}_+^d$, we have*

$$\frac{1}{d_\alpha^+} \int_{\mathbb{S}_+^{d-1}} j_\alpha(\lambda, y) d\sigma_\alpha(y) = j_{\langle\alpha\rangle+d-1}(|\lambda|).$$

Proof. See Statement 5 of [21, (3.140)]. □

Theorem 2.2. *Let $f \in L_\alpha^p(\mathbb{R}_+^d)$, $p = 1, 2$. Then*

$$\mathscr{F}_\alpha(\mathscr{M}_{r,\alpha} f)(\lambda) = j_{\langle\alpha\rangle+d-1}(r|\lambda|)\mathscr{F}_\alpha(f)(\lambda).$$

In particular, if $f \in L_\alpha^2(\mathbb{R}_+^d)$, then $\mathscr{M}_{r,\alpha} f \in L_\alpha^2(\mathbb{R}_+^d)$.

Proof. From (2.1) and (2.2) we see that $\mathscr{M}_{r,\alpha}f \in L^2_\alpha(\mathbb{R}^d_+)$. Using (2.5) we obtain

$$\begin{aligned}\mathscr{F}_\alpha(\mathscr{M}_{r,\alpha}f)(\lambda) &= \int_{\mathbb{R}^d_+} \mathscr{M}_{r,\alpha}f(x)j_\alpha(\lambda,x)\mathrm{d}\mu_\alpha(x)\\ &= \frac{1}{d^+_\alpha}\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}_+} \tau_x(f)(ry)\mathrm{d}\sigma_\alpha(y)j_\alpha(\lambda,x)\mathrm{d}\mu_\alpha(x).\end{aligned}$$

Using Fubini's theorem and (2.1), we obtain

$$\begin{aligned}\mathscr{F}_\alpha(\mathscr{M}_{r,\alpha}f)(\lambda) &= \frac{1}{d^+_\alpha}\int_{\mathbb{S}^{d-1}_+}\left(\int_{\mathbb{R}^d_+} \tau_x(f)(ry)j_\alpha(\lambda,x)\mathrm{d}\mu_\alpha(x)\right)\mathrm{d}\sigma_\alpha(y)\\ &= \frac{1}{d^+_\alpha}\int_{\mathbb{S}^{d-1}_+}\left(\int_{\mathbb{R}^d_+} \tau_{ry}(f)(x)j_\alpha(\lambda,x)\mathrm{d}\mu_\alpha(x)\right)\mathrm{d}\sigma_\alpha(y).\end{aligned}$$

By (1.2) and (2.3) we deduce that

$$\begin{aligned}\mathscr{F}_\alpha(\mathscr{M}_{r,\alpha}f)(\lambda) &= \frac{1}{d^+_\alpha}\int_{\mathbb{S}^{d-1}_+} \mathscr{F}_\alpha(\tau_{ry}f)(\lambda)\mathrm{d}\sigma_\alpha(y)\\ &= \left(\frac{1}{d^+_\alpha}\int_{\mathbb{S}^{d-1}_+} j_\alpha(\lambda,ry)\mathrm{d}\sigma_\alpha(y)\right)\mathscr{F}_\alpha(f)(\lambda).\end{aligned}$$

Using Lemma 2.2 (ii) and Lemma 2.3 we obtain

$$\begin{aligned}\mathscr{F}_\alpha(\mathscr{M}_{r,\alpha}f)(\lambda) &= \left(\frac{1}{d^+_\alpha}\int_{\mathbb{S}^{d-1}_+} j_\alpha(r\lambda,y)\mathrm{d}\sigma_\alpha(y)\right)\mathscr{F}_\alpha(f)(\lambda)\\ &= j_{\langle\alpha\rangle+d-1}(r|\lambda|)\mathscr{F}_\alpha(f)(\lambda).\end{aligned}$$

This completes the proof of the theorem. □

Finite differences of first and higher orders are defined as follows.

$$\mathscr{D}_{r,\alpha}f(x) := \left(\mathscr{M}_{r,\alpha} - I\right)f(x),$$

$$\begin{aligned}\mathscr{D}^k_{r,\alpha}f(x) &= \mathscr{D}_{r,\alpha}\left(\mathscr{D}^{k-1}_{r,\alpha}f\right)(x) = \left(\mathscr{M}_{r,\alpha} - I\right)^k f(x)\\ &= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\mathscr{M}^i_{r,\alpha}f(x), \quad k = 1,2,\ldots,\end{aligned}$$

where

$$\mathscr{M}^0_{r,\alpha}f(x) = f(x), \quad \mathscr{M}^i_{r,\alpha}f(x) = \mathscr{M}_{r,\alpha}\left(\mathscr{M}^{i-1}_{r,\alpha}f\right)(x), \quad i = 1,2,\ldots,k,$$

and I is the identity operator.

Lemma 2.4. *We have*

(i) *For* $f \in L^2_\alpha(\mathbb{R}^d_+)$ *we have*

$$\mathscr{F}_\alpha(\mathscr{D}^k_{r,\alpha}f)(\lambda) = \left(j_{\langle\alpha\rangle+d-1}(r|\lambda|) - 1\right)^k \mathscr{F}_\alpha(f)(\lambda), \quad \lambda \in \mathbb{R}^d_+,$$

and

$$\left\|\mathscr{D}^k_{r,\alpha}f\right\|^2_{L^2_\alpha(\mathbb{R}^d_+)} = \int_{\mathbb{R}^d_+}\left|1 - j_{\langle\alpha\rangle+d-1}(r|\lambda|)\right|^{2k}|\mathscr{F}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

(ii) *Let* $f \in L^1_\alpha(\mathbb{R}^d_+)$ *such that* $\mathscr{F}_\alpha(f) \in L^1_\alpha(\mathbb{R}^d_+)$, *then*

$$\mathscr{D}^k_{r,\alpha} f(x) = \int_{\mathbb{R}^d_+} (j_{\langle\alpha\rangle+d-1}(r|\lambda|) - 1)^k \mathscr{F}_\alpha(f)(\lambda) j_\alpha(\lambda, x) d\mu_\alpha(\lambda), \quad a.e. \quad x \in \mathbb{R}^d_+.$$

Proof. (i) follows from Theorem 2.1 (ii) and Theorem 2.2.
(ii) follows from Theorem 2.1 (iii) and Theorem 2.2. □

3. Generalized Titchmarsh-type theorem

In this section we define the k-Lipschitz class and then we give Titchmarsh-type theorem for the multidimensional Fourier-Bessel transform $\mathscr{F}_\alpha$.

Let $\beta \in (0, 1)$. A function $f \in L^2_\alpha\left(\mathbb{R}^d_+\right)$ is said to belong to the multidimensional Fourier-Bessel k-Lipschitz class, denoted by $Lip_k(\beta, 2)$ if:

$$\left\|\mathscr{D}^k_{r,\alpha} f\right\|_{L^2_\alpha\left(\mathbb{R}^d_+\right)} = O(r^\beta) \quad \text{as} \quad r \to 0.$$

Theorem 3.1. *Let* $f \in L^2_\alpha\left(\mathbb{R}^d_+\right)$. *Then the following assertions are equivalent*

(i) $f \in Lip_k(\beta, 2)$.

(ii) $\int_{|\lambda|\geq s} |\mathscr{F}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) = O\left(s^{-2\beta}\right) \quad$ *as* $\quad s \to \infty$.

Proof. Let $f \in L^2_\alpha\left(\mathbb{R}^d_+\right)$.

(i) $\Longrightarrow$ (ii). Suppose that $f \in Lip_k(\beta, 2)$. Then we have

$$\left\|\mathscr{D}^k_{r,\alpha} f\right\|_{L^2_\alpha\left(\mathbb{R}^d_+\right)} = O(r^\beta) \quad \text{as} \quad r \to 0.$$

By Lemma 2.4 (i), we have

$$\left\|\mathscr{D}^k_{r,\alpha} f\right\|^2_{L^2_\alpha\left(\mathbb{R}^d_+\right)} = \int_{\mathbb{R}^d_+} \left|1 - j_{\langle\alpha\rangle+d-1}(r|\lambda|)\right|^{2k} |\mathscr{F}_\alpha(f)(\lambda)|^2 \mathrm{d}\mu_\alpha(\lambda).$$

If $|\lambda| \in \left[\frac{1}{r}, \frac{2}{r}\right]$ then $r|\lambda| \geq 1$ and Lemma 2.1 (iii), we have

$$\int_{\frac{1}{r}\leq|\lambda|\leq\frac{2}{r}} \left|1 - j_{\langle\alpha\rangle+d-1}(r|\lambda|)\right|^{2k} |\mathscr{F}_\alpha(f)(\lambda)|^2 \mathrm{d}\mu_\alpha(\lambda) \geq c^{2k} \int_{\frac{1}{r}\leq|\lambda|\leq\frac{2}{r}} |\mathscr{F}_\alpha(f)(\lambda)|^2 \mathrm{d}\mu_\alpha(\lambda).$$

Then there exists a positive constant C such that

$$\begin{aligned}\int_{\frac{1}{r}\leq|\lambda|\leq\frac{2}{r}} |\mathscr{F}_\alpha(f)(\lambda)|^2 \mathrm{d}\mu_\alpha(\lambda) &\leq C \int_{\frac{1}{r}\leq|\lambda|\leq\frac{2}{r}} \left|1 - j_{\langle\alpha\rangle+d-1}(r|\lambda|)\right|^{2k} |\mathscr{F}_\alpha(f)(\lambda)|^2 \mathrm{d}\mu_\alpha(\lambda) \\ &\leq C \int_{\mathbb{R}^d_+} \left|1 - j_{\langle\alpha\rangle+d-1}(r|\lambda|)\right|^{2k} |\mathscr{F}_\alpha(f)(\lambda)|^2 \mathrm{d}\mu_\alpha(\lambda) \\ &\leq C r^{2\beta}.\end{aligned}$$

For all $s > 0$, we obtain

$$\int_{s\leq|\lambda|\leq 2s} |\mathscr{F}_\alpha(f)(\lambda)|^2 \mathrm{d}\mu_\alpha(\lambda) \leq C s^{-2\beta}.$$

So that

$$\begin{aligned}\int_{|\lambda|\geq s}|\mathscr{F}_\alpha(f)(\lambda)|^2\mathrm{d}\mu_\alpha(\lambda) &= \sum_{i=0}^{\infty}\int_{2^i s\leq|\lambda|\leq 2^{i+1}s}|\mathscr{F}_\alpha(f)(\lambda)|^2\mathrm{d}\mu_\alpha(\lambda)\\ &= O\left(s^{-2\beta}+(2s)^{-2\beta}+\ldots+(2^i s)^{-2\beta}+\ldots\right)\\ &= O\left(s^{-2\beta}\right).\end{aligned}$$

This proves that

$$\int_{|\lambda|\geq s}|\mathscr{F}_\alpha(f)(\lambda)|^2\mathrm{d}\mu_\alpha(\lambda) = O\left(s^{-2\beta}\right) \quad \text{as} \quad s\to\infty.$$

(ii) $\Longrightarrow$ (i). Suppose now that

$$\int_{|\lambda|\geq s}|\mathscr{F}_\alpha(f)(\lambda)|^2\mathrm{d}\mu_\alpha(\lambda) = O\left(s^{-2\beta}\right) \quad \text{as} \quad s\to\infty. \tag{3.1}$$

By Lemma 2.4 (i) and (2.4) we have

$$\begin{aligned}\|\mathscr{D}^k_{r,\alpha}f\|^2_{L^2_\alpha(\mathbb{R}^d_+)} &= \int_{\mathbb{R}^d_+}|1-j_{\langle\alpha\rangle+d-1}(r|\lambda|)|^{2k}|\mathscr{F}_\alpha(f)(\lambda)|^2\mathrm{d}\mu_\alpha(\lambda)\\ &= \int_0^\infty t^{2\langle\alpha\rangle+2d-1}|1-j_{\langle\alpha\rangle+d-1}(tr)|^{2k}\varphi(t)\mathrm{d}t,\end{aligned}$$

where

$$\varphi(t) = \int_{\mathbb{S}^{d-1}_+}|\mathscr{F}_\alpha(f)(ty)|^2\mathrm{d}\sigma_\alpha(y).$$

We have to show that

$$\int_0^\infty t^{2\langle\alpha\rangle+2d-1}\left|1-j_{\langle\alpha\rangle+d-1}(tr)\right|^{2k}\varphi(t)\mathrm{d}t = O\left(r^{2\beta}\right) \quad \text{as} \quad r\to 0.$$

We write

$$\int_0^\infty t^{2\langle\alpha\rangle+2d-1}\left|1-j_{\langle\alpha\rangle+d-1}(tr)\right|^{2k}\varphi(t)\mathrm{d}t = \mathrm{I}_1+\mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_0^{\frac{1}{r}} t^{2\langle\alpha\rangle+2d-1}\left|1-j_{\langle\alpha\rangle+d-1}(tr)\right|^{2k}\varphi(t)\mathrm{d}t$$

and

$$\mathrm{I}_2 = \int_{\frac{1}{r}}^\infty t^{2\langle\alpha\rangle+2d-1}\left|1-j_{\langle\alpha\rangle+d-1}(tr)\right|^{2k}\varphi(t)\mathrm{d}t.$$

From Lemma 2.1 (i), (2.4) and (3.1) we have

$$\begin{aligned}\mathrm{I}_2 &\leq 2^{2k}\int_{\frac{1}{r}}^\infty t^{2\langle\alpha\rangle+2d-1}\varphi(t)\mathrm{d}t\\ &= 2^{2k}\int_{|\lambda|\geq\frac{1}{r}}|\mathscr{F}_\alpha(f)(\lambda)|^2\mathrm{d}\mu_\alpha(\lambda)\\ &= O\left(r^{2\beta}\right) \quad \text{as} \quad r\to 0.\end{aligned}$$

To estimate I_1, we use Lemma 2.1 (i) and (ii), whence we deduce that

$$\begin{aligned} \mathrm{I}_1 &\le 2^{2k-2}\int_0^{\frac{1}{r}} t^{2\langle\alpha\rangle+2d-1}\left|1-j_{\langle\alpha\rangle+d-1}(tr)\right|^2\varphi(t)\mathrm{d}t \\ &= 2^{2k-2}r^2\int_0^{\frac{1}{r}} t^{2\langle\alpha\rangle+2d+1}\varphi(t)\mathrm{d}t. \end{aligned}$$

We define

$$g(t)=\int_t^{\infty} s^{2\langle\alpha\rangle+2d-1}\varphi(s)\mathrm{d}s=O(t^{-2\beta}) \quad \text{as} \quad t\longrightarrow\infty.$$

Then, using integration by parts, we find

$$\begin{aligned} I_1 &\le -2^{2k-2}r^2\int_0^{\frac{1}{r}} t^2 g'(t)\mathrm{d}t \\ &\le -2^{2k-2}g\left(\frac{1}{r}\right)+2^{2k-1}r^2\int_0^{\frac{1}{r}} tg(t)\mathrm{d}t \\ &= O(r^{2\beta})+O\left(r^2\int_0^{\frac{1}{r}} t^{1-2\beta}\mathrm{d}t\right)=O(r^{2\beta}). \end{aligned}$$

Finally, we conclude that

$$\left\|\mathscr{D}_{r,\alpha}^k f\right\|_{L^2_\alpha(\mathbb{R}^d_+)}^2=O(r^{2\beta}) \quad \text{as} \quad r\to 0.$$

This completes the proof of the theorem. □

Let T_{m_1} be a multiplier operator defined by the relation

$$\mathscr{F}_\alpha(T_{m_1}f)(\lambda)=m_1(\lambda)\mathscr{F}_\alpha(f)(\lambda), \tag{3.2}$$

where m_1 is a measurable function on $\mathbb{R}^d_+$ satisfying the following condition: there exist $\varepsilon, c>0$ such that, for all $|\lambda|\ge 1$,

$$|m_1(\lambda)|^2\le c|\lambda|^{-2\varepsilon}. \tag{3.3}$$

Theorem 3.2. *Let m_1 be a multiplier satisfying* (3.3), *then the operator T_{m_1} is bounded from $Lip_k(\beta,2)$ into $Lip_k(\beta+\varepsilon,2)$, for $0<\beta\le 1-\varepsilon$.*

Proof. Let $f\in Lip_k(\beta,2)$. Then by (3.2) and Theorem 3.1 we have

$$\begin{aligned} \int_{|\lambda|\ge s}|\mathscr{F}_\alpha(T_{m_1}f)(\lambda)|^2\mathrm{d}\mu_\alpha(\lambda) &= \int_{|\lambda|\ge s}|m_1(\lambda)|^2|\mathscr{F}_\alpha(f)(\lambda)|^2\mathrm{d}\mu_\alpha(\lambda) \\ &\le c\int_{|\lambda|\ge s}|\lambda|^{-2\varepsilon}|\mathscr{F}_\alpha(f)(\lambda)|^2\mathrm{d}\mu_\alpha(\lambda) \\ &\le cs^{-2\varepsilon}\int_{|\lambda|\ge s}|\mathscr{F}_\alpha(f)(\lambda)|^2\mathrm{d}\mu_\alpha(\lambda) \\ &= O(s^{-2(\beta+\varepsilon)}) \quad \text{as} \quad s\to\infty. \end{aligned}$$

Again using Theorem 3.1, we obtain

$$T_{m_1}f\in Lip_k(\beta+\varepsilon,2).$$

This completes the proof of the theorem. □

Example 3.1. (Generalized Bessel potential). Let $0<\sigma<1$, we put

$$m_{1,\sigma}(\lambda) := (1+|\lambda|^2)^{-\sigma/2}, \quad \lambda \in \mathbb{R}^d_+.$$

The operator $T_{m_{1,\sigma}} = (I-\Delta_\alpha)^{-\sigma/2}$ is a generalized Bessel potential. Then by application of Theorem 3.2, we deduce that

$$(I-\Delta_\alpha)^{-\sigma/2} : Lip_k(\beta,2) \longrightarrow Lip_k(\beta+\sigma,2)$$

is bounded, for $0<\beta\leq 1-\sigma$.

4. Investigation on Boas-type theorems

In this section, we prove two versions of Boas-type theorem for the multidimensional Fourier-Bessel transform $\mathscr{F}_\alpha$, and provide an application to multidimensional Fourier-Bessel multiplier operators.

Let f be a measurable function defined on $\mathbb{R}^d_+$ and $\gamma\in\mathbb{R}$.

Lemma 4.1. *If $0<\beta\leq\gamma$, $|\lambda|^\gamma|f(\lambda)|\in L^1_\alpha(\mathbb{R}^d_+)$ and*

$$\int_{|\lambda|\leq s} |\lambda|^\gamma |f(\lambda)|\,d\mu_\alpha(\lambda) = O(s^{\gamma-\beta}) \quad \textit{for all} \quad s>0, \tag{4.1}$$

then $f\chi_{|\lambda|\geq s}\in L^1_\alpha(\mathbb{R}^d_+)$ and

$$\int_{|\lambda|\geq s} |f(\lambda)|\,d\mu_\alpha(\lambda) = O(s^{-\beta}) \quad \textit{for all} \quad s>0. \tag{4.2}$$

Proof. By (4.1), there exists a constant $C>0$ such that for all $i\in\mathbb{Z}$, we have

$$\int_{2^i\leq|\lambda|\leq 2^{i+1}} |\lambda|^\gamma|f(\lambda)|\mathrm{d}\mu_\alpha(\lambda) \leq \int_{|\lambda|\leq 2^{i+1}} |\lambda|^\gamma|f(\lambda)|\mathrm{d}\mu_\alpha(\lambda) \leq C2^{(\gamma-\beta)(i+1)}.$$

It is clear that

$$2^{i\gamma}\int_{2^i\leq|\lambda|\leq 2^{i+1}} |f(\lambda)|\mathrm{d}\mu_\alpha(\lambda) \leq \int_{|\lambda|\leq 2^{i+1}} |\lambda|^\gamma|f(\lambda)|\mathrm{d}\mu_\alpha(\lambda) \leq C2^{(\gamma-\beta)(i+1)},$$

hence

$$\int_{2^i\leq|\lambda|\leq 2^{i+1}} |f(\lambda)|\mathrm{d}\mu_\alpha(\lambda) \leq C2^{\gamma-\beta}2^{-i\beta}. \tag{4.3}$$

Then by (4.3) we obtain

$$\begin{aligned}\int_{|\lambda|\geq 2^i} |f(\lambda)|\mathrm{d}\mu_\alpha(\lambda) &= \sum_{j=i}^{\infty}\int_{2^j\leq|\lambda|\leq 2^{j+1}} |f(\lambda)|\mathrm{d}\mu_\alpha(\lambda)\\ &\leq C2^{\gamma-\beta}\sum_{j=i}^{\infty}2^{-j\beta} = O(2^{-i\beta}).\end{aligned}$$

When $0<s<\infty$, let $i\in\mathbb{Z}$ such that $2^i\leq s<2^{i+1}$. It follows that

$$\begin{aligned}\int_{|\lambda|\geq s} |f(\lambda)|\mathrm{d}\mu_\alpha(\lambda) &\leq \int_{|\lambda|\geq 2^i} |f(\lambda)|\mathrm{d}\mu_\alpha(\lambda)\\ &\leq C2^{-i\beta} = C2^\beta 2^{-(i+1)\beta}\\ &\leq C2^\beta s^{-\beta}.\end{aligned}$$

This proves (4.2) in the general case. □

Lemma 4.2. *If* $0 < \beta \le \gamma$, $|\lambda|^\gamma |f(\lambda)| \in L^1_\alpha(\mathbb{R}^d_+)$ *and*

$$\int_{|\lambda|\le s} |\lambda|^\gamma |f(\lambda)| d\mu_\alpha(\lambda) = o(s^{\gamma-\beta}) \quad \textit{for all} \quad s > 0, \tag{4.4}$$

then $f\chi_{|\lambda|\ge s} \in L^1_\alpha(\mathbb{R}^d_+)$ *and*

$$\int_{|\lambda|\ge s} |f(\lambda)| d\mu_\alpha(\lambda) = o(s^{-\beta}) \quad \textit{for all} \quad s > 0. \tag{4.5}$$

Proof. By (4.4), for every $\varepsilon > 0$ there exists $i_0 \in \mathbb{Z}$ such that for all $i \ge i_0$ we have

$$\int_{2^i\le|\lambda|\le 2^{i+1}} |\lambda|^\gamma |f(\lambda)| \mathrm{d}\mu_\alpha(\lambda) \le \int_{|\lambda|\le 2^{i+1}} |\lambda|^\gamma |f(\lambda)| \mathrm{d}\mu_\alpha(\lambda) \le \varepsilon.2^{(\gamma-\beta)(i+1)}.$$

Let $i \ge i_0$. It is clear that

$$2^{i\gamma} \int_{2^i\le|\lambda|\le 2^{i+1}} |f(\lambda)| \mathrm{d}\mu_\alpha(\lambda) \le \int_{|\lambda|\le 2^{i+1}} |\lambda|^\gamma |f(\lambda)| \mathrm{d}\mu_\alpha(\lambda) \le \varepsilon.2^{(\gamma-\beta)(i+1)},$$

hence

$$\int_{2^i\le|\lambda|\le 2^{i+1}} |f(\lambda)| \mathrm{d}\mu_\alpha(\lambda) \le \varepsilon.2^{\gamma-\beta} 2^{-i\beta}. \tag{4.6}$$

Then by (4.6) we obtain

$$\begin{aligned}\int_{|\lambda|\ge 2^i} |f(\lambda)| \mathrm{d}\mu_\alpha(\lambda) &= \sum_{j=i}^{\infty} \int_{2^j\le|\lambda|\le 2^{j+1}} |f(\lambda)| \mathrm{d}\mu_\alpha(\lambda) \\ &\le \varepsilon.2^{\gamma-\beta} \sum_{j=i}^{\infty} 2^{-j\beta} \\ &\le C.\varepsilon.2^{-i\beta}.\end{aligned}$$

When $0 < s < \infty$, let $i \in \mathbb{Z}$ such that $2^i \le s < 2^{i+1}$. It follows that

$$\begin{aligned}\int_{|\lambda|\ge s} |f(\lambda)| \mathrm{d}\mu_\alpha(\lambda) &\le \int_{|\lambda|\ge 2^i} |f(\lambda)| \mathrm{d}\mu_\alpha(\lambda) \\ &\le C.\varepsilon.2^{-i\beta} = C.\varepsilon.2^\beta 2^{-(i+1)\beta} \\ &\le C.\varepsilon.2^\beta s^{-\beta}.\end{aligned}$$

This proves (4.5) in the general case. □

We now define generalized Lipschitz classes $Lip_k(\beta)$ and $lip_k(\beta)$.
A function $f : \mathbb{R}^d_+ \to \mathbb{R}$ is said to belong to $Lip_k(\beta)$ for $\beta > 0$ if

$$|\mathscr{D}^k_{r,\alpha} f(x)| = O(r^\beta) \quad \text{as} \quad r \to 0,$$

and is said to belong to $lip_k(\beta)$ for $\beta > 0$ if

$$|\mathscr{D}^k_{r,\alpha} f(x)| = o(r^\beta) \quad \text{as} \quad r \to 0.$$

The spaces $Lip_1(\beta)$ and $lip_1(\beta)$, for $\beta > 0$, are called respectively the Lipschitz class $Lip(\beta)$ and the little Lipschitz class $\mathrm{lip}(\beta)$. The spaces $Lip_2(\beta)$ and $lip_2(\beta)$, for $\beta > 0$, are called respectively the Zygmund class $Zyg(\beta)$ and the little Zygmund class $zyg(\beta)$.

Theorem 4.1. *Let* $f : \mathbb{R}_+^d \to \mathbb{R}$, $k \in \mathbb{N}$ *and* $0 < \beta \leq k$. *Suppose that* $f \in L_\alpha^1(\mathbb{R}_+^d)$. *If*

$$\int_{|\lambda|\leq s} |\lambda|^k |\mathscr{F}_\alpha(f)(\lambda)| d\mu_\alpha(\lambda) = O(s^{k-\beta}) \quad \text{for all} \quad s > 0, \tag{4.7}$$

then $\mathscr{F}_\alpha(f) \in L_\alpha^1(\mathbb{R}_+^d)$ *and* $f \in Lip_k(\beta)$.

Proof. From Lemma 4.1, statement (4.7) implies that

$$\int_{|\lambda|\geq s} |\mathscr{F}_\alpha(f)(\lambda)| \mathrm{d}\mu_\alpha(\lambda) < \infty.$$

Since $\mathscr{F}_\alpha(f) \in C(\mathbb{R}_+^d)$, then $\mathscr{F}_\alpha(f) \in L_\alpha^1(\mathbb{R}_+^d)$.
Now, let us prove that $f \in Lip_k(\beta)$. Let $x \in \mathbb{R}_+^d$ and $r > 0$. By Lemma 2.4 (ii) we have

$$\begin{aligned}
|\mathscr{D}_{r,\alpha}^k f(x)| &= \left| \int_{\mathbb{R}_+^d} (j_{\langle\alpha\rangle+d-1}(r|\lambda|) - 1)^k \mathscr{F}_\alpha(f)(\lambda) j_\alpha(\lambda, x) \mathrm{d}\mu_\alpha(\lambda) \right| \\
&\leq \int_{\mathbb{R}_+^d} |j_{\langle\alpha\rangle+d-1}(r|\lambda|) - 1|^k |\mathscr{F}_\alpha(f)(\lambda)| \mathrm{d}\mu_\alpha(\lambda) \\
&= \int_0^\infty t^{2\langle\alpha\rangle+2d-1} \left|1 - j_{\langle\alpha\rangle+d-1}(tr)\right|^k \phi(t) \mathrm{d}t,
\end{aligned}$$

where

$$\phi(t) = \int_{\mathbb{S}_+^{d-1}} |\mathscr{F}_\alpha(f)(ty)| \mathrm{d}\sigma_\alpha(y).$$

We write

$$|\mathscr{D}_{r,\alpha}^k f(x)| \leq J_1 + J_2, \tag{4.8}$$

where

$$J_1 = \int_0^{\frac{1}{r}} t^{2\langle\alpha\rangle+2d-1} \left|1 - j_{\langle\alpha\rangle+d-1}(tr)\right|^k \phi(t) \mathrm{d}t$$

and

$$J_2 = \int_{\frac{1}{r}}^\infty t^{2\langle\alpha\rangle+2d-1} \left|1 - j_{\langle\alpha\rangle+d-1}(tr)\right|^k \phi(t) \mathrm{d}t.$$

By Lemma 2.1 (ii) we get

$$J_1 \leq r^k \int_0^{\frac{1}{r}} t^{2\langle\alpha\rangle+2d-1} t^k \phi(t) \mathrm{d}t = r^k \int_{|\lambda|\leq\frac{1}{r}} |\lambda|^k |\mathscr{F}_\alpha(f)(\lambda)| \mathrm{d}\mu_\alpha(\lambda), \tag{4.9}$$

and by (4.7) we deduce that

$$J_1 = r^k O(r^{\beta-k}) = O(r^\beta). \tag{4.10}$$

On the other hand, by Lemma 2.1 (i), we have

$$J_2 \leq 2^k \int_{\frac{1}{r}}^\infty t^{2\langle\alpha\rangle+2d-1} \phi(t) \mathrm{d}t = 2^k \int_{|\lambda|\geq\frac{1}{r}} |\mathscr{F}_\alpha(f)(\lambda)| \mathrm{d}\mu_\alpha(\lambda). \tag{4.11}$$

By (4.7) and Lemma 4.1, we get

$$J_2 = O(r^\beta). \tag{4.12}$$

Combining (4.10) and (4.12) yields that $f \in Lip_k(\beta)$. □

Theorem 4.2. *Let $f : \mathbb{R}^d_+ \to \mathbb{R}$, $k \in \mathbb{N}$ and $0 < \beta \leq k$. Suppose that $f \in L^1_\alpha(\mathbb{R}^d_+)$. If*

$$\int_{|\lambda|\leq s} |\lambda|^k |\mathscr{F}_\alpha(f)(\lambda)| d\mu_\alpha(\lambda) = o(s^{k-\beta}) \quad as \quad s \to \infty, \tag{4.13}$$

then $\mathscr{F}_\alpha(f) \in L^1_\alpha(\mathbb{R}^d_+)$ and $f \in lip_k(\beta)$.

Proof. From Lemma 4.2, statement (4.13) implies that

$$\int_{|\lambda|\leq s} |\mathscr{F}_\alpha(f)(\lambda)| \mathrm{d}\mu_\alpha(\lambda) < \infty \quad \text{as} \quad s \to \infty.$$

Then $\mathscr{F}_\alpha(f) \in L^1_\alpha(\mathbb{R}^d_+)$.
Now, let us prove that $f \in \mathrm{lip}_k(\beta)$. Again, we start with the estimate (4.8).
By (4.9) and (4.13) we conclude that

$$J_1 = r^k o(r^{\beta-k}) = o(r^\beta) \quad \text{as} \quad r \to 0. \tag{4.14}$$

On the other hand, by Lemma 4.2, (4.11) and (4.13), we obtain

$$J_2 = o(r^\beta) \quad \text{as} \quad r \to 0. \tag{4.15}$$

Combining (4.14) and (4.15) shows that $f \in lip_k(\beta)$. □

Let m_2 be a multidimensional Fourier-Bessel multiplier function on $\mathbb{R}^d_+$ satisfying: there exist $\varepsilon, c > 0$ such that, for all $\lambda \in \mathbb{R}^d_+$,

$$|m_2(\lambda)| \leq c|\lambda|^\varepsilon. \tag{4.16}$$

Theorem 4.3. *Let m_2 be a multiplier satisfying (4.16), then*

(i) *If $f \in L^1_\alpha(\mathbb{R}^d_+)$ and satisfies (4.7), then $T_{m_2} f \in Lip_k(\beta-\varepsilon)$, for $\varepsilon < \beta \leq k$.*
(ii) *If $f \in L^1_\alpha(\mathbb{R}^d_+)$ and satisfies (4.13), then $T_{m_2} f \in lip_k(\beta-\varepsilon)$, for $\varepsilon < \beta \leq k$.*

Proof. Let $f \in L^1_\alpha(\mathbb{R}^d_+)$ and satisfies (4.7). Then, by Theorem 4.1, we have

$$\begin{aligned}
\int_{|\lambda|\leq s} |\lambda|^k |\mathscr{F}_\alpha(T_{m_2} f)(\lambda)| \mathrm{d}\mu_\alpha(\lambda) &= \int_{|\lambda|\leq s} |\lambda|^k |m_2(\lambda)| |\mathscr{F}_\alpha(f)(\lambda)| \mathrm{d}\mu_\alpha(\lambda) \\
&\leq c \int_{|\lambda|\leq s} |\lambda|^\varepsilon |\lambda|^k |\mathscr{F}_\alpha(f)(\lambda)| \mathrm{d}\mu_\alpha(\lambda) \\
&\leq c.s^\varepsilon \int_{|\lambda|\leq s} |\lambda|^k |\mathscr{F}_\alpha(f)(\lambda)| \mathrm{d}\mu_\alpha(\lambda) \\
&= O(s^{k-(\beta-\varepsilon)}) \quad \text{for all} \quad s > 0.
\end{aligned}$$

Again by applying Theorem 4.1, we get

$$T_{m_2} f \in Lip_k(\beta - \varepsilon).$$

This completes the proof of (i).
The proof of (ii) follows in the same way as (i). □

Example 4.1. (Real powers of the operator $(-\Delta_\alpha)$). Let $\sigma > 0$, we define

$$m_{2,\sigma}(\lambda) := |\lambda|^\sigma, \quad \lambda \in \mathbb{R}^d_+.$$

The operator $T_{m_{2,\sigma}} = (-\Delta_\alpha)^{\sigma/2}$ is a real power of the operator $(-\Delta_\alpha)$. Then, by applying Theorem 4.3, we deduce that

(i) If $f \in L^1_\alpha(\mathbb{R}^d_+)$ and satisfies (4.7), then $(-\Delta_\alpha)^{\sigma/2} \in Lip_k(\beta-\sigma)$, for $\sigma < \beta \leq k$.

(ii) If $f \in L^1_\alpha(\mathbb{R}^d_+)$ and satisfies (4.13), then $(-\Delta_\alpha)^{\sigma/2} \in lip_k(\beta-\sigma)$, for $\sigma < \beta \leq k$.

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