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INVESTIGATION OF A SYSTEM OF HYPERBOLIC EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. The article studies a system of hyperbolic equations defined by point and integral boundary conditions, which is a perturbation of the classical Goursat-Darboux problem. Necessary conditions for the solvability of the problem are found. The Green function of the boundary value problem is constructed and the boundary value problem is reduced to an equivalent integral equation. Using the Banach contraction mapping principle, conditions for the existence and uniqueness of a solution to the boundary value problem are found. Specific examples are given illustrating the validity of the results obtained.

1. Introduction and Problem Statement

The theory of nonlocal boundary value problems is developing intensively and it is an important section of the theory of partial differential equations. Of great interest in this area are problems with nonlocal integral conditions. Such problems serve as a convenient way to describe the conditions on the desired solution in cases when it is impossible to directly measure important physical quantities on the boundary of the region, but their average value inside is known. Integral nonlocal conditions, in a sense, can be considered as a generalization of discrete nonlocal conditions. Note that nonlocal boundary value problems are usually called problems in which conditions are specified that relate the values of the desired solution and (or) its derivatives at different points of the boundary, or at points of the boundary and at some interior points. Problems of this type arise in the mathematical modeling of various physical, chemical, biological and environmental phenomena, when instead of classical boundary conditions a certain relationship is specified between the values of the desired function on the boundary of the region and inside it. Similar situations occur, for example, in the study of: phenomena occurring in plasma, heat propagation, moisture transfer in porous media, some technological processes, problems of mathematical biology and demography [14,16,17,19].

Nonlocal condition boundary value problems arise while constructing mathematical models of processes that occur in atomic and nuclear physics, demography, heating processes and in other fields of natural science. The papers [15,

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21] study one-dimensional nonlinear hyperbolic equations given with integral and multipoint boundary conditions. Sufficient conditions for the existence and uniqueness of the problem are found.

In [12,13,20] a system of hyperbolic equations is studied within two-point and integral conditions. The Green function of the problem is structured, the boundary value problem is reduced to an equivalent integral equation, sufficient conditions for the existence and uniqueness of the problem are found. There are a lot of papers devoted to integral condition hyperbolic type equations. The papers [1–11,18] illustrate the solvability of the classic and generalized boundary value problem.

In the present work we give a Goursat-Darboux system with pointwise and integral condition. A necessary condition for the solvability of the problem is proved. The considered problem is reduced to an equivalent equation by means of equivalent transformations. Sufficient conditions for the existence and uniqueness of the solution are found by means of the Banach contraction mapping principle.

We consider a nonlocal problem with integral and point boundary conditions for a Goursat-Darboux system in the domain $Q = [0, T] \times [0, l]$:

$$z_{tx} = f\left(t, x, z\left(t, x\right)\right), \tag{1.1}$$

$$z\left(0, x\right) + \int\limits_0^T n\left(t\right) z\left(t, x\right) dt = \varphi\left(x\right), x \in [0, l], \tag{1.2}$$

$$z\left(t, 0\right) + \int\limits_0^l m\left(x\right) z\left(t, x\right) dx = \psi(t), t \in [0, T] \tag{1.3}$$

here $z\left(t, x\right) = col\left(z_1\left(t, x\right), z_2\left(t, x\right), \ldots, z_n(t, x)\right)$ is an unknown n-dimensional vector-functions; $f : Q \times R^n \to R^n$ is a given function; $\varphi\left(x\right)$, $\psi\left(t\right)$ are functions that are differentiable on $[0, T], [0, l]$ respectively. Matrix functions $n\left(t\right) \in L_\infty\left[0, T\right]$, $m(x) \in L_\infty\left[0, l\right]$ are commutative, i.e., they satisfy the relations $n\left(t\right) m\left(x\right) = m\left(x\right) n(t)$ for $(t, x) \in Q$, moreover $\|n\|_{L_\infty} < 1$, $\|m\|_{L_\infty} < 1$. Under these conditions $\left(E + \int\limits_0^T n\left(t\right) dt\right)^{-1}$ and $\left(E + \int\limits_0^l m\left(x\right) dx\right)^{-1}$ exist and

$$\left\| \left(E + \int\limits_0^T n\left(t\right) dt\right)^{-1} \right\| \le \frac{1}{1 - \|n\|}, \left\| \left(E + \int\limits_0^l m\left(x\right) dx\right)^{-1} \right\| \le \frac{1}{1 - \|m\|}.$$

By a solution of problem (1.1), (1.2) we mean a function $z(t, x) \in C(Q, R^n)$ possessing partial derivatives $z_t(t, x) \in C(Q, R^n)$, $z_x(t, x) \in C(Q, R^n)$, $z_{tx}(t, x) \in C(Q, R^n)$, which satisfies equation (1.1) and conditions (1.2). Such solutions are usually called classical.

2. Main results

In this section, it is shown that for the solvability of the problem (1.1)-(1.3) the compatibility condition of functions $\varphi\left(x\right)$ and $\psi\left(t\right)$ is satisfied.

Theorem 2.1. *For the solvability of problem (1.1)-(1.3) it is necessary that the compatibility condition*

$$\varphi(0)+\int_0^l m(x)\varphi(x)\,dx=\psi(0)+\int_0^T n(t)\psi(t)\,dt$$

is fulfilled.

Proof. We seek a solution of the equation (1.1) as follows:

$$z(t,x)=a(t)+b(x)+\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds. \tag{2.1}$$

where the functions $a(t)$ and $b(x)$ are unknown differentiable functions and are determined in the intervals $[0,T],[0,l]$ respectively. Require that the function determined by the equality (2.1) satisfy the boundary conditions (1.2) and (1.3). Then we obtain the relations

$$a(0)+b(x)+\int_0^T n(t)\left[a(t)+b(x)+\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds\right]dt=$$

$$=a(0)+\int_0^T n(t)\,a(t)\,dt+\left(E+\int_0^T n(t)\,dt\right)b(x)+$$

$$+\int_0^T n(t)\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds dt=\varphi(x),\quad x\in[0,l]. \tag{2.2}$$

$$a(t)+b(0)+\int_0^l m(x)\left[a(t)+b(x)+\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds\right]dx=$$

$$=\left(E+\int_0^l m(x)\,dx\right)a(t)+b(0)+\int_0^l m(x)\,b(x)\,dx+$$

$$+\int_0^l m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds dx=\psi(t),\quad t\in[0,T]. \tag{2.3}$$

Applying conditions (1.3) to the expressions (2.2) and conditions (1.2) to (2.3), we obtain

$$a(0)+\int_0^T n(t)\,a(t)\,dt+\left(E+\int_0^T n(t)\,dt\right)b(0)+$$

$$+\int_0^l m(x)\left[a(0)+\int_0^T n(t)\,a(t)dt+\left(E+\int_0^T n(t)\,dt\right)b(x)\right]dx+$$

$$+\int_0^T\int_0^l n(t)\,m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds dt dx=\varphi(0)+\int_0^l m(x)\,\varphi(x)\,dx,$$

$$\left(E+\int_0^l m(x)\,dx\right)a(0)+b(0)+\int_0^l m(x)\,b(x)\,dx+$$

$$+\int_0^T n(t)\left[\left(E+\int_0^l m(x)\,dx\right)a(t)+\left(b(0)+\int_0^l m(x)\,b(x)\,dx\right)\right]dt+$$

$$\int_0^T\int_0^l n(t)\,m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds dt dx=\psi(0)+\int_0^T n(t)\,\psi(t)\,dt.$$

From this we receive that

$$\left(a(0)+\int_0^T n(t)\,a(t)\,dt\right)\left(E+\int_0^l m(x)\,dx\right)+\left(E+\int_0^T n(t)\,dt\right)\times$$

$$\times\left(b(0)+\int_0^l m(x)\,b(x)\,dx\right)+\int_0^T\int_0^l n(t)\,m(x)\int_0^t\int_0^x\times$$

$$\times f(\tau,s,z(\tau,s))\,d\tau ds dt dx=\varphi(0)+\int_0^l m(x)\,f(x)\,dx,$$

$$\left(a(0)+\int_0^T n(t)\,a(t)\,dt\right)\left(E+\int_0^l m(x)\,dx\right)+\left(b(0)+\int_0^l m(x)\,b(x)\,dx\right)\times$$

$$\times\left(E+\int_0^T n(t)\,dt\right)+\int_0^T\int_0^l n(t)\,m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds dt dx=$$

$$=\psi(0)+\int_0^T n(t)\,\psi(t)\,dt.$$

The equality of the right hand side is derived from the equality of the left hand side.

Theorem 2.1 is proved. □

We construct a Green function for the problem (1.1) - (1.3). It is noted that the problem (1.1)-(1.3) is reduced to an equivalent integral equation.

Theorem 2.2. *The equivalent integral equation for the problem* (1.1)-(1.3) *is as follows*

$$z(t,x)=\left(E+\int_0^l m(x)\,dx\right)^{-1}\psi(t)+\left(E+\int_0^T n(t)\,dt\right)^{-1}\varphi(x)-$$

$$-\left(E+\int\limits_0^l m(x)\,dx\right)^{-1}\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\left(\varphi(0)+\int\limits_0^l m(x)\,\varphi(x)\,dx\right)+$$

$$+\int\limits_0^T\int\limits_0^l G(t,x,\tau,s)\,f(\tau,s,z)\,d\tau ds \tag{2.4}$$

where

$$G(t,x,\tau,s)=\left(E+\int\limits_0^l m(x)\,dx\right)^{-1}\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\times$$

$$\times\begin{cases}\left(E+\int\limits_0^\tau n(\alpha)\,d\alpha\right)\left(E+\int\limits_0^s m(\beta)\,d\beta\right), & 0\le\tau\le t,\ 0\le s\le x,\\ -\left(E+\int\limits_0^\tau n(\alpha)\,d\alpha\right)\int\limits_s^l m(\beta)\,d\beta, & 0\le\tau\le t,\ x<s\le l,\\ -\left(E+\int\limits_0^s m(\beta)\,d\beta\right)\int\limits_\tau^T n(\alpha)\,d\alpha, & t<\tau\le T,\ 0\le s\le x,\\ \int\limits_\tau^T n(\alpha)\,d\alpha\int\limits_s^l m(\beta)\,d\beta, & t<\tau\le T,\ x<s\le l.\end{cases}$$

Proof. The unknown functions $a(t)$ and $b(x)$ can be considered as solutions to a system of linear algebraic equations defined by equalities (2.2) or (2.3). It is a system of the n-th order. The sought functions $a(t)$ and $b(x)$ have dimension $2n$. It is clear that this system has an infinite set of solutions. We fix an arbitrary solution. Let

$$a(0)+\int\limits_0^T n(t)\,a(t)\,dt=0$$

be an arbitrary solution.
Then in the equalities (2.2) we find

$$b(x)=\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\varphi(x)-\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\times$$

$$\times\int\limits_0^T n(t)\int\limits_0^t\int\limits_0^x f(\tau,s,z(\tau,s))\,d\tau ds dt, \tag{2.5}$$

$$b(0)=\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\varphi(0).$$

Taking $b(x)$ and $b(0)$ into account in the equality (2.3), we obtain

$$\left(E+\int\limits_0^l m(x)\,dx\right)a(t)+\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\varphi(0)+$$

$$\int_0^l m(x)\left(E+\int_0^T n(t)\,dt\right)^{-1}\varphi(x)\,dx-\int_0^l m(x)\left(E+\int_0^T n(t)\,dt\right)^{-1}\times$$
$$\times\int_0^T n(t)\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds dt dx+$$
$$+\int_0^l m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds dx=\psi(t).$$

Hence,

$$a(t)=\left(E+\int_0^l m(x)\,dx\right)^{-1}\psi(t)-$$
$$-\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\varphi(0)-$$
$$-\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\int_0^l m(x)\varphi(x)\,dx+$$
$$+\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\times$$
$$\times\int_0^l\int_0^T m(x)n(t)\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds dt dx-$$
$$-\left(E+\int_0^l m(x)\,dx\right)^{-1}\int_0^l m(x)\int_0^t\int_0^x f(\tau,s,z(\tau,s))\,d\tau ds dx. \tag{2.6}$$

Taking into account the equalities (2.5) and (2.6) obtained for the functions $b(x)$ and $a(t)$ in the equality (2.1), we have

$$z(t,x)=\left(E+\int_0^l m(x)\,dx\right)^{-1}\psi(t)+\left(E+\int_0^T n(t)\,dt\right)^{-1}\varphi(x)-$$
$$-\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\left[\varphi(0)+\int_0^l m(x)\varphi(x)\,dx\right]+$$
$$+\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\int_0^l\int_0^T m(x)n(t)\times$$

$$\times \int_0^t \int_0^x f(\tau, s, z(\tau, s))\, d\tau ds dt dx -$$

$$-\left(E + \int_0^l m(x)\, dx\right)^{-1} \int_0^l m(x) \int_0^t \int_0^x f(\tau, s, z(\tau, s))\, d\tau ds dx -$$

$$-\left(E + \int_0^T n(t)\, dt\right)^{-1} \int_0^T n(t) \int_0^t \int_0^x f(\tau, s, z(\tau, s))\, d\tau ds dt +$$

$$+\int_0^t \int_0^x f(\tau, s, z(\tau, s))\, d\tau ds, \; (t, x) \in Q. \tag{2.7}$$

We perform the same transformations in the equality (2.7).

$$\int_0^T n(t) \int_0^t \int_0^x f(\tau, s, z(\tau, s))\, d\tau ds dt =$$

$$= \int_0^T \int_0^x \left(\int_t^T n(\tau)\, d\tau\right) f(\tau, s, z(\tau, s))\, d\tau ds,$$

$$\int_0^l m(x) \int_0^t \int_0^x f(\tau, s, z(\tau, s))\, d\tau ds dx =$$

$$= \int_0^l \int_0^t \left(\int_x^l m(s)\, ds\right) f(\tau, x, z(t, x))\, d\tau dx,$$

$$\int_0^l \int_0^T m(x)\, n(t) \int_0^t \int_0^x f(\tau, s, z(\tau, s))\, d\tau ds dt dx =$$

$$= \int_0^T \int_0^l \left(\int_t^T n(\tau)\, d\tau \int_x^l m(s) ds\right) f(t, x, z(t, x))\, dt dx.$$

Taking into consideration these expressions in the equality (2.7), we can write

$$z(t, x) = \left(E + \int_0^l m(x)\, dx\right)^{-1} \psi(t) + \left(E + \int_0^T n(t)\, dt\right)^{-1} \varphi(x) -$$

$$-\left(E + \int_0^l m(x)\, dx\right)^{-1} \left(E + \int_0^T n(t)\, dt\right)^{-1} \left[\varphi(0) + \int_0^l m(x)\, \varphi(x)\, dx\right] +$$

$$+\left(E + \int_0^l m(x)\, dx\right)^{-1} \left(E + \int_0^T n(t)\, dt\right)^{-1} \times$$

$$\times \int_0^T \int_0^l \left[\int_t^T n(\tau)\, d\tau \int_x^l m(s) ds \right] f(t,x,z(t,x))\, dt dx -$$

$$- \left(E + \int_0^l m(x)\, dx \right)^{-1} \int_0^l \int_0^t \left(\int_x^l m(s)\, ds \right) f(\tau,x,z(\tau,x))\, d\tau dx -$$

$$- \left(E + \int_0^T n(t)\, dt \right)^{-1} \int_0^T \int_0^x \left(\int_t^T n(\tau)\, d\tau \right) f(t,s,z(t,s))\, dt ds +$$

$$+ \int_0^t \int_0^x f(\tau,s,z(\tau,s))\, d\tau ds, \quad (t,x) \in Q. \tag{2.8}$$

From the equality (2.8) we receive

$$z(t,x) = \left(E + \int_0^l m(x)\, dx \right)^{-1} \psi(t) + \left(E + \int_0^T n(t)\, dt \right)^{-1} \varphi(x) -$$

$$- \left(E + \int_0^l m(x)\, dx \right)^{-1} \left(E + \int_0^T n(t)\, dt \right)^{-1} \left(\varphi(0) + \int_0^l m(x) \varphi(x)\, dx \right) +$$

$$+ \int_0^t \int_0^x \left[E - \left(E + \int_0^T n(t)\, dt \right)^{-1} \int_t^T n(\alpha)\, d\alpha - \right.$$

$$- \left(E + \int_0^l m(x)\, dx \right)^{-1} \int_s^l m(\beta)\, d\beta + \left(E + \int_0^l m(x)\, dx \right)^{-1} \times$$

$$\times \left(E + \int_0^l m(x)\, dx \right)^{-1} \left(E + \int_0^T n(t)\, dt \right)^{-1} \times$$

$$\left. \times \int_\tau^T n(\alpha)\, d\alpha \int_s^l m(\beta)\, d\beta \right] f(\tau,s,z(\tau,s))\, d\tau ds +$$

$$+ \int_0^t \int_x^l \left[\left(E + \int_0^l m(x)\, dx \right)^{-1} \left(E + \int_0^T n(t)\, dt \right)^{-1} \times \right.$$

$$\left. \times \int_\tau^T n(\alpha)\, d\alpha \int_s^l m(\beta)\, d\beta - \left(E + \int_0^l m(x)\, dx \right)^{-1} \int_s^l m(\beta)\, d\beta \right] \times$$

$$\times f(\tau,s,z(\tau,s))\,d\tau ds+\int\limits_t^T\int\limits_0^x\left[\left(E+\int\limits_0^l m(x)\,dx\right)^{-1}\times\right.$$

$$\times\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\times\int\limits_\tau^T n(\alpha)\,d\alpha\int\limits_s^l m(\beta)\,d\beta-$$

$$\left.-\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\int\limits_\tau^T n(\alpha)\,d\alpha\right] f(\tau,s,z(\tau,s))\,d\tau ds+$$

$$+\int\limits_t^T\int\limits_x^l\left[\left(E+\int\limits_0^l m(x)\,dx\right)^{-1}\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\times\right.$$

$$\left.\times\int\limits_\tau^T n(\alpha)\,d\alpha\int\limits_s^l m(\beta)\,d\beta\right] f(\tau,s,z(\tau,s))\,d\tau ds,\quad (t,x)\in Q. \tag{2.9}$$

Taking these equalities into account, we can write

$$E-\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\int\limits_\tau^T n(\alpha)\,d\alpha-\left(E+\int\limits_0^l m(x)\,dx\right)^{-1}\int\limits_s^l m(\beta)\,d\beta+$$

$$+\left(E+\int\limits_0^l m(x)\,dx\right)^{-1}\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\int\limits_\tau^T n(\alpha)\,d\alpha\int\limits_s^l m(\beta)\,d\beta=$$

$$=\left(E+\int\limits_0^l m(x)\,dx\right)^{-1}\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\times$$

$$\times\left(E+\int\limits_0^\tau n(\alpha)\,d\alpha\right)\left(E+\int\limits_0^s m(\beta)\,d\beta\right),$$

$$\left(E+\int\limits_0^l m(x)\,dx\right)^{-1}\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\int\limits_\tau^T n(\alpha)\,d\alpha\int\limits_s^l m(\beta)\,d\beta-$$

$$-\left(E+\int\limits_0^l m(x)\,dx\right)^{-1}\int\limits_s^l m(\beta)\,d\beta=$$

$$=-\left(E+\int\limits_0^l m(x)\,dx\right)^{-1}\left(E+\int\limits_0^T n(t)\,dt\right)^{-1}\times$$

$$\times \left[\left(E + \int_0^{\tau} n(\alpha)\, d\alpha \right) \int_s^l m(\beta)\, d\beta \right],$$

$$\left(E + \int_0^l m(x)\, dx \right)^{-1} \left(E + \int_0^T n(t)\, dt \right)^{-1} \int_{\tau}^{T} n(\alpha)\, d\alpha \int_s^l m(\beta)\, d\beta -$$

$$-\left(E + \int_0^T n(t)\, dt \right)^{-1} \int_{\tau}^{T} n(\alpha)\, d\alpha = -\left(E + \int_0^l m(x)\, dx \right)^{-1} \times$$

$$\times \left(E + \int_0^T n(t)\, dt \right)^{-1} \left[\left(E + \int_0^s m(\beta)\, d\beta \right) \int_{\tau}^{T} n(\alpha)\, d\alpha \right].$$

As a result we derive

$$z(t,x) = \left(E + \int_0^l m(x)\, dx \right)^{-1} \psi(t) + \left(E + \int_0^T n(t)\, dt \right)^{-1} \varphi(x) -$$

$$-\left(E + \int_0^l m(x)\, dx \right)^{-1} \left(E + \int_0^T n(t)\, dt \right)^{-1} \left[\varphi(0) + \int_0^l m(x)\, \varphi(x)\, dx \right] +$$

$$+ \int_0^t \int_0^x \left(E + \int_0^l m(x)\, dx \right)^{-1} \left(E + \int_0^T n(t)\, dt \right)^{-1} \times$$

$$\times \left(E + \int_0^t n(\alpha)\, d\alpha \right) \left(E + \int_0^s m(\beta)\, d\beta \right) f(\tau, s, z(\tau, s))\, d\tau ds -$$

$$- \int_0^t \int_x^l \left(E + \int_0^l m(x)\, dx \right)^{-1} \left(E + \int_0^T n(t)\, dt \right)^{-1} \times$$

$$\times \left[\left(E + \int_0^{\tau} n(\alpha)\, d\alpha \right) \int_s^l m(\beta)\, d\beta \right] f(\tau, s, z(\tau, s))\, d\tau ds -$$

$$- \int_t^T \int_0^x \left(E + \int_0^l m(x)\, dx \right)^{-1} \left(E + \int_0^T n(t)\, dt \right)^{-1}$$

$$\left[\left(E + \int_0^s m(\beta)\, d\beta \right) \int_{\tau}^{T} n(\alpha)\, d\alpha \right] f(\tau, s, z(\tau, s))\, d\tau ds +$$

$$+ \int_t^T \int_x^l \left(E + \int_0^l m(x)\, dx \right)^{-1} \left(E + \int_0^T n(t)\, dt \right)^{-1} \times$$

$$\times\left[\int_{\tau}^{T} n(\alpha)\,d\alpha \int_{s}^{l} m(\beta)\,d\beta\right] f(\tau,s,z(\tau,s))\,d\tau ds,\quad (t,x)\in Q. \tag{2.10}$$

Thus, we proved the first part of the theorem, as we determined the matrix-function $G(t,x,\tau,s)$. Now we calculate the derivative of function $z(t,x)$ determined by the equality (2.10) with respect to t and x

$$z_{tx}(t,x) = \frac{\partial^2}{\partial t\partial x}\left[\left(E+\int_0^l m(x)\,dx\right)^{-1}\psi(t) + \left(E+\int_0^T n(t)\,dt\right)^{-1}\varphi(x) -\right.$$

$$\left.-\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\left[\varphi(0)+\int_0^l m(x)\varphi(x)\,dx\right]\right]+$$

$$+\frac{\partial^2}{\partial t\partial x}\left[\int_0^{\tau}\int_0^{x}\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\times\right.$$

$$\times\left(E+\int_0^{\tau} n(\alpha)\,d\alpha\right)\left(E+\int_0^{s} m(\beta)\,d\beta\right) f(\tau,s,z(\tau,s))\,d\tau ds] -$$

$$-\frac{\partial^2}{\partial t\partial x}\left[\int_0^{t}\int_x^{l}\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\times\right.$$

$$\left.\times\left[\left(E+\int_0^{\tau} n(\alpha)\,d\alpha\right)\int_s^l m(\beta)\,d\beta\right] f(\tau,s,z(\tau,s))\,d\tau ds\right] -$$

$$-\frac{\partial^2}{\partial t\partial x}\left[\int_t^{T}\int_0^{x}\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\times\right.$$

$$\times\left[\left(E+\int_0^{s} m(\beta)\,d\beta\right)\int_{\tau}^T n(\alpha)\,d\alpha\right] f(\tau,s,z(\tau,s))\,d\tau ds] +$$

$$+\frac{\partial^2}{\partial t\partial x}\int_t^{T}\int_x^{l}\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\times$$

$$\times\left[\int_{\tau}^{T} n(\alpha)\,d\alpha\int_s^l m(\beta)\,d\beta\right] f(\tau,s,z(\tau,s))\,d\tau ds =$$

$$=\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\left[E+\int_0^t n(\alpha)\,d\alpha+\right.$$

$$+\int_0^x m(\beta)d\beta+\int_0^t n(\alpha)\,d\alpha\int_0^x m(\beta)\,d\beta+\int_x^l m(\beta)\,d\beta+$$
$$+\int_0^t n(\alpha)\,d\alpha\int_x^l m(\beta)\,d\beta+\int_t^T n(\alpha)\,d\alpha+$$
$$\left.+\int_0^x m(\beta)\,d\beta\int_t^T n(\alpha)\,d\alpha+\int_t^T n(\alpha)\,d\alpha\int_x^l m(\beta)\,d\beta\right]\times$$
$$\times f(t,x,z(t,x))=f(t,x,z(t,x)).$$

Let us show that the function determined by the equality (2.8) satisfies nonlocal boundary conditions (1.2) and (1.3)

$$\left(E+\int_0^l m(x)\,dx\right)^{-1}\psi(0)+\left(E+\int_0^T n(t)\,dt\right)^{-1}\varphi(x)-$$
$$-\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\left[\varphi(0)+\int_0^l m(x)\,\varphi(x)\,dx\right]+$$
$$+\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\times$$
$$\times\int_0^T\int_0^l\left[\int_t^T n\;(\tau)d\tau\int_x^l m(s)\,ds\right]f(t,x,z(t,x))\,dtdx-$$
$$-\left(E+\int_0^T n(t)\,dt\right)^{-1}\int_0^T\int_0^x\left(\int_t^T n(\tau)d\tau\right)f(t,s,z(t,s))\,dtds+$$
$$+\int_0^T n(t)\left[\left(E+\int_0^l m(x)\,dx\right)^{-1}\psi(t)+\left(E+\int_0^T n(t)\,dt\right)^{-1}\varphi(x)-\right.$$
$$-\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\left[\varphi(0)+\int_0^l m(x)\varphi(x)\,dx\right]+$$
$$+\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\times$$
$$\times\int_0^T\int_0^l\left[\int_\tau^T n(\tau)\,d\tau\int_x^l m(s)\,ds\right]f(t,x,z(t,x))\,dtdx-$$

$$-\left(E+\int_0^l m(x)\,dx\right)^{-1}\int_0^l\int_0^t\left(\int_x^l m(s)\,ds\right)f(\tau,x,z(\tau,x))\,d\tau dx-$$

-

$$-\left(E+\int_0^T n(t)\,dt\right)^{-1}\int_0^T\int_0^x\left[\int_t^T n(\tau)\,d\tau\right]f(t,s,z(t,s))\,dtds+$$

$$\left.+\int_0^t\int_0^x f(t,s,z(t,s))dtds\right]dt=$$

$$=\left(E+\int_0^l m(x)\,dx\right)^{-1}\left[\psi(0)+\int_0^T n(t)\,\psi(t)\,dt\right]-$$

$$-\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\varphi(0)\left(E+\int_0^T n(t)\,dt\right)+$$

$$+\left(E+\int_0^T n(t)\,dt\right)^{-1}\varphi(x)\left(E+\int_0^T n(t)\,dt\right)-\left(E+\int_0^T n(t)\,dt\right)^{-1}\times$$

$$\times\left(E+\int_0^T n(t)\,dt\right)\int_0^T\int_0^x\left(\int_t^T n(t)\,dt\right)f(t,s,z(t,s))\,dtds-$$

$$-\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\times$$

$$\times\left(E+\int_0^T n(t)\,dt\right)\int_0^l m(x)\,\varphi(x)\,dx+$$

$$+\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)\times$$

$$\times\int_0^T\int_0^l\left[\int_t^T n(\tau)\,d\tau\int_x^l m(s)\,ds\right]f(\tau,x,z(\tau,x))\,d\tau dx-$$

$$-\left(E+\int_0^l m(x)\,dx\right)^{-1}\int_0^T n(t)\int_0^l\int_0^t\left(\int_x^l m(s)\,ds\right)f(\tau,x,z(\tau,x))\,d\tau dxdt+$$

$$+\int_0^T\int_0^x\left(\int_t^T n(\tau)\,d\tau\right)f(t,s,z(t,s))\,dtds=$$

$$= \left(E + \int_0^l m(x)\,dx\right)^{-1} \left[\psi(0) + \int_0^T n(t)\,\psi(t)\,dt\right] -$$

$$- \left(E + \int_0^l m(x)\,dx\right)^{-1} \varphi(0) + \varphi(x) - \int_0^T \int_0^x \left(\int_t^T n(\tau)\,d\tau\right) f(t,s,z(t,s))\,dtds -$$

$$- \left(E + \int_0^l m(x)\,dx\right)^{-1} \int_0^l m(x)\,\varphi(x)\,dx + \left(E + \int_0^l m(x)\,dx\right)^{-1} \times$$

$$\times \int_0^T \int_0^l \left[\int_t^T n(\tau)\,d\tau \int_x^l m(s)\,ds\right] f(t,x,z(t,x))\,dtdx -$$

$$- \left(E + \int_0^l m(x)\,dx\right)^{-1} \int_0^T n(t) \int_0^l \int_0^t \left(\int_x^l m(s)\,ds\right) f(\tau,x,z(\tau,x))\,d\tau dx dt +$$

$$+ \int_0^T \int_0^x \left(\int_t^T n(\tau)\,d\tau\right) f(t,s,z(t,s))\,dtds =$$

$$= \left(E + \int_0^l m(x)\,dx\right)^{-1} \left[\left(\psi(0) + \int_0^T n(t)\,\psi(t)\,dt\right) - \right.$$

$$\left. \left(\varphi(0) + \int_0^l m(x)\,\varphi(x)\,dx\right)\right] + \varphi(x) = \varphi(x).$$

In a similar way, we can show that the pointwise and integral boundary condition

$$z(t,0) + \int_0^l m(x)\,z(t,x)\,dx = \psi(t), \quad t \in [0,T]$$

is satisfied.

Theorem 2.2 is proved. □

3. Existence and uniqueness

It is seen from the proved theorem that the problem (1.1)-(1.3) is equivalent to the integral equation

$$z(t,x) = \left(E + \int_0^l m(x)\,dx\right)^{-1} \psi(t) + \left(E + \int_0^T n(t)\,dt\right)^{-1} \varphi(x) -$$

$$- \left(E + \int_0^l m(x)\,dx\right)^{-1} \left(E + \int_0^T n(t)\,dt\right)^{-1} \left[\varphi(0) + \int_0^l m(x)\,\varphi(x)\,dx\right] +$$

$$+\int_0^T\int_0^l G(t,x,\tau,s)\,f(\tau,s,z)\,d\tau ds. \tag{3.1}$$

In order to prove the existence and uniqueness of the solution of the problem (1.1)-(1.3) we determine the operator $P : C(Q;R^n) \to C(Q;R^n)$ as follows:

$$(Pz)(t,x) = \left(E+\int_0^l m(x)\,dx\right)^{-1}\psi(t)+\left(E+\int_0^T n(t)\,dt\right)^{-1}\varphi(x)-$$

$$-\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\left(\varphi(0)+\int_0^l m(x)\,\varphi(x)\,dx\right)+$$

$$+\int_0^T\int_0^l G(t,x,\tau,s)\,f(\tau,s,z)\,d\tau ds.$$

It is known that solving the problem (1.1)-(1.3) or the integral equation (3.1) is equivalent to finding the fixed point of the operator P. In other words, the problem (1.1)-(1.3) has a solution if and only if the operator P has a fixed point.

Theorem 3.1. *Assume that the following conditions are valid:*

$$|f(t,x,z_2-f(t,x,z_1)|\le M\,|z_2-z_1|\,,\forall\,(t,x)\in Q,\quad z_1,z_2\in R^n\ ,\quad M\ge 0 \tag{3.2}$$

and

$$L = lTSM < 1, \tag{3.3}$$

where

$$S = \underset{Q\times Q}{max}\,\|G(t,x,\tau,s\|\ .$$

Then the problem (1.1)-(1.3) *has a unique solution in* Q.

Proof. Denote

$$N = \max_Q\left|\left(E+\int_0^l m(x)\,dx\right)^{-1}\psi(t)+\left(E+\int_0^T n(t)\,dt\right)^{-1}\varphi(x)-\right.$$

$$\left.-\left(E+\int_0^l m(x)\,dx\right)^{-1}\left(E+\int_0^T n(t)\,dt\right)^{-1}\left(\varphi(0)+\int_0^l m(x)\,\varphi(x)\,dx\right)\right|,$$

$$\max_{(t,x)\in Q}|f(t,x,0)| = M_f$$

and choose $r\ge\frac{N+M_fTS}{1-L}$. We show that the relation $PB_r\subset B_r$ is valid, where

$$B_r = \{x\in C(Q,R^n) : \|z\|\le r\}.$$

For arbitrary $z\in B_r$

$$\|Pz(t,x)\|\le N+$$

$$+\int_0^T\int_0^l |G(t,x,\tau,s)|\left(|f(\tau,s,z(\tau,s)-f(\tau,s,0)|+|f(\tau,s,0)|\right)d\tau ds\le$$

$$\leq N + S\int_0^T\int_0^l (M|z| + M_f)\,dtdx \leq N + SMrTl + M_fTlS \leq$$

$$\leq \frac{N + M_fTS}{1-L} \leq r.$$

On the other hand, from condition (3.2) we obtain that for arbitrary $z_1, z_2 \in B_r$ the relation

$$|Pz_2 - Pz_1| \leq \int_0^T\int_0^l |G(t,x,\tau,s)|\,(|f(\tau,s,z_2(\tau,s) - f(\tau,s,z_1(\tau,s)|) \leq$$

$$\leq S\int_o^T\int_0^l M|z_2(t,x) - z_1(t,x)|\,dtdx \leq MSTl\max_Q |z_2(t,x) - z_1(t,x)| \leq$$

$$\leq MSTl\,\|z_2 - z_1\|$$

is valid. Hence we receive

$$\|Pz_2 - Pz_1\| \leq L\,\|z_2 - z_1\|\,.$$

Taking the condition (3.3) into account we obtain that the operator P is a contraction. Thus, problem (1.1)-(1.3) has a unique solution. □

4. Application of the obtained results

Example 4.1. To illustrate the results obtained, consider the following nonlocal boundary value problem for a system of hyperbolic equations

$$\begin{cases} z_{1tx}(t,x) = 0,1\ \ sinz_2(t,x)\,, \\ z_{2tx}(t,x) = \frac{|z_1(t,x)|}{10(1+|z_2(t,x)|)}, \end{cases} \quad (t,x) \in [0,1]\times[0,1]\,. \tag{4.1}$$

$$\begin{cases} z_1(0,x) + \int\limits_0^1 \frac{1}{2}tz_1(t,x)\,dt = x^2, \\ z_2(0,x) = x, \end{cases} \quad x \in [0,1]\,. \tag{4.2}$$

$$\begin{cases} z_1(t,0) + \int\limits_0^1 \frac{1}{2}xz_1(t,x)\,dx = t^2, \\ z_2(t,0) = t, \end{cases} \quad t \in [0,1]\,. \tag{4.3}$$

Obviously,

$$n(t) = \begin{pmatrix} \frac{1}{2}t & 0 \\ 0 & 0 \end{pmatrix},\ m(x) = \begin{pmatrix} \frac{1}{2}x & 0 \\ 0 & 0 \end{pmatrix}$$

$$E + \int_0^1 n(t)\,dt = \begin{pmatrix} 1.25 & 0 \\ 0 & 0 \end{pmatrix},\ E + \int_0^1 m(x)\,dx = \begin{pmatrix} 1.25 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$G(t,x,t,s)=\begin{cases}\begin{pmatrix}0.64\left(1+\frac{\tau^2}{4}\right)\left(1+\frac{s^2}{4}\right) & 0\\ 0 & 1\end{pmatrix}, 0\le\tau\le t,\ 0\le s\le x,\\ -\begin{pmatrix}0.16\left(1+\frac{\tau^2}{4}\right)\left(1-s^2\right) & 0\\ 0 & 1\end{pmatrix}, 0\le\tau\le t,\ x<s\le 1,\\ -\begin{pmatrix}0.16\ \left(1-\tau^2\right)\left(1+\frac{s^2}{4}\right) & 0\\ 0 & 1\end{pmatrix}, t\le\tau\le 1,\ 0<s\le x,\\ \begin{pmatrix}0.04\left(1-\tau^2\right)\left(1-s^2\right) & 0\\ 0 & 0\end{pmatrix},\ t\le\tau\le 1,\ x<s\le 1.\end{cases}$$

Now we will estimate the parameters of the problem:

the norm of the Green function $\max\|G(t,x,\tau,s\|\le 1$;

the Lipschitz constant $M=0.1$;

contraction parameter $L=1\cdot 0.1\cdot 1\cdot 1=0.1<1$.

Thus, all the conditions of Theorem 3.1 are satisfied and therefore the boundary value problem (4.1)-(4.3) has a unique solution.

Example 4.2. Now let us look at the system of differential equations (4.1) in the square $\left[0,\ \frac{\pi}{2}\right]\times\left[0,\ \frac{\pi}{2}\right]$ with the following boundary conditions:

$$\begin{pmatrix}z_1(0,x)\\ z_2(t,x)\end{pmatrix}+\int_0^{\frac{\pi}{2}}\begin{pmatrix}\sin t & 0\\ 0 & \sin t\end{pmatrix}\begin{pmatrix}z_1(t,x)\\ z_2(t,x)\end{pmatrix}dt=\begin{pmatrix}0\\ 0\end{pmatrix},\ x\in\left(0,\frac{\pi}{2}\right),\quad (4.4)$$

$$\begin{pmatrix}z_1(t,0)\\ z_2(t,0)\end{pmatrix}+\int_0^{\frac{\pi}{2}}\begin{pmatrix}\cos x & 0\\ 0 & \cos x\end{pmatrix}\begin{pmatrix}z_1(t,x)\\ z_2(t,x)\end{pmatrix}dx=\begin{pmatrix}0\\ 0\end{pmatrix},\ t\in\left(0,\frac{\pi}{2}\right),\quad (4.5)$$

It is clear that,

$$n(t)=\begin{pmatrix}\sin t & 0\\ 0 & \sin t\end{pmatrix}, m(x)=\begin{pmatrix}\cos x & 0\\ 0 & \cos x\end{pmatrix}$$

and

$$E+\int_0^{\frac{\pi}{2}}n(t)\,dt=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}, E+\int_0^{\frac{\pi}{2}}m(x)\,dx=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}.$$

$$G(t,x,t,s)=$$

$$=\begin{cases}\begin{pmatrix}\frac{(1+\sin t)(2-\cos s)}{4} & 0\\ 0 & \frac{(1+\sin t)(2-\cos s)}{4}\end{pmatrix}, 0\le t\le t,\ 0\le s\le x,\\ -\begin{pmatrix}\frac{(1+\sin t)\cos s}{4} & 0\\ 0 & \frac{(1+\sin t)\cos s}{4}\end{pmatrix},\ 0\le t\le t,\ x<s\le\frac{\pi}{2},\\ -\begin{pmatrix}\frac{(1-\sin t)(2-\cos s)}{4} & 0\\ 0 & \frac{(1-\sin t)(2-\cos s)}{4}\end{pmatrix},\ t<t\le\frac{\pi}{2}, 0\le s\le x,\\ \begin{pmatrix}\frac{(1-\sin t)\cos s}{4} & 0\\ 0 & \frac{(1-\sin t)\cos s}{4}\end{pmatrix},\ t<t\le\frac{\pi}{2},\ x<s\le\frac{\pi}{2}.\end{cases}$$

As seen $S=max\|G(t,x,t,s\|\le 1$ and $L=1\cdot 0.1\cdot\frac{\pi^2}{4}=\frac{\pi^2}{40}<1$. Then the equation (4.1) with boundary conditions (4.4), (4,5) has a unique solution.

5. Conclusion

We obtain the following results:

- The Green function was structured to investigate the boundary value problem;

- The conditions satisfying the initial data were found for the unique solvability of the problem;

- A theorem on the existence and uniqueness of the solution based on the contraction mapping principle was proved;

- The results obtained in a specific example were illustrated.

It is shown that all the conditions imposed in the initial data of the problem are important for a unique solvability of the problem.

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