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APPROXIMATE OBLIQUE DUAL $g-$FRAMES FOR CLOSED SUBMODULES OF HILBERT C^*-MODULES

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Abstract. In this paper, we introduce the concept of approximate oblique dual $g-$frames in Hilbert C^*-modules and give some properties. We provide a characterization of approximate oblique dual $g-$frames. Finally, we discuss the perturbation problem of approximate oblique dual $g-$frames.

1. Introduction and Preliminaries

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [8] in 1952 to study some problems in nonharmonic Fourier series. The topic remained dormant for 30 years until it was brought back to life in the context of data processing (see [6]). After this, the subject advanced rapidly and soon became one of the most active areas of research in applied mathematics. Frames were originally used in signal and image processing, and later in sampling theory, data compression, time-frequency analysis coding theory, and Fourier series. Today, there are ever increasing applications of frames to problems in pure and applied mathematics, computer science, engineering, medicine, and physics with new applications arising regularly.

In 2005 Sun [20] introduced $g-$frames in Hibert space.

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Frank and Larson presented a general approach to the frame theory in Hilbert C^*-modules [21]. They showed that every countably generated Hilbert C^*-module over a unital C^*-algebra admits a frame. It was also shown in [7] that every Hilbert C^*-module is countably generated in the set of adjointable operators admits a frame of multipliers.

Furthermore, $g-$frames in Hilbert C^*-modules were introduced in [13] as generalization of the concept frames in Hilbert C^*-modules.

Dual frames are important to reconstruction of vector (or signals) in terms of the frame elements. In 2008, D. Han, W. Jing, D. Larson and R. N. Mohapatra are introduced the concept of dual frames in Hilbert C^*-modules [10]. Dual $g-$frames were introduced by A. Alijani and M. A. Dehghan [1] in 2012. The approximate dual frames in Hilbert C^*-modules is started by Lj. Arambasic [2] in

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2019. For more detailed information on frame theory, the reader is recommended to consult: [3, 9, 12, 14, 16, 17, 18, 19].

The Gabor transform as the set of all time-frequency shifts of a single vector in C^n has deep implication in signal processing [4]. From the correspondence of fusion frame in C^n and frame in Hilbert C^*-module $B(C^n)$ [15], we can study the signal processing in Hilbert C^*-module $B(C^n)$, also we can construct $g-$frame from frame in Hilbert C^*-module.

In this paper, we generalize several results previously established in Hilbert spaces [5] to the setting of Hilbert C^*-modules. The content of the present paper is as follows:

In Section 1, we state some of the definitions and basic properties of Hilbert C^*-modules. In Section 2, we introduce the concept of approximate oblique $g-$dual frame in Hilbert C^*-module and give some properties. In Section 3, we give a characterization of approximate oblique $g-$dual in Hilbert C^*-module. In Section 4, we study the problem of perturbation for the concept of approximate oblique $g-$dual in Hilbert C^*-module.

We start by introducing some notation, followed by a brief review of definitions and a presentation of the properties that will be used later.

Let $\mathcal{A}$ be a unital C^*-algebra, and $\mathbb{I}$ be finite or countable index set. Let $\mathcal{H}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$, and for each $i \in \mathbb{I}$, $\mathcal{H}_i$ be a closed submodules of $\mathcal{H}$. For each $i \in \mathbb{I}, \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i)$ is the collection of all adjointable $\mathcal{A}-$linear maps from $\mathcal{H}$ to $\mathcal{H}_i$ and $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$ is denoted by $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$.

For an operator $S \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, let $ranS$, $kerS$, $S^\dagger$ and S^* be the range space, the null space, pseudo-inverse and the adjoint of S, respectively. By Kasparov Stabilization Theorem [22], for any countably generated Hilbert $\mathcal{A}-$module $\mathcal{H}$, we have $\mathcal{H} \oplus \ell^2(\mathcal{A}) \cong \ell^2(\mathcal{A})$, where

$$\ell^2(\mathcal{A}) = \left\{ (a_i)_{i\in\mathbb{N}} : a_i \in \mathcal{A}, \sum_{i=1}^{\infty} a_i^* a_i \text{ converges in} || \cdot ||_{\mathcal{A}} \right\}.$$

Let $\{\mathcal{H}_i : i \in \mathbb{I}\}$ be submodules of $\mathcal{H}$, the space $\left(\sum_{i\in\mathbb{I}} \oplus\mathcal{H}_i\right)_{\ell^2}$ defined by

$$\left(\sum_{i\in\mathbb{I}} \oplus\mathcal{H}_i\right)_{\ell^2} = \left\{ x = (x_i)_{i\in\mathbb{I}} : x_i \in \mathcal{H}_i, \sum_{i\in\mathbb{I}} \langle x_i, x_i\rangle \text{ is norm convergent in } \mathcal{A} \right\}$$

is a Hilbert $\mathcal{A}-$module with inner product defined by $\langle x, y\rangle_{\mathcal{A}} = \sum_{i\in\mathbb{I}} \langle x_i, y_i\rangle$. A closed submodule $\mathcal{W}$ of $\mathcal{H}$ is topologically complemented if for some closed submodule $\mathcal{V}$ of $\mathcal{H}$ we have $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}$ and $P_{\mathcal{W}} : \mathcal{H} \to \mathcal{W}$ is the projection onto $\mathcal{W}$. We say $\mathcal{W}$ is orthogonally complemented if $\mathcal{H} = \mathcal{W} \oplus \mathcal{W}^\perp$ and in this case $P_{\mathcal{W}} \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}, \mathcal{W})$.

If $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$, the oblique projection onto $\mathcal{W}$ along $\mathcal{V}^\perp$, is the unique operator that satisfies $P_{\mathcal{W},\mathcal{V}^\perp} f = f$ for all $f \in \mathcal{W}$ and $P_{\mathcal{W},\mathcal{V}^\perp} f = 0$ for all $f \in \mathcal{V}^\perp$. Equivalently, $ran\left(P_{\mathcal{W},\mathcal{V}^\perp}\right) = \mathcal{W}$ and $ker\left(P_{\mathcal{W},\mathcal{V}^\perp}\right) = \mathcal{V}^\perp$.

Definition 1.1 ([11])**.** Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}-$module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}-$module if $\mathcal{H}$ is equipped with an $\mathcal{A}-$valued inner product $\langle ., .\rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In other words,

(1) $\langle x,x\rangle_{\mathcal{A}} \geqslant 0$ for all $x\in\mathcal{H}$ and $\langle x,x\rangle_{\mathcal{A}}=0$ if and only if $x=0$;
(2) $\langle ax+y,z\rangle_{\mathcal{A}} = a\langle x,z\rangle_{\mathcal{A}}+\langle y,z\rangle_{\mathcal{A}}$ for all $a\in\mathcal{A}$ and $x,y,z\in\mathcal{H}$;
(3) $\langle x,y\rangle_{\mathcal{A}}=\langle y,x\rangle_{\mathcal{A}}^{*}$ for all $x,y\in\mathcal{H}$.

For $x\in\mathcal{H}$, we define $\|x\|=\|\langle x,x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $\|.\|$, it is called a Hilbert $\mathcal{A}-$module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a|=(a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}-$valued norm on $\mathcal{H}$ is defined by $|x|=\langle x,x\rangle_{\mathcal{A}}^{\frac{1}{2}}$ for $x\in\mathcal{H}$.

Definition 1.2. We call a sequence $\Lambda=\{\Lambda_i\in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H},\mathcal{H}_i): i\in\mathbb{I}\}$ a $g-$frame in Hilbert $\mathcal{A}-$module $\mathcal{H}$ with respect to $\{\mathcal{H}_i: i\in\mathbb{I}\}$ if there exist two positive constants $0<A\leq B\leq\infty$, such that for all $x\in\mathcal{H}$,

$$A\langle x,x\rangle_{\mathcal{A}} \leq \sum_{i\in\mathbb{I}}\langle\Lambda_i x,\Lambda_i x\rangle_{\mathcal{A}}\leq B\langle x,x\rangle_{\mathcal{A}}.$$

The numbers A and B are called lower and upper bounds of the $g-$frame, respectively. If $A=B=\lambda$, the $g-$frame is $\lambda-$tight. If $A=B=1$, it is called a $g-$Parseval frame. If the sum is convergent in norm, the $g-$frame is called standard. We use T_Λ, T_Λ^* and S_Λ to denote the analysis operator, synthesis operator and the frame operator respectively;

(1) The analysis operator is defined by $T_\Lambda:\mathcal{H}\to\left(\sum_{i\in\mathbb{I}}\oplus\mathcal{H}_i\right)_{\ell^2}, T_\Lambda x=\{\Lambda_i x\}_{i\in\mathbb{I}}$.
(2) The synthesis operator is defined by

$$T_\Lambda^*:\left(\sum_{i\in\mathbb{I}}\oplus\mathcal{H}_i\right)_{\ell^2}\to\mathcal{H}, T_\Lambda^* y=\sum_{i\in\mathbb{I}}\Lambda_i^* y_i,\quad y=\{y_i\}_{i\in\mathbb{I}}\in\left(\sum_{i\in\mathbb{I}}\oplus\mathcal{H}_i\right)_{\ell^2}.$$

(3) The $g-$frame operator S_Λ is defined as $S_\Lambda=T_\Lambda^*T_\Lambda:\mathcal{H}\to\mathcal{H}$ is given by $S_\Lambda x=\sum_{i\in\mathbb{I}}\Lambda_i^*\Lambda_i x$, for each $x\in\mathcal{H}$. The $g-$frame operator is positive, invertible, and the follow reconstruction formula holds

$$x=S_\Lambda S_\Lambda^{-1}x=\sum_{i\in\mathbb{I}}\Lambda_i^*\Lambda_i S_\Lambda^{-1}x=S_\Lambda^{-1}S_\Lambda x=\sum_{i\in\mathbb{I}}S_\Lambda^{-1}\Lambda_i^*\Lambda_i x.$$

Given closed submodules $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{H}$. We assume that this equalitie $\mathcal{H}=\mathcal{W}\oplus\mathcal{V}^\perp$ holds throughout the remainder of the article. A sequence $\Lambda=\{\Lambda_i\in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H},\mathcal{H}_i): i\in\mathbb{I}\}$ is a $g-$frame for $\mathcal{W}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ if and only if $\overline{\mathrm{span}}\{\Lambda_i^*\mathcal{H}_i\}_{i\in\mathbb{I}}=\mathcal{W}$. If Λ is a $g-$Bessel sequence for $\mathcal{W}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, then Λ is a $g-$Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ and $\overline{\mathrm{span}}\{\Lambda_i^*\mathcal{H}_i\}_{i\in\mathbb{I}}\subset\mathcal{W}$.

2. Approximate Oblique Dual $g-$Frames in Hilbert C^*-modules

Now we introduce the concept of approximate dual, oblique dual and approximate oblique dual in Hilbert C^*-modules.

Definition 2.1. Let $\Lambda=\{\Lambda_i\in End_{\mathcal{A}}^*(\mathcal{H},\mathcal{H}_i)_{i\in\mathbb{I}}\}$ and $\Theta=\{\Theta_i\in End_{\mathcal{A}}^*(\mathcal{H},\mathcal{H}_i)_{i\in\mathbb{I}}\}$ be $g-$Bessel sequences for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$. Then Λ and Θ are approximate duals if

$$\|I-T_\Lambda^*T_\Theta\|<1\quad or\quad \|I-T_\Theta^*T_\Lambda\|<1.$$

Definition 2.2. Let Λ be a $g-$frame for $\mathcal{W}$ and Θ a $g-$Bessel sequence for $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$. Θ is a dual of Λ if $T_\Lambda^* T_\Theta|_{\mathcal{W}} = I|_{\mathcal{W}}$. Θ is an oblique dual of Λ if Θ is a $g-$frame for $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ that is dual to Λ, and Λ is dual to Θ.

Lemma 2.1. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ a Hilbert $\mathcal{A}-$module. Assume that $\mathcal{W}$ and $\mathcal{V}$ are closed submodules of $\mathcal{H}$, then we hawe*
(1) $(P_{\mathcal{W},\mathcal{V}^\perp})^* = P_{\mathcal{V},\mathcal{W}^\perp}$.
(2) $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp \iff \mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$.

Proof. (1) Let $P := P_{\mathcal{W},\mathcal{V}^\perp} : \mathcal{H} \to \mathcal{W}$ satisfy

$$P_{\mathcal{W},\mathcal{V}^\perp} f = w \quad \text{for all } f \in \mathcal{W}, \quad \text{and} \quad P_{\mathcal{W},\mathcal{V}^\perp} f = 0 \quad \text{for all } f \in \mathcal{V}^\perp.$$

Let $Q := P_{\mathcal{V},\mathcal{W}^\perp} : \mathcal{H} \to \mathcal{V}$ satisfy

$$P_{\mathcal{V},\mathcal{W}^\perp} f = f \quad \text{for all } f \in \mathcal{V}, \quad \text{and} \quad P_{\mathcal{V},\mathcal{W}^\perp} f = 0 \quad \text{for all } f \in \mathcal{W}^\perp.$$

For any $x, y \in \mathcal{H}$, we have

$$x = w_x + v_x^\perp, \quad \text{where } w_x \in \mathcal{W},\ v_x^\perp \in \mathcal{V}^\perp,$$

and

$$y = w_y^\perp + v_y, \quad \text{where } w_y^\perp \in \mathcal{W}^\perp,\ v_y \in \mathcal{V}.$$

By definition,

$$Px = P(w_x + v_x^\perp) = P(w_x) + P(v_x^\perp) = w_x + 0 = w_x,$$

and

$$Qy = Q(w_y^\perp + v_y) = Q(w_y^\perp) + Q(v_y) = 0 + v_y = v_y.$$

Now compute the inner product:

$$\langle Px, y\rangle = \langle w_x, w_y^\perp + v_y\rangle = \langle w_x, w_y^\perp\rangle + \langle w_x, v_y\rangle.$$

Since $w_x \in \mathcal{W}$ and $w_y^\perp \in \mathcal{W}^\perp$, we have $\langle w_x, w_y^\perp\rangle = 0$. Hence, $\langle Px, y\rangle = \langle w_x, v_y\rangle$. Similarly,

$$\langle x, Qy\rangle = \langle w_x + v_x^\perp, v_y\rangle = \langle w_x, v_y\rangle + \langle v_x^\perp, v_y\rangle = \langle w_x, v_y\rangle,$$

since $v_x^\perp \in V^\perp$ and $v_y \in V$. Thus, $\langle Px, y\rangle = \langle x, Qy\rangle$, which proves that $P_{\mathcal{W},\mathcal{V}^\perp}$ is adjointable and that $(P_{\mathcal{W},\mathcal{V}^\perp})^* = P_{\mathcal{V},\mathcal{W}^\perp}$.

(2) $\Longrightarrow$ Suppose that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$. Let $P := P_{\mathcal{W},\mathcal{V}^\perp} : \mathcal{H} \to \mathcal{W}$ be the projection onto $\mathcal{W}$ along $\mathcal{V}^\perp$. Then, $\mathrm{Ran}(P) = \mathcal{W}$ and $\mathrm{Ker}(P) = \mathcal{V}^\perp$. Since P is adjointable, we have, $\mathrm{Ker}(P^*) = \mathrm{Ran}(P)^\perp$ and $\mathrm{Ran}(P^*) \subseteq \mathrm{Ker}(P)^\perp$. Because $\mathrm{Ran}(P^*)$ is closed (as the image of an adjointable projection), we obtain $\mathrm{Ran}(P^*) = \mathrm{Ker}(P)^\perp$. Hence,

$$\mathrm{Ker}(P^*) = \mathrm{Ran}(P)^\perp = \mathcal{W}^\perp, \quad \text{and} \quad \mathrm{Ran}(P^*) = \mathrm{Ker}(P)^\perp = (\mathcal{V}^\perp)^\perp = \mathcal{V},$$

since $\mathcal{V}$ is closed. Therefore, $\mathcal{H} = \mathrm{Ran}(P^*) \oplus \mathrm{Ker}(P^*) = \mathcal{V} \oplus \mathcal{W}^\perp$.

$\Longleftarrow$ Conversely, suppose that $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$. Let $Q := P_{\mathcal{V},\mathcal{W}^\perp} : \mathcal{H} \to \mathcal{V}$ be the projection onto $\mathcal{V}$ along $\mathcal{W}^\perp$. Then, $\mathrm{Ran}(Q) = \mathcal{V}$ and $\mathrm{Ker}(Q) = \mathcal{W}^\perp$. Since, $P^* = (P_{\mathcal{W},\mathcal{V}^\perp})^* = P_{\mathcal{V},\mathcal{W}^\perp} = Q$, we have

$$\mathrm{Ran}(Q^*) = \mathrm{Ran}(P) = \mathcal{W}, \quad \text{and} \quad \mathrm{Ker}(Q^*) = \mathrm{Ker}(P) = \mathcal{V}^\perp.$$

Consequently, $\mathcal{H} = \mathrm{Ran}(Q^*) \oplus \mathrm{Ker}(Q^*) = \mathcal{W} \oplus \mathcal{V}^\perp$. $\square$

We know that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^{\perp}$. This is equivalent to $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^{\perp}$. Then Θ and Λ are oblique duals if and only if they are $g-$frames for closed submodules $\mathcal{W}$ and $\mathcal{V}$, respectively. And $P_{\mathcal{W},\mathcal{V}^{\perp}} = T_{\Lambda}^{*}T_{\Theta}$ or equivalently, $P_{\mathcal{V},\mathcal{W}^{\perp}} = T_{\Theta}^{*}T_{\Lambda}$.

Definition 2.3. Let Λ and Θ be $g-$frames for $\mathcal{W}$ and $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, respectively. Let $\epsilon \geq 0$. We say Λ and Θ are $\epsilon-$approximate oblique duals if

$$\|P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\Lambda}^{*}T_{\Theta}\| \leq \epsilon \quad or \quad \|P_{\mathcal{V},\mathcal{W}^{\perp}} - T_{\Theta}^{*}T_{\Lambda}\| \leq \epsilon.$$

Lemma 2.2. *(1) If $\epsilon < 1$, Λ and Θ are $\epsilon-$approximate oblique duals, then $T_{\Lambda}^{*}T_{\Theta} \mid_{\mathcal{W}}$ is an invertible operator on $\mathcal{W}$.*
(2) If $\epsilon < 1$, then it is sufficient that Λ and Θ are $g-$Bessel sequences for $\mathcal{W}$ and $\mathcal{V}$, respectively, for Λ and Θ to be $\epsilon-$approximate oblique duals.

Proof. (1) If $\epsilon < 1$, For $f \in \mathcal{W}$, we have $P_{\mathcal{W},\mathcal{V}^{\perp}}f = f$, then

$$\begin{aligned}\|I - T_{\Lambda}^{*}T_{\Theta}\| &= \|P_{\mathcal{W},\mathcal{V}^{\perp}} - P_{\mathcal{W},\mathcal{V}^{\perp}} + I - T_{\Lambda}^{*}T_{\Theta}\| \\ &\leq \|P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\Lambda}^{*}T_{\Theta}\| + \|P_{\mathcal{W},\mathcal{V}^{\perp}} - I\| \leq \epsilon < 1.\end{aligned}$$

Hence $T_{\Lambda}^{*}T_{\Theta}|_{\mathcal{W}}$ is an invertible operator on $\mathcal{W}$ by Neumann's Theorem.
(2) If $\epsilon < 1$, then $T_{\Lambda}^{*}T_{\Theta}|_{\mathcal{W}}$ is an invertible operator on $\mathcal{W}$. For each $f \in \mathcal{W}$, we get

$$f = T_{\Lambda}^{*}T_{\Theta}\left(T_{\Lambda}^{*}T_{\Theta} \mid \mathcal{W}\right)^{-1} f = \sum_{i\in\mathbb{I}} \Lambda_i^{*}\Theta_i \left(T_{\Lambda}^{*}T_{\Theta} \mid \mathcal{W}\right)^{-1} f.$$

Furthermore,

$$\begin{aligned}\|f\|^4 &= \|\langle f, f\rangle_{\mathcal{A}}\|^2 \\ &= \left\|\sum_{i\in\mathbb{I}} \left\langle \Lambda_i^{*}\Theta_i \left(T_{\Lambda}^{*}T_{\Theta} \mid \mathcal{W}\right)^{-1} f, f\right\rangle_{\mathcal{A}}\right\|^2 \\ &= \left\|\sum_{i\in\mathbb{I}} \left\langle \Theta_i \left(T_{\Lambda}^{*}T_{\Theta} \mid \mathcal{W}\right)^{-1} f, \Lambda_i f\right\rangle_{\mathcal{A}}\right\|^2 \\ &\leq \left\|\sum_{i\in\mathbb{I}} \left\langle \Theta_i \left(T_{\Lambda}^{*}T_{\Theta} \mid \mathcal{W}\right)^{-1} f, \Theta_i \left(T_{\Lambda}^{*}T_{\Theta} \mid \mathcal{W}\right)^{-1} f\right\rangle_{\mathcal{A}}\right\| \left\|\sum_{i\in\mathbb{I}} \langle \Lambda_i f, \Lambda_i f\rangle_{\mathcal{A}}\right\|.\end{aligned}$$

Then Λ is a $g-$frame for $\mathcal{W}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$. Similarly, Θ is a $g-$frame for $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$.

□

Let $\Lambda = \{\Lambda_i \in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ be a $g-$frame for $\mathcal{W}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$. Suppose that we have access to the samples $T_{\Lambda}f = \{\Lambda_i f\}_{i\in\mathbb{I}}$ of an unknown signal $f \in \mathcal{H}$. Our objective is to reconstruct f from these samples by using a $g-$frame $\Theta = \{\Theta_i \in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ for $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ ensuring that the reconstruction $\hat{f} \in \mathcal{V}$ is a best approximation of f. To develop the reconstruction algorithm, we begin by imposing the following basic requirements. Let $\epsilon \geq 0$.

(1) Uniqueness of reconstruction: If $f, g \in \mathcal{V}$ and $T_{\Lambda}f = T_{\Theta}g$, then $f = g$.
(2) $\epsilon-$consistent reconstruction: For every $f \in \mathcal{H}$, $\left\|T_{\Lambda}\hat{f} - T_{\Lambda}f\right\| \leq \epsilon\|f\|$.

The first condition guarantees uniqueness in the sampling process, ensuring that no two distinct signals in $\mathcal{V}$ share identical samples. To meet this condition, it is necessary that $\mathcal{V}\cap\mathcal{W}^{\perp}=\{0\}$. If the second requirement is also satisfied, the samples of the reconstructed $\hat{f}$ are close to the samples of the original signal f. In this case, we say that $\hat{f}$ is an $\epsilon-$consistent reconstruction of $f\in\mathcal{H}$. The following result demonstrates the connection between $\epsilon-$consistent reconstruction and oblique projections.

Theorem 2.1. *Let $\Lambda=\{\Lambda_i\in\operatorname{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_i):i\in\mathbb{I}\}$ be a $g-$frame for $\mathcal{W}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}, f\in\mathcal{H}$ and $\epsilon\geq 0$. The following hold:*

(1) If $\hat{f}$ is an $\epsilon-$consistent reconstruction of f in $\mathcal{V}$, then

$$\left\|\hat{f}-P_{\mathcal{V},\mathcal{W}^{\perp}}f\right\|\leq\epsilon\left\|P_{\mathcal{V},\mathcal{W}^{\perp}}\right\|\left\|T_{\Lambda}^{\dagger}\right\|\|f\|.$$

(2) If $\left\|\hat{f}-P_{\mathcal{V},\mathcal{W}^{\perp}}f\right\|\leq\frac{\epsilon}{\|T_{\Lambda}\|}\|f\|$, then $\hat{f}$ is an $\epsilon-$consistent reconstruction of f in $\mathcal{V}$.

Proof. (1) Let $\Lambda=\{\Lambda_i\in\operatorname{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_i):i\in\mathbb{I}\}$ be a $g-$frame for $\mathcal{W}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$. Let $\widetilde{\Lambda}_i=\Lambda_i S_{\Lambda}^{\dagger}$ and $\Theta=\left\{\Theta_i=\widetilde{\Lambda}_i P_{\mathcal{W},\mathcal{V}^{\perp}}\in\operatorname{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_i)\right\}_{i\in\mathbb{I}}$, then Θ is a $g-$Bessel sequence for $\mathcal{H}$. For $f\in\mathcal{W}$, we have $P_{\mathcal{W},\mathcal{V}^{\perp}}f=f$, hence

$$f=\sum_{i\in\mathbb{I}}\Lambda_i^*\Lambda_i S_{\Lambda}^{\dagger}f=\sum_{i\in\mathbb{I}}\Lambda_i^*\Lambda_i S_{\Lambda}^{\dagger}P_{\mathcal{W},\mathcal{V}^{\perp}}f=\sum_{i\in\mathbb{I}}\Lambda_i^*\Theta_i f.$$

Then Θ is a dual of Λ. Similarly, for $f\in\mathcal{V}$, we have $P_{\mathcal{V},\mathcal{W}^{\perp}}f=f$, which means

$$\begin{aligned}f&=\sum_{i\in\mathbb{I}}\Lambda_i^*\Lambda_i S_{\Lambda}^{\dagger}f=\sum_{i\in\mathbb{I}}\Lambda_i^*\Lambda_i S_{\Lambda}^{\dagger}P_{\mathcal{V},\mathcal{W}^{\perp}}f\\&=\sum_{i\in\mathbb{I}}P_{\mathcal{V},\mathcal{W}^{\perp}}S_{\Lambda}^{\dagger}\Lambda_i^*\Lambda_i f=\sum_{i\in\mathbb{I}}P_{\mathcal{V},\mathcal{W}^{\perp}}(\Lambda_i S_{\Lambda}^{\dagger})^*\Lambda_i f\\&=\sum_{i\in\mathbb{I}}P_{\mathcal{V},\mathcal{W}^{\perp}}\widetilde{\Lambda_i}^*\Lambda_i f=\sum_{i\in\mathbb{I}}(P_{\mathcal{W},\mathcal{V}^{\perp}})^*\widetilde{\Lambda_i}^*\Lambda_i f\\&=\sum_{i\in\mathbb{I}}(\widetilde{\Lambda}_i P_{\mathcal{W},\mathcal{V}^{\perp}})^*\Lambda_i f=\sum_{i\in\mathbb{I}}\Theta_i^*\Lambda_i f.\end{aligned}$$

Then Λ is a dual of Θ. Consequently, Θ is an oblique dual of Λ and $T_{\Theta}^*T_{\Lambda}=P_{\mathcal{V},\mathcal{W}^{\perp}}$. We also obtain $T_{\Theta}^*=P_{\mathcal{V},\mathcal{W}^{\perp}}T_{\Lambda}^{\dagger}$. Then for $f\in\mathcal{H}$,

$$\begin{aligned}\left\|\hat{f}-P_{\mathcal{V},\mathcal{W}^{\perp}}f\right\|&=\left\|T_{\Theta}^*T_{\Lambda}\hat{f}-T_{\Theta}^*T_{\Lambda}P_{\mathcal{V},\mathcal{W}^{\perp}}f\right\|\\&\leq\|T_{\Theta}^*\|\left\|T_{\Lambda}\hat{f}-T_{\Lambda}f\right\|\\&\leq\epsilon\|T_{\Theta}^*\|\|f\|\\&\leq\epsilon\left\|P_{\mathcal{V},\mathcal{W}^{\perp}}\right\|\left\|T_{\Lambda}^{\dagger}\right\|\|f\|.\end{aligned}$$

(2) Suppose that, for all $f\in\mathcal{H}$, $\left\|\hat{f}-P_{\mathcal{V},\mathcal{W}^{\perp}}f\right\|\leq\frac{\epsilon}{\|T_{\Lambda}\|}\|f\|$. Since $f\in\mathcal{H}$, then $f=u_1+u_2$ such that $u_1\in\mathcal{V}$ and $u_2\in\mathcal{W}^{\perp}$. We have Λ is a $g-$frame for $\mathcal{W}$,

we get $T_\Lambda|_{W^\perp} = 0$, so

$$T_\Lambda(f) = T_\Lambda(u_1 + u_2) = T_\Lambda(u_1) = T_\Lambda P_{\mathcal{V},\mathcal{W}^\perp}(u_1) = T_\Lambda P_{\mathcal{V},\mathcal{W}^\perp}(f).$$

Therefore, we obtain

$$\begin{aligned}\|T_\Lambda(\hat{f}) - T_\Lambda(f)\| &= \|T_\Lambda(\hat{f}) - T_\Lambda P_{\mathcal{V},\mathcal{W}^\perp}(f)\| \\ &\leq \|T_\Lambda\|\|\hat{f} - P_{\mathcal{V},\mathcal{W}^\perp}(f)\| \leq \epsilon\|f\|.\end{aligned}$$

Then $\hat{f}$ is an $\epsilon-$consistent reconstruction of f in $\mathcal{V}$.

□

Corollary 2.1. *Let $\Lambda = \{\Lambda_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ be a $g-$frame for $\mathcal{W}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, $\Theta = \{\Theta_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ be a $g-$frame for $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ and $\epsilon \geq 0$. The following hold:*

(1) If $\hat{f} = T^*_\Theta T_\Lambda f$ is an $\frac{\epsilon}{\left\|P_{\mathcal{V},\mathcal{W}^\perp}\right\|\left\|T^\dagger_\Lambda\right\|}-$consistent reconstruction of f in $\mathcal{V}$ for $f \in \mathcal{H}$, then Λ and Θ are $\epsilon-$approximate oblique duals.

(2) If Λ and Θ are $\epsilon-$approximate oblique duals, then $\hat{f} = T^*_\Theta T_\Lambda f$ is an $\epsilon\,\|T_\Lambda\|-$consistent reconstruction of f in $\mathcal{V}$ for all $f \in \mathcal{H}$.

Proof. (1) Since

$$\|P_{\mathcal{V},\mathcal{W}\perp} - T^*_\Theta T_\Lambda\| = \sup_{\|f\|\leq 1} \left\|P_{\mathcal{V},\mathcal{W}\perp} f - \hat{f}\right\| \leq \sup_{\|f\|\leq 1} (\epsilon\|f\|) \leq \epsilon,$$

then Λ and Θ are $\epsilon-$approximate oblique duals.

(2) Suppose that Λ and Θ are $\epsilon-$approximate oblique duals, which means that

$$\|P_{\mathcal{V},\mathcal{W}\perp} f - T^*_\Theta T_\Lambda f\| = \left\|P_{\mathcal{V},\mathcal{W}\perp} f - \hat{f}\right\| \leq \frac{\epsilon\,\|T_\Lambda\|}{\|T_\Lambda\|}\|f\| = \epsilon\|f\|.$$

Then $\hat{f}$ is an $\epsilon\,\|T_\Lambda\|-$consistent reconstruction of f in $\mathcal{V}$.

□

3. Characterizations of Approximate Oblique Duals

This section focuses on the characterizations of approximate oblique dual $g-$frames for closed submodules.

Remark 3.1. Let $\Lambda = \{\Lambda_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ and $\Theta = \{\Theta_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ be g-frames for $\mathcal{W}$ and $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, respectively. Let $\epsilon \geq 0$, then the following statements are equivalent:

(1) Λ and Θ are $\epsilon-$approximate oblique duals.
(2) $\left\|P_{\mathcal{W},\mathcal{V}^\perp} f - \sum_{i\in\mathbb{I}} \Lambda^*_i \Theta_i f\right\| \leq \epsilon\|f\|$ for all $f \in \mathcal{H}$.
(3) $\left\|P_{\mathcal{V},\mathcal{W}^\perp} f - \sum_{i\in\mathbb{I}} \Theta^*_i \Lambda_i f\right\| \leq \epsilon\|f\|$ for all $f \in \mathcal{H}$.

Proposition 3.1. *Let $\Lambda = \{\Lambda_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ be a $g-$Bessel sequence for $\mathcal{W}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ and $\Theta = \{\Theta_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ be a $g-$Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ such that for all $f \in \mathcal{W}$*

$$\|f - \sum_{i\in\mathbb{I}} \Lambda^*_i \Theta_i f\| \leq \epsilon\|f\|.$$

If $\epsilon \left\|P_{\mathcal{W},\mathcal{V}^\perp}\right\| < 1$, *then* $\left\{\Theta_i P_{\mathcal{W},\mathcal{V}^\perp}\right\}_{i\in\mathbb{I}}$ *is a* $g-$*frame for* $\mathcal{V}$ *with respect to* $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, Λ *is a g-frame for* $\mathcal{W}$ *with respect to* $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, *and* $\left\{\Theta_i P_{\mathcal{W},\mathcal{V}^\perp}\right\}_{i\in\mathbb{I}}$ *and* Λ *are* $\epsilon\left\|P_{\mathcal{W},\mathcal{V}\perp}\right\|-$*approximate oblique duals.*

Proof. For all $f\in\mathcal{H}$, we get

$$\begin{aligned}\|P_{\mathcal{W},\mathcal{V}\perp}f - T_\Lambda^* T_\Theta P_{\mathcal{W},\mathcal{V}\perp}f\| &= \left\|P_{\mathcal{W},\mathcal{V}^\perp}f - \sum_{i\in\mathbb{I}}\Lambda_i^*\Theta_i P_{\mathcal{W},\mathcal{V}^\perp}f\right\| \\ &\le \epsilon\|P_{\mathcal{W},\mathcal{V}^\perp}\|\|f\| < \|f\|.\end{aligned}$$

Then Λ is a $g-$frame for $\mathcal{W}$ and $\left\{\Theta_i P_{\mathcal{W},\mathcal{V}^\perp}\right\}_{i\in\mathbb{I}}$ is a $g-$frame for $\mathcal{V}$. We also have Λ and $\left\{\Theta_i P_{\mathcal{W},\mathcal{V}^\perp}\right\}_{i\in\mathbb{I}}$ are $\epsilon\left\|P_{\mathcal{W},\mathcal{V}^\perp}\right\|-$approximate oblique duals. □

Theorem 3.1. *Let* $\Lambda = \{\Lambda_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_i) : i\in\mathbb{I}\}$ *and* $\Theta = \{\Theta_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_i) : i\in\mathbb{I}\}$.

(1) Let Λ *and* Θ *be* $g-$*Bessel sequences for* $\mathcal{W}$ *and* $\mathcal{V}$ *with respect to* $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, *respectively. Suppose that* Λ *and* Θ *are approximate duals.*
Then $\left\{\Gamma = \Theta_i (T_\Lambda^* T_\Theta)^{-1}\right\}_{i\in\mathbb{I}}$ *is an approximate dual of* Λ *for* $\mathcal{V}$.

(2) Let Λ *and* Θ *be* $g-$*frame for* $\mathcal{W}$ *and* $\mathcal{V}$ *with respect to* $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, *respectively. Suppose that* Λ *and* Θ *are* $\epsilon-$*approximate duals.*
Then Λ *and* Θ *are oblique duals of* $\widetilde{\Theta} = \left\{\Theta_i (I - P_{\mathcal{V},\mathcal{W}\perp} + T_\Theta^* T_\Lambda)^{-1}\right\}_{i\in\mathbb{I}}$ *and* $\widetilde{\Lambda} = \left\{\Lambda_i (I - P_{\mathcal{W},\mathcal{V}\perp} + T_\Lambda^* T_\Theta)^{-1}\right\}_{i\in\mathbb{I}}$, *respectively.*

(3) Let Λ *and* Θ *be* $g-$*frames for* $\mathcal{W}$ *and* $\mathcal{V}$ *with respect to* $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, *respectively. Let* $0\le\epsilon<1$ *such that* Λ *and* Θ *are* $\epsilon-$*approximate duals. Then* $\widetilde{\Theta}$ *is an* $\frac{\epsilon}{1-\epsilon}-$*approximate oblique dual of* $\widetilde{\Lambda}$ *for* $\mathcal{V}$.

Proof. (1) Suppose that Λ and Θ are approximate duals, then $\|I - T_\Theta^* T_\Lambda\| \le 1$ and the inverse of $T_\Lambda^* T_\Theta$ can be written via a Neumann series as

$$(T_\Lambda^* T_\Theta)^{-1} = (I - (I - T_\Lambda^* T_\Theta))^{-1} = \sum_{n=0}^{N} (I - T_\Lambda^* T_\Theta)^n .$$

We prove that Γ is a $g-$Bessel sequence for $\mathcal{H}$.
Let $\Gamma_i^{(N)} = \sum_{n=0}^{N}\Theta_i (I - T_\Lambda^* T_\Theta)^n$ for each $N\in\mathbb{N}$. Define the operator $L_N : \mathcal{H}\to\mathcal{H}$ by $L_N = \sum_{n=0}^{N}(I - T_\Lambda^* T_\Theta)^n$. Thus, L_N is an adjointable operator and $\Gamma_i^{(N)} = \Theta_i L_N, \forall i\in\mathbb{I}$. The sequence $\left\{\Gamma_i^{(N)}\right\}_{i\in\mathbb{I}}$ is obtained from the $g-$Bessel sequence $\{\Theta_i\}_{i\in\mathbb{I}}$ by an adjointable operator, it follows that it is a $g-$Bessel sequence for $\mathcal{H}$. Suppose that T_Γ is the analysis operator of $\left\{\Gamma_i^{(N)}\right\}_{i\in\mathbb{I}}$. For $f\in\mathcal{H}$,

$$T_\Lambda^* T_\Theta L_N f = \sum_{i\in\mathbb{I}} \Lambda_i^*\Theta_i L_N f = \sum_{i\in\mathbb{I}}\Lambda_i^*\Gamma_i^{(N)} f = T_\Lambda^* T_\Gamma f.$$

Furthermore,

$$\begin{aligned}
T_\Gamma^* T_\Lambda f = L_N^* T_\Theta^* T_\Lambda f &= \sum_{n=0}^{N} \left(I - T_\Theta^* T_\Lambda\right)^n T_\Theta^* T_\Lambda f \\
&= \sum_{n=0}^{N} \left(I - T_\Theta^* T_\Lambda\right)^n \left(I - \left(I - T_\Theta^* T_\Lambda\right)\right) f \\
&= \sum_{n=0}^{N} \left(I - T_\Theta^* T_\Lambda\right)^n f - \sum_{n=0}^{N} \left(I - T_\Theta^* T_\Lambda\right)^{n+1} f \\
&= f - \left(I - T_\Theta^* T_\Lambda\right)^{N+1} f.
\end{aligned}$$

Consequently,

$$\|I - T_\Gamma^* T_\Lambda\| = \left\|\left(I - T_\Theta^* T_\Lambda\right)^{N+1}\right\| \leq \|I - T_\Theta^* T_\Lambda\|^{N+1} \leq 1.$$

Given that Θ is a $g-$Bessel sequence for $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, then Γ is a $g-$Bessel sequence for $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, which implies that Γ is an approximate dual of Λ for $\mathcal{V}$.

(2) Suppose that Λ and Θ are $\epsilon-$approximate duals, then

$$\|P_{\mathcal{W},\mathcal{V}^\perp} - T_\Lambda^* T_\Theta\| \leq \epsilon < 1,$$

and the inverse of $P_{\mathcal{W},\mathcal{V}^\perp} - T_\Lambda^* T_\Theta$ can be written via a Neumann series as

$$\left(I - \left(P_{\mathcal{W},\mathcal{V}^\perp} - T_\Lambda^* T_\Theta\right)\right)^{-1} = \sum_{n=0}^{\infty} \left(P_{\mathcal{W},\mathcal{V}^\perp} - T_\Lambda^* T_\Theta\right)^n.$$

Define the operator $J_N : \mathcal{H} \to \mathcal{H}$ by $J_N = \sum_{n=0}^{N} \left(P_{\mathcal{W},\mathcal{V}^\perp} - T_\Lambda^* T_\Theta\right)^n$. Let $\delta_i^{(N)} = \sum_{n=0}^{N} \Theta_i \left(P_{\mathcal{W},\mathcal{V}^\perp} - T_\Lambda^* T_\Theta\right)^n$, for each $N \in \mathbb{N}, i \in \mathbb{I}$. So, J_N is an adjointable operator and $\delta_i^{(N)} = \Theta_i J_N, \forall i \in \mathbb{I}$. The sequence $\left\{\delta_i^{(N)}\right\}_{i\in\mathbb{I}}$ is obtained from the $g-$Bessel sequence $\{\Theta_i\}_{i\in\mathbb{I}}$ by an adjointable operator, then it is a g-Bessel sequence for $\mathcal{H}$. Let T_δ is the analysis operator of $\left\{\delta_i^{(N)}\right\}_{i\in\mathbb{I}}$. For $f \in \mathcal{H}$,

$$T_\Lambda^* T_\Theta J_N f = \sum_{i\in\mathbb{I}} \Lambda_i^* \Theta_i J_N f = \sum_{i\in\mathbb{I}} \Lambda_i^* \delta_i^{(N)} f = T_\Lambda^* T_\delta f.$$

Then

$$\begin{aligned}
T_\delta^* T_\Lambda f = J_N^* T_\Theta^* T_\Lambda f &= \sum_{n=0}^{N} \left(P_{\mathcal{V},\mathcal{W}^\perp} - T_\Theta^* T_\Lambda\right)^n T_\Theta^* T_\Lambda P_{\mathcal{V},\mathcal{W}^\perp} f \\
&= \sum_{n=0}^{N} \left(P_{\mathcal{V},\mathcal{W}^\perp} - T_\Theta^* T_\Lambda\right)^n \left(P_{\mathcal{V},\mathcal{W}^\perp} - \left(P_{\mathcal{V},\mathcal{W}^\perp} - T_\Theta^* T_\Lambda\right)\right) P_{\mathcal{V},\mathcal{W}^\perp} f \\
&= \sum_{n=0}^{N} \left(P_{\mathcal{V},\mathcal{W}^\perp} - T_\Theta^* T_\Lambda\right)^n P_{\mathcal{V},\mathcal{W}^\perp} f - \sum_{n=0}^{N} \left(P_{\mathcal{V},\mathcal{W}^\perp} - T_\Theta^* T_\Lambda\right)^{n+1} P_{\mathcal{V},\mathcal{W}^\perp} f \\
&= P_{\mathcal{V},\mathcal{W}^\perp} f - \left(P_{\mathcal{V},\mathcal{W}^\perp} - T_\Theta^* T_\Lambda\right)^{N+1} P_{\mathcal{V},\mathcal{W}^\perp} f.
\end{aligned}$$

For $f \in \mathcal{W}^{\perp}$ we have $T_{\Lambda} f = 0$, then $\left(P_{\mathcal{V},\mathcal{W}^{\perp}} - T_{\Theta}^{*} T_{\Lambda}\right) f = 0$. Hence $\mathcal{W}^{\perp} \subset \ker\left(P_{\mathcal{V},\mathcal{W}^{\perp}} - T_{\Theta}^{*} T_{\Lambda}\right)$. Therefore,

$$\begin{aligned} P_{\mathcal{V},\mathcal{W}^{\perp}} f - T_{\delta}^{*} T_{\Lambda} f &= \left(P_{\mathcal{V},\mathcal{W}^{\perp}} - T_{\Theta}^{*} T_{\Lambda}\right)^{N+1} P_{\mathcal{V},\mathcal{W}^{\perp}} f \\ &= \left(P_{\mathcal{V},\mathcal{W}^{\perp}} - T_{\Theta}^{*} T_{\Lambda}\right)^{N+1} f. \end{aligned}$$

Consequently,

$$\begin{aligned} \left\| P_{\mathcal{V},\mathcal{W}^{\perp}} - T_{\delta}^{*} T_{\Lambda} \right\| &= \left\| \left(P_{\mathcal{V},\mathcal{W}^{\perp}} - T_{\Theta}^{*} T_{\Lambda}\right)^{N+1} \right\| \\ &\leq \left\| P_{\mathcal{V},\mathcal{W}^{\perp}} - T_{\Theta}^{*} T_{\Lambda} \right\|^{N+1} \leq \epsilon^{N+1} \to 0. \end{aligned}$$

Hence,

$$\begin{aligned} \left\| P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\Lambda}^{*} T_{\widetilde{\Theta}} \right\| &= \left\| P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\Lambda}^{*} T_{\Theta} J_N + T_{\Lambda}^{*} T_{\Theta} J_N - T_{\Lambda}^{*} T_{\widetilde{\Theta}} \right\| \\ &\leq \left\| P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\Lambda}^{*} T_{\Theta} J_N \right\| + \left\| T_{\Lambda}^{*} T_{\Theta} J_N - T_{\Lambda}^{*} T_{\widetilde{\Theta}} \right\| \\ &\leq \epsilon^{N+1} + \left\| T_{\Lambda}^{*} \right\| \left\| T_{\Theta} J_N - T_{\widetilde{\Theta}} \right\| \to 0 \quad as \quad N \to \infty. \end{aligned}$$

Consequently, $T_{\Lambda}^{*} T_{\widetilde{\Theta}} = P_{\mathcal{W},\mathcal{V}^{\perp}}$, this is equivalent to $T_{\widetilde{\Theta}}^{*} T_{\Lambda} = P_{\mathcal{V},\mathcal{W}^{\perp}}$. For any $f \in \mathcal{V}$, we have $f = \sum_{i \in \mathbb{I}} \widetilde{\Theta}_i^{*} \Lambda_i f$. Then

$$\begin{aligned} \|f\|^4 = \left\| \sum_{i \in \mathbb{I}} \left\langle \widetilde{\Theta}_i^{*} \Lambda_i f, f \right\rangle_{\mathcal{A}} \right\|^2 &= \left\| \sum_{i \in \mathbb{I}} \left\langle \Lambda_i f, \widetilde{\Theta}_i f \right\rangle_{\mathcal{A}} \right\|^2 \\ \leq \left\| \sum_{i \in \mathbb{I}} \left\langle \Lambda_i f, \Lambda_i f \right\rangle_{\mathcal{A}} \right\| &\left\| \sum_{i \in \mathbb{I}} \left\langle \widetilde{\Theta}_i f, \widetilde{\Theta}_i f \right\rangle_{\mathcal{A}} \right\|. \end{aligned}$$

This shows that $\widetilde{\Theta}$ is a $g-$frame for $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. Then it is an oblique dual of Λ. Then Λ and $\widetilde{\Theta}$ are oblique duals. The other can be proved similarly.

(3) We have Θ and Λ are $\epsilon-$approximate oblique duals, and $0 \leq \epsilon < 1$. Then by (2), $\widetilde{\Lambda}$ is an oblique dual of Θ. Then we have

$$\begin{aligned} P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\widetilde{\Lambda}}^{*} T_{\widetilde{\Theta}} &= P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\widetilde{\Lambda}}^{*} T_{\Theta} J_N \\ &= P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\widetilde{\Lambda}}^{*} T_{\Theta} \left(\sum_{n=0}^{\infty} \left(P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\Lambda}^{*} T_{\Theta}\right)^n \right) \\ &= P_{\mathcal{W},\mathcal{V}^{\perp}} - P_{\mathcal{W},\mathcal{V}^{\perp}} \left(I + \sum_{n=1}^{\infty} \left(P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\Lambda}^{*} T_{\Theta}\right)^n \right). \end{aligned}$$

Let $f \in \mathcal{W}^{\perp}$. Since $T_{\Theta} f = 0$ and $P_{\mathcal{W},\mathcal{V}^{\perp}} f = 0$, we have $\left(P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\Lambda}^{*} T_{\Theta}\right) f = 0$. As a result, $\mathcal{W}^{\perp} \subset ker\left(\sum_{n=1}^{\infty} \left(P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\Lambda}^{*} T_{\Theta}\right)^n\right)$, which is equivalent to,

$$ran \left(\sum_{n=1}^{\infty} \left(P_{\mathcal{W},\mathcal{V}^{\perp}} - T_{\Lambda}^{*} T_{\Theta}\right)^n \right) \subset \mathcal{W}.$$

Consequently,

$$\begin{aligned}\left\|P_{\mathcal{W},\mathcal{V}^{\perp}}-T_{\widetilde{\Lambda}}^{*}T_{\widetilde{\Theta}}\right\| &= \left\|\sum_{n=1}^{\infty}\left(P_{\mathcal{W},\mathcal{V}^{\perp}}-T_{\Lambda}^{*}T_{\Theta}\right)^{n}\right\| \\ &\leq \sum_{n=1}^{\infty}\left\|P_{\mathcal{W},\mathcal{V}^{\perp}}-T_{\Lambda}^{*}T_{\Theta}\right\|^{n} \leq \frac{\epsilon}{1-\epsilon}.\end{aligned}$$

Then, $\widetilde{\Theta}$ is an $\frac{\epsilon}{1-\epsilon}-$approximate oblique dual of $\widetilde{\Lambda}$ for $\mathcal{V}$.

□

4. Perturbations of Approximate Oblique Dual $g-$frames for Closed Submodules

In this section we introduced the concept of $(\lambda,\mu,\gamma)-$perturbation and gives a generalization of Paley-Wiener theorem on perturbation of bases.

Definition 4.1. Let $\Gamma=\{\Gamma_i\in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H},\mathcal{H}_i): i\in\mathbb{I}\}$ be a $g-$frame for $\mathcal{W}$, $0\leq \lambda,\mu<1$ and $\gamma>0$. Let $\Theta=\{\Theta_i\in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H},\mathcal{H}_i): i\in\mathbb{I}\}$ be an operator sequence of $\mathcal{V}$. We say that Θ is a $(\lambda,\mu,\gamma)-$perturbation of Γ if we have

$$\left\|\sum_{i\in\mathbb{J}}\left(\Gamma_i^{*}-\Theta_i^{*}\right)g_i\right\|\leq\lambda\left\|\sum_{i\in\mathbb{J}}\Gamma_i^{*}g_i\right\|+\mu\left\|\sum_{i\in\mathbb{J}}\Theta_i^{*}g_i\right\|+\gamma\left\|\sum_{i\in\mathbb{J}}\langle g_i,g_i\rangle\right\|^{\frac{1}{2}},$$

for all $\{g_i\}_{i\in\mathbb{I}}\in\left(\sum_{i\in\mathbb{I}}\oplus\mathcal{H}_i\right)_{\ell^2}$.

Theorem 4.1. *Let $\Lambda=\left\{\Lambda_i\in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H},\mathcal{H}_i)_{i\in\mathbb{I}}\right\}$ be a $g-$frame for $\mathcal{W}$ with upper bound b and $\Gamma=\left\{\Gamma_i\in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H},\mathcal{H}_i)_{i\in\mathbb{I}}\right\}$ be an oblique dual of Λ for $\mathcal{V}$ with upper bound d. Let $0\leq\epsilon<1$, suppose that $\Theta=\left\{\Theta_i\in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H},\mathcal{H}_i)_{i\in\mathbb{I}}\right\}$ is an operator sequence of $\mathcal{V}$ and $\overline{span}\{\Theta_i^{*}(\mathcal{H}_i)\}_{i\in\mathbb{I}}\subset\mathcal{V}$, such that Θ is a $(\lambda,\mu,\gamma)-$perturbation of Γ, where $\gamma>0, 0\leq\lambda,\mu<1$ and*

$$\lambda\sqrt{bd}+\mu\sqrt{bd}\left(1+\frac{\lambda+\mu+\frac{\gamma}{\sqrt{d}}}{1-\mu}\right)+\gamma\sqrt{b}\leq\epsilon.$$

Then Θ is an $\epsilon-$approximate oblique dual of Λ for $\mathcal{V}$.

Proof. Given that Θ is a $(\lambda,\mu,\gamma)-$perturbation of Γ. Then for each $\{g_i\}_{i\in\mathbb{J}}\in\left(\sum_{i\in\mathbb{I}}\oplus\mathcal{H}_i\right)_{\ell^2}$

$$\left\|\sum_{i\in\mathbb{J}}\left(\Gamma_i^{*}-\Theta_i^{*}\right)g_i\right\|\leq\lambda\left\|\sum_{i\in\mathbb{J}}\Gamma_i^{*}g_i\right\|+\mu\left\|\sum_{i\in\mathbb{J}}\Theta_i^{*}g_i\right\|+\gamma\left\|\sum_{i\in\mathbb{J}}\langle g_i,g_i\rangle\right\|^{\frac{1}{2}}.$$

Given that Γ is a $g-$frame for $\mathcal{V}$ with upper bound d, it follows that $\|T_{\Gamma}^{*}\|\leq\sqrt{d}$. For $\{g_i\}_{i\in\mathbb{I}}\in\left(\sum_{i\in\mathbb{I}}\oplus\mathcal{H}_i\right)_{\ell^2}$, we get $\|\{g_i\}\|^2=\|\langle\{g_i\},\{g_i\}\rangle_{\mathcal{A}}\|=\|\sum_{i\in\mathbb{I}}\langle g_i,g_i\rangle_{\mathcal{A}}\|$, and $\left\|\sum_{i\in\mathbb{I}}\Gamma_i^* g_i\right\| = \|T_\Gamma^*(\{g_i\}_{i\in\mathbb{I}})\| \leq \sqrt{d}\left\|\sum_{i\in\mathbb{I}}\langle g_i, g_i\rangle\right\|^{\frac{1}{2}}$. For $g_i \in \mathcal{H}_i$, we have

$$\begin{aligned}\left\|\sum_{i\in\mathbb{I}}\Theta_i^* g_i\right\| &\leq \left\|\sum_{i\in\mathbb{I}}(\theta_i^* - \Gamma_i^*)g_i\right\| + \left\|\sum_{i\in\mathbb{I}}\Gamma_i^* g_i\right\| \\ &\leq (1+\lambda)\left\|\sum_{i\in\mathbb{I}}\Gamma_i^* g_i\right\| + \mu\left\|\sum_{i\in\mathbb{I}}\Theta_i^* g_i\right\| + \gamma\left\|\sum_{i\in\mathbb{I}}\langle g_i, g_i\rangle\right\|^{\frac{1}{2}} \\ &\leq \left(\frac{(1+\lambda)\sqrt{d}+\gamma}{1-\mu}\right)\left\|\sum_{i\in\mathbb{I}}\langle g_i, g_i\rangle\right\|^{\frac{1}{2}}.\end{aligned}$$

Therefore, $\sum_{i\in\mathbb{I}}\Theta_i^* g_i$ is convergent in $\mathcal{H}$, then Θ is a $g-$Bessel sequence for $\mathcal{H}$ with bound $\left(\frac{(1+\lambda)\sqrt{d}+\gamma}{1-\mu}\right)^2$, and

$$\left\|\sum_{i\in\mathbb{I}}\left(\Gamma_i^* - \Theta_i^*\right)g_i\right\| \leq \lambda\left\|\sum_{i\in\mathbb{I}}\Gamma_i^* g_i\right\| + \mu\left\|\sum_{i\in\mathbb{I}}\Theta_i^* g_i\right\| + \gamma\left\|\sum_{i\in\mathbb{I}}\langle g_i, g_i\rangle\right\|^{\frac{1}{2}}.$$

On the other hand, for each $f \in \mathcal{H}$

$$\begin{aligned}\left\|\left(P_{\mathcal{V},\mathcal{W}^\perp} - T_\Theta^* T_\Lambda\right)f\right\| &= \|T_\Gamma^* T_\Lambda f - T_\Theta^* T_\Lambda f\| \\ &\leq \|T_\Gamma^* - T_\Theta^*\|\,\|T_\Lambda f\| \\ &\leq (\lambda\|T_\Gamma^*\| + \mu\|T_\Theta^*\| + \gamma)\,\|T_\Lambda f\| \\ &\leq \left(\lambda\sqrt{bd} + \mu\sqrt{bd}\left(1 + \frac{\lambda+\mu+\frac{\gamma}{\sqrt{d}}}{1-\mu}\right) + \gamma\sqrt{b}\right)\|f\| \\ &\leq \epsilon\|f\|.\end{aligned}$$

Since $0 \leq \epsilon \leq 1$, it follows that Θ is a $g-$frame for $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$. Hence, we conclude that Θ is an $\epsilon-$approximate oblique dual of Λ for $\mathcal{V}$. □

Corollary 4.1. *Let $\Lambda = \{\Lambda_i \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ be a $g-$frame for $\mathcal{W}$ and $\Gamma = \{\Gamma_i \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ be an oblique dual of Λ for $\mathcal{V}$ with upper bound b. Let $0 \leq \epsilon \leq 1$, suppose that $\Theta = \{\Theta_i \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ is an operator sequence with $\overline{span}\{\Theta_i^*(\mathcal{H}_i)\}_{i\in\mathbb{I}} \subset \mathcal{V}$ such that for each $g_i \in \mathcal{H}_i$,*

$$\left\|\sum_{i\in\mathbb{I}}\left(\Gamma_i^* - \Theta_i^*\right)g_i\right\| \leq \gamma\left(\sum_{i\in\mathbb{I}}\|g_i\|^2\right)^{\frac{1}{2}},$$

where $\sqrt{b}\gamma \leq \epsilon$. Then Θ is an $\epsilon-$approximate oblique dual of Λ for $\mathcal{V}$.

Proof. Suppose that for each $g_i \in \mathcal{H}_i$,

$$\left\|\sum_{i\in\mathbb{I}}\left(\Gamma_i^* - \Theta_i^*\right)g_i\right\| \leq \gamma\left(\sum_{i\in\mathbb{I}}\|g_i\|^2\right)^{\frac{1}{2}}.$$

Given that Λ is a $g-$frame for $\mathcal{W}$ with upper bound b, it follows that $\|T_\Lambda^*\| \leq \sqrt{b}$. For any $g = \{g_i\}_{i\in\mathbb{I}} \in \left(\sum_{i\in\mathbb{I}}\oplus\mathcal{H}_i\right)_{\ell^2}$. Take $g = T_\Lambda f$ for $f \in \mathcal{H}$. Then

$$\left\|\left(P_{\mathcal{V},\mathcal{W}^\perp} - T_\Lambda^* T_\Theta\right)f\right\| \leq \gamma|\,\|T_\Lambda^* f\| \leq \sqrt{b}\gamma\|f\| \leq \epsilon\|f\|.$$

Since $0 \leq \epsilon \leq 1$ and $\overline{\operatorname{span}}\{\Theta_i^*\mathcal{H}_i\}_{i\in\mathbb{I}} \subset \mathcal{V}$, it follows that Θ is a $g-$frame for $\mathcal{V}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$. Then Θ is an $\epsilon-$approximate oblique dual of Λ for $\mathcal{V}$. $\square$

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