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# DISTRIBUTION OF EIGENVALUES OF AN EVEN-ORDER DIFFERENTIAL OPERATOR WITH OSCILLATING COEFFICIENTS

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**Abstract**. The paper considers a class of non-semi-bounded singular differential operators of even order with irregular growth of the potential at infinity. The asymptotics of the fundamental system of solutions and the defect indices are investigated. Asymptotic formulas for the eigenvalue distribution function are obtained in terms of the regular part of the operator's potential.

## 1. Introduction

This work is devoted to the study of the asymptotic behavior of the spectrum of a non-semi-bounded differential operator $L_0$, generated in $L_2(-\infty,+\infty)$ by the differential expression

$$ly := (-1)^n y^{(2n)} - q_1(x)y, \quad n \geq 1, \quad x \in (-\infty,+\infty). \tag{1.1}$$

In the monograph [3], asymptotic formulas were obtained for the function $N(\lambda)$ – the eigenvalue distribution function of self-adjoint extensions of the minimal differential operator $L_0$, generated in $L_2(-\infty,+\infty)$ by a differential expression of the form (1.1) in the case when $q_1(x)$ is a "regular" function in the sense of Titchmarsh–Levitain. Here and henceforth, the regularity of the function $q_1(x)$ is understood as follows:

– the function $q_1(x)$ is twice continuously differentiable;
– the functions $q_1'(x)$, $q_1''(x)$ do not change sign for sufficiently large $x$, $|x| > R$, where $R > 0$;
– $q_1(x) \to +\infty$ as $|x| \to +\infty$;
– $q_1'(x) = o(q_1^{\gamma}(x))$, $|x| \to +\infty$, $0 < \gamma < \frac{5}{4}$.

The aim of this work is to obtain asymptotic formulas for the function $N(\lambda)$ in the case when the function $q_1(x)$ does not satisfy the Titchmarsh–Levitain regularity conditions and is an oscillating function. Examples of such irregular functions include, for instance, functions of the form $q_1(x) = q(x) + h(x)$, where $q(x)$ is "regular", and $h(x)$ contains oscillations of the form $h(x) = \sum a_k(x) \cdot$

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$S_k(\phi_k(x))$, where $S_k(t)$ are periodic functions, and $a_k(x)$, $\phi_k(x)$ are sufficiently smooth monotonic functions.

Asymptotic formulas (both in $x$ and $\lambda$) for the fundamental system of solutions (FSS) of equations of the form

$$-y'' - (q(x) + h(x)) = \lambda y$$

with such potentials were studied in [4]. In the works [6] and [7], a new method was proposed for constructing such asymptotic formulas as $|x| \to +\infty$ for fourth and sixth order equations.

In this work, we show that the proposed method allows obtaining asymptotic formulas for the FSS of equation (1.1) as $\lambda \to \infty$, $\lambda \in \Gamma$, where $\Gamma = \{\lambda = \sigma + i\tau,\ \sigma > 0,\ \xi \leq \tau \leq \sigma^\gamma,\ \xi > 0,\ 0 < \gamma < 1\}$ uniformly in $x \in (-\infty, +\infty)$, which, in turn, yields asymptotic formulas for the function $N(\lambda)$ as $\lambda \to \pm\infty$.

## 2. Asymptotic Formulas for the Fundamental System of Solutions of the Equation

The main result of the work is the following

**Theorem 2.1.** *Let the function $q(x)$ satisfy the following conditions:*

(1) $q(x) \to +\infty$ *as* $|x| \to +\infty$;
(2) $q'(x)$, $q''(x)$ *do not change sign for sufficiently large* $x$, $|x| \geq R$, *where* $R > 0$;
(3) $q'(x) = o(q^\zeta(x))$, $|x| \to +\infty$, $0 < \zeta < \frac{2n+1}{2n}$.

*Let the real function $h(x) \in L_{1,loc}$ be such that*

(4) $\int_x^\infty \phi(t,\lambda)\, dt = o(1)$

*as $\lambda \to \infty$, $\lambda \in \Gamma := \{\lambda = \sigma + i\tau, \sigma > 0, \xi \leq \tau \leq \sigma^\gamma, \xi > 0, 0 < \gamma < 1\}$, uniformly in $x \in (-\infty, +\infty)$, where*

$$\phi(x,\lambda) = \frac{h(x)}{2n\,(q(x)+\lambda)^{\frac{2n-1}{2n}}}.$$

*Furthermore, let the functions*

$$\omega(x,\lambda) = \frac{q'(x)}{4n(q(x)+\lambda)}, \quad \phi_1(x,\lambda) = \int_x^\infty \phi(t,\lambda)\, dt,$$
$$\phi_2(x,\lambda) = \int_x^\infty \mu(t,\lambda)\phi_1(t,\lambda)\, dt,$$

*where $\mu(t,\lambda)$ main value of the root $\sqrt[2n]{q(x)+\lambda}$ $(-\frac{\pi}{2n} < arg \sqrt[2n]{q(x)+\lambda} \leq \frac{\pi}{2n})$, satisfy the conditions*

$$\int_x^\infty \omega(t,\lambda)\phi_1(t,\lambda)\, dt = o(1), \quad \int_x^\infty \omega(t,\lambda)\phi_2(t,\lambda)\, dt = o(1),$$
$$\int_x^\infty (q(t)+\lambda)^{\frac{1}{2n}}\phi_2(t,\lambda)\, dt = o(1) \tag{2.1}$$

*as $\lambda \to \infty$, $\lambda \in \Gamma$, uniformly in $x \in (-\infty, +\infty)$.*

*Then the equation*

$$(-1)^n y^{(2n)} - (q(x)+h(x))y = \lambda y, \quad n \geq 1 \tag{2.2}$$

*has $2n$ linearly independent solutions, for which, as $\lambda\to\infty$, $\lambda\in\Gamma$, the following asymptotic formulas hold:*

$$Y_j(x,\lambda)=\begin{pmatrix} y_j\\ y_j^{[1]}\\ y_j^{[2]}\\ \dots\\ y_j^{[2n-1]}\end{pmatrix}= \tag{2.3}$$
$$=\frac{e^{\varepsilon_j\int\limits_0^x \sqrt[2n]{q(t)+\lambda}\,dt}}{(q(x)+\lambda)^{\frac{2n-1}{4n}}}\begin{pmatrix} 1+o(1)\\ \varepsilon_j\,(q(x)+\lambda)^{\frac{1}{2n}}\,(1+o(1))\\ \varepsilon_j^2\,(q(x)+\lambda)^{\frac{1}{n}}\,(1+o(1))\\ \dots\\ \varepsilon_j^{2n-1}\,(q(x)+\lambda)^{\frac{2n-1}{2n}}\,(1+o(1))\end{pmatrix}$$

*uniformly in $x\in(-\infty,+\infty)$, where $j=1,\dots,2n$, $\varepsilon_j$ are all distinct $2n$-th roots of $(-1)^n$, and $y^{[k]}(x)$ are the quasi-derivatives of the function $y(x)$ (see* [5]*, Chapter V, §15, p. 181).*

We proceed from equation (2.2) to a system of linear equations. Denote $Y=(y,y^{[1]},\dots,y^{[2n-1]})^T$. Then equation (2.2) can be written as a system

$$Y'=(A_0(x,\lambda)+A_1(x))Y, \tag{2.4}$$

where

$$A_0(x,\lambda)=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & -1\\ q(x)+\lambda & 0 & 0 & \dots & 0\end{pmatrix},\quad A_1(x)=\begin{pmatrix} 0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 0\\ h(x) & 0 & \dots & 0\end{pmatrix}.$$

The matrices $A_0(x,\lambda)$ and $A_1(x)$ are square matrices of order $2n$, and in $A_0(x,\lambda)$, the coefficients equal $-1$ starting from the $n$-th row (see [5], Chapter VII, §22, p. 313).

We find the eigenvalues of the matrix $A_0(x,\lambda)$,

$$\det\left(A_0(x,\lambda)-\mu(x,\lambda)E\right)=0,$$

where $E$ is the identity matrix of the same size as $A_0(x,\lambda)$. From the last equation, we obtain

$$\mu^{2n}(x,\lambda)=(-1)^n(q(x)+\lambda). \tag{2.5}$$

Denote the main value of the root $\mu(x,\lambda):=\sqrt[2n]{q(x)+\lambda}$, and let $\varepsilon_k$ denote all distinct $2n$-th roots of $(-1)^n$, $\varepsilon_{k+1}=\cos\frac{\pi(n+2k)}{2n}+i\sin\frac{\pi(n+2k)}{2n}$, $k=0,1,\dots,2n-1$.

The eigenvalues of the matrix $A_0(x,\lambda)$ are denoted by $\mu_i(x,\lambda)=\varepsilon_i\mu(x,\lambda)$, $i=1,2,\dots,2n$.

Let

$$T=\begin{pmatrix} 1 & 1 & 1 & \dots & 1\\ \mu_1 & \mu_2 & \mu_3 & \dots & \mu_{2n}\\ \mu_1^2 & \mu_2^2 & \mu_3^2 & \dots & \mu_{2n}^2\\ \dots & \dots & \dots & \dots & \dots\\ \mu_1^{2n-1} & \mu_2^{2n-1} & \mu_3^{2n-1} & \dots & \mu_{2n}^{2n-1} \end{pmatrix}$$

be a square matrix of order $2n$.

Then the matrix $T$ transforms the matrix $A_0$ into diagonal form $T^{-1}A_0T = \Lambda$, $\quad \Lambda=\mu\Lambda_0$, where $\Lambda_0=\begin{pmatrix} \varepsilon_1 & 0 & 0 & \dots & 0\\ 0 & \varepsilon_2 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & \varepsilon_{2n} \end{pmatrix}$.

The following substitution

$$Y=TZ,\quad Z=(z_1,z_2,\dots,z_{2n})^T \tag{2.6}$$

transforms the system (2.4) into

$$Z'=\left(T^{-1}A_0T+T^{-1}A_1T-T^{-1}T'\right)Z, \tag{2.7}$$

where

$$T^{-1}A_0T=\mu\Lambda_0,$$

$$T^{-1}A_1T=\frac{h(x)}{2n\,(q(x)+\lambda)^{\frac{2n-1}{2n}}}H_0,\quad H_0=\begin{pmatrix} \varepsilon_1 & \varepsilon_1 & \varepsilon_1 & \dots & \varepsilon_1\\ \varepsilon_2 & \varepsilon_2 & \varepsilon_2 & \dots & \varepsilon_2\\ \dots & \dots & \dots & \dots & \dots\\ \varepsilon_{2n} & \varepsilon_{2n} & \varepsilon_{2n} & \dots & \varepsilon_{2n} \end{pmatrix},$$

$$T^{-1}T'=\frac{q'(x)}{4n(q(x)+\lambda)}F_0,\quad F_0=\begin{pmatrix} 2n-1 & f_{1,2} & f_{1,3} & \dots & f_{1,2n}\\ f_{2,1} & 2n-1 & f_{2,3} & \dots & f_{2,2n}\\ \dots & \dots & \dots & \dots & \dots\\ f_{2n,1} & f_{2n,2} & f_{2n,3} & \dots & 2n-1 \end{pmatrix},$$

where $f_{i,j}$ are complex numbers, and $f_{i,j}=\overline{f_{j,i}}$.

We rewrite the system (2.7) as

$$Z'=\left(\mu(x,\lambda)\Lambda_0+\phi(x,\lambda)H_0-\omega(x,\lambda)F_0\right)Z, \tag{2.8}$$

where

$$\phi(x,\lambda)=\frac{h(x)}{2n\,(q(x)+\lambda)^{\frac{2n-1}{2n}}},\quad \omega(x,\lambda)=\frac{q'(x)}{4n(q(x)+\lambda)}.$$

We impose the condition on the function $\phi(x,\lambda)$

$$\int_x^\infty \phi(t,\lambda)\,dt=o(1) \tag{2.9}$$

as $\lambda\to\infty$, $\lambda\in\Gamma$, uniformly in $x\in(-\infty,\infty)$, and denote

$$\phi_1(x,\lambda):=\int_x^\infty \phi(t,\lambda)\,dt,$$

where, here and henceforth, the integral is considered for both positive and negative $x$.

The following substitution

$$Z=e^{-\phi_1(x,\lambda)H_0}U \tag{2.10}$$

transforms the system (2.8) into

$$U' = e^{\phi_1(x,\lambda)H_0}\left(\mu(x,\lambda)\Lambda_0 - \omega(x,\lambda)F_0\right)e^{-\phi_1(x,\lambda)H_0}U.$$

We apply the Hausdorff formula [1] to the right-hand side of the last equation

$$e^{g(x)A}Be^{-g(x)A} = B + g(x)[A,B] + \frac{g^2(x)}{2}[A,[A,B]] + \dots,$$

where $A, B$ are square matrices, and $[A,B] = AB - BA$ is the matrix commutator. We obtain

$$\begin{aligned}\mu(x,\lambda)e^{\phi_1(x,\lambda)H_0}\Lambda_0 e^{-\phi_1(x,\lambda)H_0} =& \mu(x,\lambda)\Lambda_0 + \mu(x,\lambda)\phi_1(x,\lambda)[H_0,\Lambda_0] \\ &+ \mu(x,\lambda)\frac{\phi_1^2(x,\lambda)}{2}[H_0,[H_0,\Lambda_0]] + \dots\end{aligned}$$

and

$$\begin{aligned}\omega(x,\lambda)e^{\phi_1(x,\lambda)H_0}F_0 e^{-\phi_1(x,\lambda)H_0} =& \omega(x,\lambda)F_0 + \omega(x,\lambda)\phi_1(x,\lambda)[H_0,F_0] \\ &+ \omega(x,\lambda)\frac{\phi_1^2(x,\lambda)}{2}[H_0,[H_0,F_0]] + \dots\end{aligned}$$

$$H_{11} = [H_0,\Lambda_0] = \begin{pmatrix} 0 & h_{1,2}^{(1)} & h_{1,3}^{(1)} & \dots & h_{1,2n}^{(1)} \\ h_{2,1}^{(1)} & 0 & h_{2,3}^{(1)} & \dots & h_{2,2n}^{(1)} \\ \dots & \dots & \dots & \dots & \dots \\ h_{2n,1}^{(1)} & h_{2n,2}^{(1)} & h_{2n,3}^{(1)} & \dots & 0 \end{pmatrix}, \ [H_0,[H_0,\Lambda_0]] = \theta,$$

where $h_{i,j}^{(1)}$ are complex numbers, and $\theta$ is the zero matrix of order $2n$.

$$F_{11} = [H_0,F_0] = (2-4n)\begin{pmatrix} \varepsilon_1 & \varepsilon_1 & \dots & \varepsilon_1 \\ \varepsilon_2 & \varepsilon_2 & \dots & \varepsilon_2 \\ \dots & \dots & \dots & \dots \\ \varepsilon_{2n} & \varepsilon_{2n} & \dots & \varepsilon_{2n} \end{pmatrix} = (2-4n)H_0,$$

$$[H_0,[H_0,F_0]] = \theta.$$

From the last relations, we obtain a new system

$$\begin{aligned}U' = \big(&\mu(x,\lambda)\Lambda_0 + \mu(x,\lambda)\phi_1(x,\lambda)H_{11} - \omega(x,\lambda)F_0 \\ &-\omega(x,\lambda)\phi_1(x,\lambda)F_{11}\big)U.\end{aligned} \tag{2.11}$$

For the function $\phi_1(x,\lambda)$, we require the condition

$$\int_x^\infty \mu(t,\lambda)\phi_1(t,\lambda)\,dt = o(1) \tag{2.12}$$

as $\lambda \to \infty$, $\lambda \in \Gamma$, uniformly in $x \in (-\infty,\infty)$, and denote $\phi_2(x,\lambda) := \int_x^\infty \mu(t,\lambda)\phi_1(t,\lambda)dt$.

We make the following substitution

$$U = e^{-\phi_2(x,\lambda)H_{11}}Q, \tag{2.13}$$

and the system (2.11) transforms into

$$\begin{aligned}Q' = e^{\phi_2(x,\lambda)H_{11}}\Big(&\mu(x,\lambda)\Lambda_0 - \omega(x,\lambda)F_0 \\ &- \omega(x,\lambda)\phi_1(x,\lambda)F_{11}\Big)e^{-\phi_2(x,\lambda)H_{11}}Q.\end{aligned} \tag{2.14}$$

Next, we require the conditions (2.1) to hold as $\lambda \to \infty$, $\lambda \in \Gamma$, uniformly in $x \in (-\infty, \infty)$.

Using the Hausdorff formula and conditions (2.9) and (2.12), we obtain

$$\begin{aligned}\mu(x,\lambda)e^{\phi_2(x,\lambda)H_{11}}\Lambda_0 e^{-\phi_2(x,\lambda)H_{11}} =& \mu(x,\lambda)\Lambda_0 + \mu(x,\lambda)\phi_2(x,\lambda)[H_{11},\Lambda_0] \\ &+ \mu(x,\lambda)\frac{\phi_2^2(x,\lambda)}{2}[H_{11},[H_{11},\Lambda_0]] + \dots \\ &= \mu(x,\lambda)\Lambda_0 + \mu(x,\lambda)\phi_2(x,\lambda)H_{21},\end{aligned}$$

$$H_{21} = [H_{11},\Lambda_0] = \begin{pmatrix} 0 & h^{(2)}_{1,2} & h^{(2)}_{1,3} & \dots & h^{(2)}_{1,2n} \\ h^{(2)}_{2,1} & 0 & h^{(2)}_{2,3} & \dots & h^{(2)}_{2,2n} \\ \dots & \dots & \dots & \dots & \dots \\ h^{(2)}_{2n,1} & h^{(2)}_{2n,2} & h^{(2)}_{2n,3} & \dots & 0 \end{pmatrix}, \ [H_{11},[H_{11},\Lambda_0]] = \theta,$$

where $h^{(2)}_{i,j}$ are complex numbers.

$$\begin{aligned}\omega(x,\lambda)e^{\phi_2(x,\lambda)H_{11}}F_0 e^{-\phi_2(x,\lambda)H_{11}} =& \omega(x,\lambda)F_0 + \omega(x,\lambda)\phi_2(x,\lambda)[H_{11},F_0] \\ &+ \omega(x,\lambda)\frac{\phi_2^2(x,\lambda)}{2}[H_{11},[H_{11},F_0]] + \dots \\ &= \omega(x,\lambda)F_0 + \omega(x,\lambda)\phi_2(x,\lambda)F_{21},\end{aligned}$$

$$F_{21} = [H_{11},F_0] = \begin{pmatrix} 0 & f^{(2)}_{1,2} & f^{(2)}_{1,3} & \dots & f^{(2)}_{1,2n} \\ f^{(2)}_{2,1} & 0 & f^{(2)}_{2,3} & \dots & f^{(2)}_{2,2n} \\ \dots & \dots & \dots & \dots & \dots \\ f^{(2)}_{2n,1} & f^{(2)}_{2n,2} & f^{(2)}_{2n,3} & \dots & 0 \end{pmatrix}, \ [H_{11},[H_{11},F_0]] = \theta,$$

where $f^{(2)}_{i,j}$ are complex numbers.

$$\begin{aligned}\omega(x,\lambda)\phi_1(x,\lambda)e^{\phi_2(x,\lambda)H_{11}}F_{11}e^{-\phi_2(x,\lambda)H_{11}} &= \omega(x,\lambda)\phi_1(x,\lambda)F_{11} + \\ &\omega(x,\lambda)\phi_1(x,\lambda)\phi_2(x,\lambda)[H_{11},F_{11}] + \dots \\ &= \omega(x,\lambda)\phi_1(x,\lambda)F_{11}, \\ [H_{11},F_{11}] = \theta.\end{aligned}$$

After these transformations, the system (2.14) takes the form

$$Q' = (\mu(x,\lambda)\Lambda_0 - \omega(x,\lambda)F_0 + S(x,\lambda))\,Q, \tag{2.15}$$

$$S(x,\lambda) = \mu(x,\lambda)\phi_2(x,\lambda)H_{21} - \omega(x,\lambda)\phi_2(x,\lambda)F_{21} - \omega(x,\lambda)\phi_1(x,\lambda)F_{11}. \tag{2.16}$$

Denote

$$\widetilde{\Lambda}_0 = \mu(x,\lambda)\Lambda_0 - (2n-1)\omega(x,\lambda)E, \quad \widetilde{F}_0 = F_0 - (2n-1)E, \tag{2.17}$$

then the following equality holds

$$\mu(x,\lambda)\Lambda_0 - \omega(x,\lambda)F_0 = \widetilde{\Lambda}_0 - \omega(x,\lambda)\widetilde{F}_0.$$

Taking into account the last equality, we rewrite equation (2.15) as

$$Q' = \left(\widetilde{\Lambda}_0 - \omega(x,\lambda)\widetilde{F}_0 + S(x,\lambda)\right)Q. \tag{2.18}$$

Let $T_0$ be a constant matrix, which is a solution to the equation

$$\widetilde{\Lambda}_0 T_0 - T_0 \widetilde{\Lambda}_0 = \mu(x,\lambda)\widetilde{F}_0.$$

Then the matrix $T_1(x,\lambda) = \frac{\omega(x,\lambda)}{\mu(x,\lambda)} T_0$ satisfies the equality

$$\widetilde{\Lambda}_0 T_1(x,\lambda) = T_1(x,\lambda)\widetilde{\Lambda}_0 + \omega(x,\lambda)\widetilde{F}_0. \tag{2.19}$$

In the system (2.18), we make the substitution

$$Q = (E + T_1(x,\lambda))W. \tag{2.20}$$

Then

$$\begin{aligned}(E + T_1(x,\lambda))W' =& \big(\widetilde{\Lambda}_0 + \widetilde{\Lambda}_0 T_1(x,\lambda) - \omega(x,\lambda)\widetilde{F}_0\big)W \\ &+ \big(S(x,\lambda)(E + T_1(x,\lambda)) - \omega(x,\lambda)\widetilde{F}_0 T_1(x,\lambda) - T_1'(x,\lambda)\big)W.\end{aligned}$$

By virtue of (2.19), from the last equality it follows

$$\begin{aligned}(E + T_1(x,\lambda))W' =& \big(E + T_1(x,\lambda)\big)\widetilde{\Lambda}_0 W \\ &+ \big(S(x,\lambda)(E + T_1(x,\lambda)) - \omega(x,\lambda)\widetilde{F}_0 T_1(x,\lambda) - T_1'(x,\lambda)\big)W.\end{aligned}$$

From this equality it follows

$$\begin{aligned}W' =& \widetilde{\Lambda}_0 W + \\ &\big(E + T_1(x,\lambda)\big)^{-1}\big(S(x,\lambda)(E + T_1(x,\lambda)) - \omega(x,\lambda)\widetilde{F}_0 T_1(x,\lambda) - T_1'(x,\lambda)\big)W.\end{aligned}$$

Denote

$$S_1(x,\lambda) = \big(E + T_1(x,\lambda)\big)^{-1}\big(S(x,\lambda)(E + T_1(x,\lambda)) - \omega(x,\lambda)\widetilde{F}_0 T_1(x,\lambda) - T_1'(x,\lambda)\big).$$

Then we obtain the system

$$W' = \widetilde{\Lambda}_0 W + S_1(x,\lambda)W. \tag{2.21}$$

Setting in (2.21) for each fixed $i$, $i = 1, 2, \ldots, 2n$

$$W = e^{\int_0^x \left(\mu_i(t,\lambda) - (2n-1)\omega(t,\lambda)\right) dt} V, \tag{2.22}$$

where $V$ is a new unknown vector function and taking into account the equalities (2.17), we arrive at a system of first-order equations

$$\frac{d}{dx} v_k(x,\lambda) = \psi_k(x,\lambda)v_k(x,\lambda) + \sum_{j=1}^{2n} s_{k,j}^{(1)}(x,\lambda)v_j(x,\lambda), \;\; k = 1, 2, \ldots, 2n, \tag{2.23}$$

where the matrix $S_1(x,\lambda) = \left\{s_{k,j}^{(1)}(x,\lambda)\right\}_{k,j=1}^{2n}$, and $\psi_k(x,\lambda) = \mu_k(x,\lambda) - \mu_i(x,\lambda)$.

We formulate and prove an auxiliary statement.

**Lemma 2.1.** *As $\lambda \to \infty$, $\lambda \in \Gamma$, the following estimate holds uniformly in $x \in (-\infty, \infty)$*

$$\int_{-\infty}^{+\infty} \|S_1(x,\lambda)\|\, dx = o(1), \;\; \|S_1(x,\lambda)\| = \sum_{k=1}^{2n}\sum_{j=1}^{2n} \left|s_{k,j}^{(1)}(x,\lambda)\right|. \tag{2.24}$$

*Proof.* First, as $\lambda \to \infty$, $\lambda \in \Gamma$, we estimate $|\lambda + q(x)|$:

$$|\lambda + q(x)| = \sqrt{(\sigma + q(x))^2 + \tau^2} = (\sigma + q(x))\sqrt{1 + \frac{\tau^2}{(\sigma + q(x))^2}} =$$
$$= (\sigma + q(x))\left(1 + \frac{1}{2}\left(\frac{\tau}{\sigma + q(x)}\right)^2 + O\left(\frac{\tau}{\sigma + q(x)}\right)^2\right).$$

Since

$$\frac{\tau}{\sigma + q(x)} \le \frac{\sigma^\gamma}{\sigma + q(x)} \le \sigma^{\gamma - 1}, \quad 0 < \gamma < 1,$$

we have $|\lambda + q(x)| = (\sigma + q(x))(1 + o(1))$.

By virtue of equation (2.16) and conditions (2.1), it follows that

$$\int_{-\infty}^{+\infty} \|S(x, \lambda)\|\, dx = \sum_{k=1}^{2n}\sum_{j=1}^{2n} \int_{-\infty}^{+\infty} |s_{k,j}(x, \lambda)|\, dx = o(1). \tag{2.25}$$

We estimate the elements of the matrix $T_1 = \frac{\omega(x,\lambda)}{\mu(x,\lambda)} T_0$ as $\lambda \to \infty$, $\lambda \in \Gamma$, where $T_0$ is a matrix with constant complex coefficients. From the estimate

$$\left|\frac{\omega(x,\lambda)}{\mu(x,\lambda)}\right| = \left|\frac{q'(x)}{4n(q(x) + \lambda)^{\frac{2n+1}{2n}}}\right| \le C\frac{q^\zeta(x)}{(q(x)+\sigma)^{\frac{2n+1}{2n}}} \le C\frac{(q(x)+\sigma)^\zeta}{(q(x)+\sigma)^{\frac{2n+1}{2n}}}$$
$$\le C(q(x)+\sigma)^{\zeta - \frac{2n+1}{2n}} = o(1), \quad 0 < \zeta < \frac{2n+1}{2n},$$

as $\sigma \to +\infty$, it follows that $\|T_1\| = \left|\frac{\omega(x,\lambda)}{\mu(x,\lambda)}\right| \|T_0\| = o(1)$.

Next, we estimate the following term

$$\int_{-\infty}^{+\infty} \|\omega(x,\lambda)\widetilde{F}_0 T_1(x,\lambda)\|\, dx = \|\widetilde{F}_0 T_0\| \int_{-\infty}^{+\infty} \left|\frac{\omega^2(x,\lambda)}{\mu(x,\lambda)}\right| dx,$$

where the matrix $F_0 T_0$ has constant coefficients,

$$\int_{-\infty}^{+\infty} \left|\frac{\omega^2(x,\lambda)}{\mu(x,\lambda)}\right| dx = \int_{-\infty}^{+\infty} \left|\frac{(q'(x))^2}{16n^2(q(x)+\lambda)^{\frac{4n+1}{2n}}}\right| dx \le$$
$$C\int_{-\infty}^{+\infty} \frac{q^\zeta(x) q'(x)}{(q(x)+\sigma)^{\frac{4n+1}{2n}}}\, dx \le C\int_{-\infty}^{+\infty} \frac{(q(x)+\sigma)^\zeta q'(x)}{(q(x)+\sigma)^{\frac{4n+1}{2n}}}\, dx =$$
$$C\int_{-\infty}^{+\infty} \big(q(x)+\sigma\big)^{\zeta - \frac{4n+1}{2n}} q'(x)\, dx,$$

where $C$ is a constant independent of $x$ and $\lambda$. Let $R$ be an arbitrary fixed number, then

$$\int_{-\infty}^{+\infty} \big(q(x)+\sigma\big)^{\zeta - \frac{4n+1}{2n}} q'(x)\, dx = \left\{\int_{-\infty}^{-R} + \int_{-R}^{R} + \int_{R}^{+\infty}\right\} \big(q(x)+\sigma\big)^{\zeta - \frac{4n+1}{2n}} q'(x)\, dx.$$

By virtue of the continuity of $q'(x)$ and as $\lambda \to \infty$, $\lambda \in \Gamma$, for $0 < \zeta < \frac{2n+1}{2n}$, we have

$$\int_{-R}^{R} \big(q(x)+\sigma\big)^{\zeta - \frac{4n+1}{2n}} q'(x)\, dx = \int_{q(-R)+\sigma}^{q(R)+\sigma} t^{\zeta - \frac{4n+1}{2n}}\, dt = o(1),$$

as $\sigma \to +\infty$. Similarly,

$$\int_R^{+\infty} \big(q(x)+\sigma\big)^{\zeta-\frac{4n+1}{2n}} q'(x)\, dx = \int_{q(R)+\sigma}^{+\infty} t^{\zeta-\frac{4n+1}{2n}}\, dt = o(1),$$
$$\int_{-\infty}^{-R} \big(q(x)+\sigma\big)^{\zeta-\frac{4n+1}{2n}} q'(x)\, dx = \int_{+\infty}^{q(-R)+\sigma} t^{\zeta-\frac{4n+1}{2n}}\, dt = o(1),$$

as $\sigma \to +\infty$. From the last relations, we obtain the estimate

$$\int_{-\infty}^{+\infty} \|\omega(x,\lambda)\widetilde{F}_0 T_1(x,\lambda)\|\, dx = o(1), \quad \sigma \to +\infty. \tag{2.26}$$

From the definition of the matrix $T_1$, it follows that

$$T_1'(x,\lambda) = \left( \frac{q''(x)}{4n(q(x)+\lambda)^{\frac{2n+1}{2n}}} - \frac{(2n+1)(q'(x))^2}{8n^2(q(x)+\lambda)^{\frac{4n+1}{2n}}} \right) T_0.$$

Then as $\lambda \to \infty$, $\lambda \in \Gamma$

$$\int\limits_{-\infty}^{+\infty} \|T_1'(x,\lambda)\|\, dx = \|T_0\| \int\limits_{-\infty}^{+\infty} \left| \frac{q''(x)}{4n(q(x)+\lambda)^{\frac{2n+1}{2n}}} - \frac{(2n+1)(q'(x))^2}{8n^2(q(x)+\lambda)^{\frac{4n+1}{2n}}} \right| dx,$$

$$\int_{-\infty}^{+\infty} \left| \frac{q''(x)}{4n(q(x)+\lambda)^{\frac{2n+1}{2n}}} \right| dx \le C \int_{-\infty}^{+\infty} \frac{q''(x)}{(q(x)+\sigma)^{\frac{2n+1}{2n}}}\, dx.$$

Integrating by parts, we obtain

$$\int\limits_{-\infty}^{+\infty} \frac{q''(x)}{(q(x)+\sigma)^{\frac{2n+1}{2n}}}\, dx = \frac{q'(x)}{(q(x)+\sigma)^{\frac{2n+1}{2n}}}\Bigg|_{-\infty}^{+\infty} + \frac{2n+1}{2n} \int\limits_{-\infty}^{+\infty} \frac{(q'(x))^2}{(q(x)+\sigma)^{\frac{4n+1}{2n}}}\, dx.$$

Then

$$\int_{-\infty}^{+\infty} \|T_1'(x,\lambda)\|\, dx \le \frac{Cq'(x)}{(q(x)+\sigma)^{\frac{2n+1}{2n}}}\Bigg|_{-\infty}^{+\infty} + C\int_{-\infty}^{+\infty} \frac{(q'(x))^2}{(q(x)+\sigma)^{\frac{4n+1}{2n}}}\, dx,$$

$$\begin{aligned}
\frac{Cq'(x)}{(q(x)+\sigma)^{\frac{2n+1}{2n}}}\Bigg|_{-\infty}^{+\infty} &\le \frac{Cq^{\zeta}(x)}{(q(x)+\sigma)^{\frac{2n+1}{2n}}}\Bigg|_{-\infty}^{+\infty} \le \frac{C(q(x)+\sigma)^{\zeta}}{(q(x)+\sigma)^{\frac{2n+1}{2n}}}\Bigg|_{-\infty}^{+\infty} = \\
&= C(q(x)+\sigma)^{\zeta-\frac{2n+1}{2n}}\Bigg|_{-\infty}^{+\infty} = C(q(x)+\sigma)^{\zeta-\frac{2n+1}{2n}}\Bigg|_{-\infty}^{-R} \\
&+ C(q(x)+\sigma)^{\zeta-\frac{2n+1}{2n}}\Bigg|_{-R}^{R} + C(q(x)+\sigma)^{\zeta-\frac{2n+1}{2n}}\Bigg|_{R}^{+\infty} = o(1),
\end{aligned}$$

as $\sigma \to +\infty$, $0 < \zeta < \frac{2n+1}{2n}$, and $q(x) \to +\infty$ as $|x| \to +\infty$, $R$ is an arbitrary fixed number, and $C$ is a constant independent of $x$ and $\lambda$. Similarly,

$$\int\limits_{-\infty}^{+\infty} \frac{(q'(x))^2}{(q(x)+\sigma)^{\frac{4n+1}{2n}}}\,dx \le \int\limits_{-\infty}^{+\infty} \frac{Cq^{\zeta}(x)q'(x)}{(q(x)+\sigma)^{\frac{4n+1}{2n}}}\,dx \le \int\limits_{-\infty}^{+\infty} \frac{C(q(x)+\sigma)^{\zeta}q'(x)}{(q(x)+\sigma)^{\frac{4n+1}{2n}}}\,dx$$

$$= \int\limits_{-\infty}^{+\infty} C\big(q(x)+\sigma\big)^{\zeta-\frac{4n+1}{2n}} q'(x)\,dx = \int\limits_{-\infty}^{-R} C\big(q(x)+\sigma\big)^{\zeta-\frac{4n+1}{2n}} q'(x)\,dx +$$

$$\int\limits_{-R}^{R} C\big(q(x)+\sigma\big)^{\zeta-\frac{4n+1}{2n}} q'(x)\,dx + \int\limits_{R}^{+\infty} C\big(q(x)+\sigma\big)^{\zeta-\frac{4n+1}{2n}} q'(x)\,dx$$

$$= \int\limits_{+\infty}^{q(-R)+\sigma} Ct^{\zeta-\frac{4n+1}{2n}}\,dt + \int\limits_{q(-R)+\sigma}^{q(R)+\sigma} Ct^{\zeta-\frac{4n+1}{2n}}\,dt + \int\limits_{q(R)+\sigma}^{+\infty} Ct^{\zeta-\frac{4n+1}{2n}}\,dt = o(1),$$

as $\sigma \to +\infty$, $0 < \zeta < \frac{2n+1}{2n}$.

Thus, as $\lambda \to \infty$, $\lambda \in \Gamma$, and $0 < \zeta < \frac{2n+1}{2n}$, we obtain the estimate

$$\int_{-\infty}^{+\infty} \|T_1'(x,\lambda)\|\,dx = o(1). \tag{2.27}$$

From the estimate of the norm of the matrix $T_1(x,\lambda)$, it follows that $\|T_1(x,\lambda)\|$ can be made arbitrarily small as $\lambda \to \infty$, $\lambda \in \Gamma$, and hence less than $1/2$. Therefore, the matrix $E + T_1(x,\lambda)$ is invertible, and the following estimates hold:

$$\|E + T_1(x,\lambda)\| \le C, \quad \|(E + T_1(x,\lambda))^{-1}\| \le C.$$

From the last two inequalities and the estimates (2.25), (2.26), (2.27), the validity of the estimate (2.24) follows. The lemma is proved. □

Next, we complete the proof of Theorem 2.1.

From the proved lemma, it follows that all conditions of Lemma 1.1 (p. 166, §1, Chapter V) from [3] for the system (2.23) are satisfied. Therefore, for the solutions $v_k(x,\lambda)$, the following asymptotic formulas hold uniformly with respect to $x \in (-\infty,+\infty)$:

$$v_i(x,\lambda) = 1 + o(1),$$
$$v_k(x,\lambda) = o(1), \quad k \ne i, \quad k = 1,2,\dots,2n.$$

Applying the statement of this lemma for each fixed $k$, as $|\lambda| \to \infty$, $\lambda \in \Gamma$, and returning through the equalities

$$e^{\int_0^x \big(\mu_i(t,\lambda)-(2n-1)\omega(t,\lambda)\big)\,dt} = \frac{e^{\int_0^x \mu_i(t,\lambda)\,dt}}{(q(x)+\lambda)^{\frac{2n-1}{4n}}}$$

and (2.22), (2.20), (2.13), (2.10), (2.6) to the vector $Y$, taking into account the estimates (2.1) and (2.12), we obtain the asymptotic formulas (2.3) as $\lambda \to \infty$, $\lambda \in \Gamma$, uniformly in $x \in (-\infty,+\infty)$. Theorem 2.1 is proved.

## 3. Green's Function of the Real Self-Adjoint Extension of the Operator $L_0$ and Its Properties

The asymptotic formulas obtained in Theorem 2.1 allow us to address the question of the defect indices of the minimal differential operator $L_0$ generated in $L_2(-\infty,+\infty)$ by the differential expression

$$ly := (-1)^n y^{(2n)} - (q(x)+h(x))y, \quad x \in (-\infty,+\infty).$$

**Theorem 3.1.** *Suppose all conditions of Theorem 2.1 are satisfied, and let*

$$|q(x)| > c|x|^{\frac{2n}{2n-1}+\varepsilon}, \quad c>0, \quad \varepsilon>0, \quad x\in(-\infty,+\infty). \tag{3.1}$$

*Then the defect indices of the operator on the positive and negative half-axes are* $(n+1,n+1)$*, and on the entire axis, they are* $(2,2)$*.*

*Proof.* Indeed, for a fixed $\lambda$, the roots of the characteristic equation (2.5) as $|x|\to\infty$ behave like the roots of the equation

$$\mu^{2n} = (-1)^n q(x).$$

Set $\mu = \sigma\sqrt[2n]{q(x)}$ in this equation. Then we obtain the equation

$$\sigma^{2n} = (-1)^n.$$

It is easy to verify that among the roots of the last equation, there are exactly two imaginary roots, both for even $n$ and for odd $n$. Half of the remaining roots have a positive real part, and the other half have a negative real part. From condition (3.1), it follows that both solutions corresponding to purely imaginary roots belong to $L_2(-\infty,\infty)$. Solutions corresponding to roots with positive real parts belong to $L_2(-\infty,0)$, and those corresponding to roots with negative real parts belong to $L_2(0,\infty)$. The theorem is proved. □

Consider the minimal non-semi-bounded differential operator $L_0$, generated by the differential expression

$$ly := (-1)^n y^{(2n)} - (q(x)+h(x))y, \quad x \in (-\infty,+\infty).$$

We enumerate $\varepsilon_j$ and the corresponding $2n$ linearly independent solutions of the equation $ly=\lambda y$ as follows:

$$\begin{aligned} \varepsilon_1 &= i, \quad \varepsilon_{n+1} = -i,\\ \varepsilon_{j+n} &= -\varepsilon_j \text{ and } \operatorname{Re}\varepsilon_j < 0 \text{ for } j=2,\dots,n,\\ \varepsilon_{j-n} &= -\varepsilon_j \text{ and } \operatorname{Re}\varepsilon_j > 0 \text{ for } j=n+2,\dots,2n, \end{aligned} \tag{3.2}$$

hence,

$$\begin{aligned} y_1(x,\lambda),\ y_{n+1}(x,\lambda) &\in L_2(-\infty,+\infty),\\ y_2(x,\lambda),\ y_3(x,\lambda),\dots,y_n(x,\lambda) &\in L_2(0,+\infty),\\ y_{n+2}(x,\lambda),\ y_{n+3}(x,\lambda),\dots,y_{2n}(x,\lambda) &\in L_2(-\infty,0). \end{aligned}$$

Repeating the arguments given in [3] (Chapter IX, §2), we show that the Green's function of the real self-adjoint extension of the operator $L_0$ is

$$K(x,\eta,\lambda)=\begin{cases} y_1(x,\lambda)h_1(\eta,\lambda)+y_{n+1}(x,\lambda)h_{n+1}(\eta,\lambda)\\ -\frac{1}{2n}\sum_{j=n+2}^{2n}\varepsilon_j y_j(x,\lambda)y_{j-n}(\eta,\lambda), \quad x\le\eta,\\ y_1(x,\lambda)\left(\frac{1}{2n}\varepsilon_1 y_{n+1}(\eta,\lambda)+h_1(\eta,\lambda)\right)\\ +y_{n+1}(x,\lambda)\left(\frac{1}{2n}\varepsilon_{n+1}y_1(\eta,\lambda)+h_{n+1}(\eta,\lambda)\right)\\ +\frac{1}{2n}\sum_{j=2}^{n}\varepsilon_j y_j(x,\lambda)y_{j+n}(\eta,\lambda), \quad x>\eta,\end{cases} \tag{3.3}$$

$$h_1(\eta,\lambda)=y_1(\eta,\lambda)o(1)+y_{n+1}(\eta,\lambda)o(1),$$
$$h_{n+1}(\eta,\lambda)=y_1(\eta,\lambda)\frac{i}{2n}(1+o(1))+y_{n+1}(\eta,\lambda)o(1).$$

## 4. Asymptotic Formulas for the Eigenvalue Distribution Function $N(\lambda)$

Henceforth, we assume that the following conditions are satisfied:

$$|\operatorname{Re}\mu_s(x,\lambda)|\le C\left|\operatorname{Re} i\sqrt[2n]{q(x)+\lambda}\right|,\ s=1,n+1,\ \lambda\in\Gamma, \tag{4.1}$$

$$\left|\int_{-\infty}^{\infty}F(x,\lambda)\,dx\right|\ge C\int_{-\infty}^{\infty}\big(\sigma+q(x)\big)^{-\frac{2n-1}{2n}}\,dx,\ \lambda\in\Gamma,\ \lambda\to\infty, \tag{4.2}$$

where

$$F(x,\lambda)=\frac{1}{2n\big(q(x)+\lambda\big)^{\frac{2n-1}{2n}}}\left(i-\sum_{j=n+2}^{2n}\varepsilon_j\right). \tag{4.3}$$

We show that as $\lambda\in\Gamma$, $\lambda\to\infty$,

$$\int_{-\infty}^{\infty}K(x,x,\lambda)\,dx\sim\int_{-\infty}^{\infty}F(x,\lambda)\,dx. \tag{4.4}$$

Indeed, due to the uniformity in $x\in(-\infty,\infty)$ of the quantity $o(1)$ in the previous estimates, for any preassigned $\delta>0$, there exists $N>0$ such that for $\lambda\in\Gamma$, $|\lambda|>N$, $o(1)<\delta$. Then

$$\left|\int_{-\infty}^{\infty}\frac{1}{\big(q(x)+\lambda\big)^{\frac{2n-1}{2n}}}o(1)\,dx\right|\le\delta\left|\int_{-\infty}^{\infty}\frac{1}{\big(q(x)+\lambda\big)^{\frac{2n-1}{2n}}}\,dx\right|.$$

Hence, as $\lambda\in\Gamma$, $\lambda\to\infty$,

$$\int_{-\infty}^{\infty}\frac{1}{\big(q(x)+\lambda\big)^{\frac{2n-1}{2n}}}o(1)\,dx=o\left(\int_{-\infty}^{\infty}\frac{1}{\big(q(x)+\lambda\big)^{\frac{2n-1}{2n}}}\,dx\right). \tag{4.5}$$

Similarly to [3] (Chapter IX, §3), it can be shown that for $\lambda\in\Gamma$, $\lambda\to\infty$, $s=1,n+1$, the following estimates hold:

$$\int_{-\infty}^{\infty}\frac{e^{\int_0^x\mu_s(t,\lambda)\,dt}}{\big(q(x)+\lambda\big)^{\frac{2n-1}{2n}}}o(1)\,dx=o\left(\int_{-\infty}^{\infty}\frac{1}{\big(q(x)+\lambda\big)^{\frac{2n-1}{2n}}}\,dx\right), \tag{4.6}$$

$$\int_{-\infty}^{\infty} \frac{e^{\int_0^x \mu_s(t,\lambda)\,dt}}{\left(q(x)+\lambda\right)^{\frac{2n-1}{2n}}}\,dx = o\left(\int_{-\infty}^{\infty} \frac{1}{\left(q(x)+\lambda\right)^{\frac{2n-1}{2n}}}\,dx\right). \tag{4.7}$$

Taking into account (4.5) and (4.6), this proves (4.4).

From the method of obtaining the function

$$\rho(\lambda) = \int_{-\infty}^{\infty} F(x,\lambda)\,dx \tag{4.8}$$

it is clear that $\rho(\lambda)$ is an analytic function in the $\lambda$-plane with a cut along the real axis, and $\overline{\rho(\lambda)} = \rho(\overline{\lambda})$, and the function $\operatorname{Im}\rho(\sigma+i\tau)$ has a limit as $\tau \to +0$.

Denote by $\delta(t)$ the function

$$\delta(t) = \frac{1}{\pi}\lim_{\tau\to+0}\int_0^t \operatorname{Im}\rho(\sigma+i\tau)\,d\sigma. \tag{4.9}$$

All distinct $2n$-th roots of $(-1)^n$ were denoted by $\varepsilon_j$, i.e., the roots of the equation $\varepsilon^{2n} = (-1)^n$. Consider two cases. The first case is $n = 2k-1$. Then we obtain the equation $\varepsilon^{4k-2}+1=0$, which can be rewritten as $(\varepsilon^{2k-1}+i)(\varepsilon^{2k-1}-i)=0$. If $\varepsilon$ is a root of this equation, then $\overline{\varepsilon}$ is also a root of this equation. The second case is $n=2k$. In this case, the equation $\varepsilon^{4k}=1$ has roots $\mu_k = e^{i\frac{2\pi m}{4k}}$, $k=0,1,\dots,4k-1$. The root $\pi\frac{k-1}{2k}$ corresponds to the root $\pi\frac{3k+1}{2k} = \pi\left(2-\frac{k-1}{2k}\right)$. The corresponding exponentials are symmetric with respect to the real axis, so the term $\sum_{j=n+2}^{2n}\varepsilon_j$ in the function $F(x,\lambda)$ will not contain imaginary parts. Then

$$\delta(t) = \frac{1}{2n\pi}\int_0^t\int_{-\infty}^{\infty}\frac{d\sigma\,dx}{\left(q(x)+\sigma\right)^{\frac{2n-1}{2n}}} =$$
$$= \begin{cases} \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\left(\left(q(x)+\sigma\right)^{\frac{1}{2n}} - \left(q(x)\right)^{\frac{1}{2n}}\right)dx, & t>0,\\ \frac{1}{2n\pi}\int\limits_0^t\int\limits_{q(x)+\sigma>0}\frac{d\sigma\,dx}{\left(q(x)+\sigma\right)^{\frac{2n-1}{2n}}}, & t<0. \end{cases} \tag{4.10}$$

By the Stieltjes inversion formula

$$\rho(t) = \int_{-\infty}^{\infty}\frac{d\delta(\lambda)}{\lambda - t}. \tag{4.11}$$

From the obtained asymptotic formulas (2.3), it also follows that the spectrum of an arbitrary self-adjoint extension of the operator $L_0$ is discrete and consists of two series: $\{\lambda_n^+,\ \lambda_n^-\}$ such that

$$\lambda_n^+ \to +\infty,\ \text{as } n\to\infty,$$
$$\lambda_n^- \to -\infty,\ \text{as } n\to\infty.$$

**Theorem 4.1.** *Suppose all conditions of Theorem 2.1 and conditions (3.1), (4.1), and (4.2) are satisfied. Set*

$$N(\lambda) = \begin{cases} \sum_{\lambda_n^+<\lambda} 1, & \lambda\ge 0,\\ -\sum_{\lambda_n^->\lambda} 1, & \lambda<0. \end{cases} \tag{4.12}$$

*Let the function $\delta(\lambda)$ satisfy the Tauberian condition*

$$\begin{cases} c_1 \le \left|\frac{\delta(-\lambda)}{\delta(\lambda)}\right| \le c_2, \\ \alpha\delta(\lambda) < \lambda\delta'(\lambda) < \beta\delta(\lambda). \end{cases} \tag{4.13}$$

*Then, as $\lambda \to \pm\infty$, the following asymptotic formulas hold:*

$$N(\lambda) \sim \delta(\lambda). \tag{4.14}$$

*Proof.* On the one hand, by the Carleman formula, we have

$$\int_{-\infty}^{\infty} K(x,x,z)\,dx = \int_{-\infty}^{\infty} \frac{dN(\lambda)}{\lambda - z},$$

which, by virtue of equalities (4.8), (4.11), and (4.4), leads to the formula

$$\int_{-\infty}^{\infty} \frac{dN(\lambda)}{\lambda - z} \sim \int_{-\infty}^{\infty} \frac{d\delta(\lambda)}{\lambda - z} \tag{4.15}$$

as $z \in \Gamma$ and $z \to \infty$.

On the other hand, the conditions (4.13) allow the application of the Tauberian theorem ([3], Theorem 4.1, Chapter X), according to which the formulas (4.14), i.e., the statement of Theorem 4.1, follow from the equality (4.15). □

**Remark.** Similarly to what was done in the works [6] and [7], it is shown that for the operator $L$ with the model potential $q_1(x) = -x^\alpha + cx^\alpha \sin(x^\beta)$ under the conditions $\frac{2n}{2n-1} < \alpha$, $\frac{\alpha}{2n} + 2 < \beta$, the conditions of Theorems 2.1 and 3.1 regarding the asymptotics of the fundamental system of solutions and the defect indices are satisfied. Moreover, since the regular part of the potential $q_1(x)$ satisfies the conditions (4.1) and (4.3), the asymptotics of the eigenvalue distribution function $N(\lambda)$ can be represented in the form (4.10).

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