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ON THE COMMUTATIVITY OF WEIGHTED COMPOSITION AND COMPOSITION DIFFERENTIATION OPERATORS ON THE HARDY SPACE

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Abstract. This work provides a complete characterization of when pairs of fundamental operators, composition operators C_ϕ, multiplication operators M_ψ, differentiation operator D and their products commute on the Hardy space $\mathcal{H}^2(\mathbb{D})$. Using systematic functional equation analysis, we prove that in most of the cases commutation typically occurs only in trivial cases with inducing symbols ψ constant with $\phi(z) = z$. Among various operator pairs studied, DM_ψ and $M_\psi D$ exhibit the most flexibility (admitting all linear multipliers $\psi(z) = az + b$), while $M_\psi C_\phi$ and DM_ψ are most restrictive (essentially only identity solutions). The commutativity of weighted composition operators pairs, J-symmetric weighted composition operators, pairs consisting of weighted composition and composition differentiation operators, and pairs consisting of weighted composition and the adjoint of weighted composition operators is also characterized.

1. Introduction

A fundamental problem in operator theory is to characterize the conditions under which two given operators commute. More precisely, given two bounded linear operators $T, S : \mathcal{H} \to \mathcal{H}$ acting on a Hilbert space $\mathcal{H}$, one seeks to determine when the operator equality

$$TS = ST$$

holds. Such commutation relations have deep implications in both the algebraic and spectral analysis of operators.

The study of commutativity is central for multiple reasons:

- The set of all bounded linear operators $\mathcal{B}(\mathcal{H})$ forms a non-commutative algebra. Families of commuting operators generate commutative subalgebras, which are often easier to analyze and classify. Understanding when two operators commute allows one to identify maximal abelian subalgebras (MASAs).

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- Commuting normal operators can be simultaneously diagonalized via the spectral theorem. More generally, commutativity is closely related to the possibility of decomposing the Hilbert space into invariant subspaces that reduce both operators simultaneously. Such decompositions provide powerful tools to study operators via their joint spectra.
- For commuting operators, one can define a joint spectral measure that enables a multivariable functional calculus, generalizing the single-operator case. This enriches spectral analysis and has applications in quantum mechanics, where observables correspond to commuting self-adjoint operators.
- In applied mathematical fields including control theory, ergodic theory, and dynamical systems, the commutativity of operators reflects the presence of symmetries and invariances within the underlying systems. Detecting and characterizing these commuting pairs can reveal structural properties fundamental to the behavior of the system.

Thus, characterizing the commutativity of a class of operators is a theoretical problem with deep implications towards understanding the operator algebra generated by these operators, along with their invariant subspaces and spectral properties. In this paper, we investigate the commutativity of weighted differentiation composition operators acting on the Hardy space. Our aim is to relate the analytic properties of the inducing maps to algebraic relations in operator compositions. Identifying when certain classes of operators commute involves blending complex function theory with operator theory and operator algebra. To proceed, we establish key notations and recall essential definitions and important results.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. We denote by $\mathcal{H}(\mathbb{D})$ the space of functions analytic on $\mathbb{D}$. Among subspaces of $\mathcal{H}(\mathbb{D})$, the *Hardy space* $\mathcal{H}^2(\mathbb{D})$ holds particular significance. This space consists of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which the sequence of Taylor coefficients $(a_n)_{n=0}^{\infty}$ is square summable, i.e., $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. Equipped with the norm

$$\|f\| = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2},$$

$\mathcal{H}^2(\mathbb{D})$ is a Hilbert space with the inner product $\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ belong to $\mathcal{H}^2(\mathbb{D})$.

It is well-established that $\mathcal{H}^2(\mathbb{D})$ itself is a reproducing kernel Hilbert space. The reproducing kernel for evaluation at any point $w \in \mathbb{D}$ is explicitly given by $K_w(z) = 1/(1 - \overline{w}z)$. Moreover, the set of such kernel functions $\{K_w : w \in \mathbb{D}\}$ spans a dense subset in $\mathcal{H}^2(\mathbb{D})$ (see [3, 13, 17, 15, 14]).

Given a holomorphic self-map $\phi \in \mathcal{H}(\mathbb{D})$ with $\phi(\mathbb{D}) \subseteq \mathbb{D}$, the corresponding *composition operator* $C_\phi : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$ is defined by $(C_\phi f)(z) := f(\phi(z))$, $f \in \mathcal{H}^2(\mathbb{D})$, $z \in \mathbb{D}$. For a fixed holomorphic function $\psi \in \mathcal{H}(\mathbb{D})$, the *multiplication operator* $M_\psi : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$ is given by $(M_\psi f)(z) := \psi(z)f(z)$. The *differentiation operator* $D : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ is defined by $(Df)(z) := f'(z)$, where f' denotes the complex derivative of f. Combining composition, multiplication and differentiation in different order extended this to *weighted differentiation composition operator*. It is a classical result stemming from Littlewood's subordination principle (see, e.g., [3]) that every analytic self-map ϕ of $\mathbb{D}$ induces a bounded composition operator C_ϕ on $\mathcal{H}^2(\mathbb{D})$. The boundedness and compactness properties of weighted differentiation composition operator on Hardy and weighted Bergman spaces have been extensively studied; see, for example, [12] and [16].

These operators arise naturally in diverse areas such as dynamical systems, model theory, and control theory. For more details, we refer to [4, 1].

Recall that a conjugation on a complex Hilbert space $\mathcal{H}$ is an antilinear, isometric involution $J : \mathcal{H} \to \mathcal{H}$, that is $J^2 = I$ and $\langle Jf, Jg\rangle = \langle g, f\rangle$ for all $f, g \in \mathcal{H}$. In the Hardy space $\mathcal{H}^2(\mathbb{D})$, the operator defined by $(Jf)(z) := \overline{f(\overline{z})}$, $z \in \mathbb{D}$ is a natural example of a conjugation. An operator T on $\mathcal{H}^2(\mathbb{D})$ is said to be *J-symmetric* or equivalently complex symmetric, if $T = JT^*J$. See [5, 6, 9, 7, 11] and references therein for more about these type of operators.

Before we proceed to our main results, we state and recall the Denjoy–Wolff Theorem which guarantees the unique fixed point α to which the iterates of the analytic self-map ϕ converge (except the trivial elliptic automorphism case).

Theorem 1.1 (Denjoy–Wolff Theorem). *Let ϕ be an analytic self-map of $\mathbb{D}$ that is not an elliptic automorphism (i.e., not a rotation of the form $\phi(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$).*

Then there exists unique point $\alpha \in \overline{\mathbb{D}}$ (called the Denjoy–Wolff point*) such that:*

(1) *The sequence of iterates $\{\phi^n\}$ defined by $\phi^n = \phi \circ \phi^{n-1}$ with $\phi^0 = \mathrm{id}$ converges locally uniformly in $\mathbb{D}$ to the constant function α:*

$$\lim_{n\to\infty} \phi^n(z) = \alpha \quad \text{for all } z \in \mathbb{D}.$$

(2) *The Denjoy–Wolff point α lies either:*
  - *in $\mathbb{D}$, in which case $\phi(\alpha) = \alpha$ and $0 < |\phi'(\alpha)| < 1$, or*
  - *on the boundary $\partial\mathbb{D}$, where ϕ has an angular (nontangential) limit α and the angular derivative $\phi'(\alpha)$ exists with $0 < \phi'(\alpha) \leq 1$.*

(3) *No other fixed point in $\overline{\mathbb{D}}$ has derivative less than or equal to 1 in magnitude.*

2. Main Results

Theorem 2.1. *Suppose that the operators C_ϕ and M_ψ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then the following are equivalent:*

(1) *C_ϕ and M_ψ commute;*
(2) *C_ϕ^* and M_ψ commute;*
(3) *ϕ and ψ satisfy the functional equation*

$$\psi = \psi \circ \phi.$$

Moreover, the possible solutions to $\psi = \psi \circ \phi$ are characterized as follows:

(i) *If ϕ is not an elliptic automorphism, then ψ is constant on $\mathbb{D}$.*
(ii) *If ϕ is a rotation $\phi(z) = e^{i\theta}z$, then ψ must satisfy the rotation invariance*

$$\psi(z) = \psi(e^{i\theta}z) \quad \text{for all } z \in \mathbb{D}.$$

In this case,

If $\frac{\theta}{2\pi}$ is irrational, then ψ is constant.

If $\frac{\theta}{2\pi} = \frac{p}{q}$ is rational in lowest terms, then ψ is a function of z^q, i.e.,

$$\psi(z) = \sum_{k=0}^{\infty} a_{kq} z^{kq}.$$

Proof. Suppose that C_ϕ and M_ψ commute; that is, $C_\phi M_\psi = M_\psi C_\phi$. Applying both sides to an arbitrary function f and evaluating at $z \in \mathbb{D}$, we obtain

$$\psi(z)\, f(\phi(z)) = \psi(\phi(z))\, f(\phi(z)).$$

Since $f(\phi(z))$ can be chosen arbitrarily, it follows that

$$\psi(z) = \psi(\phi(z)) \quad \text{for all } z \in \mathbb{D},$$

or equivalently,

$$\psi \circ \phi = \psi.$$

Again taking inner products with an arbitrary $f \in H^2$:

$$\langle C_\phi^* M_\psi K_w, f\rangle = \langle M_\psi K_w, C_\phi f\rangle = \langle \psi K_w, f \circ \phi\rangle.$$

Using the reproducing property,

$$\langle \psi k_w, f \circ \phi\rangle = \psi(w)\overline{f(\phi(w))}.$$

On the other hand,

$$\langle M_\psi C_\phi^* k_w, f\rangle = \langle \psi k_{\phi(w)}, f\rangle = \psi(\phi(w))\overline{f(\phi(w))}.$$

Since this holds for all f, it follows that

$$\psi(w) = \psi(\phi(w)) \quad \forall w \in \mathbb{D}.$$

This proves that (1), (2) and (3) are equivalent.
If ϕ is not an elliptic automorphism, then the Denjoy–Wolff theorem implies $\phi^n(w)$ converge to unique point $\alpha \in \overline{\mathbb{D}}$. Iterating the functional equation gives

$$\psi(w) = \psi(\phi(w)) = \psi(\phi^2(w)) = \cdots = \psi(\phi^n(w)).$$

Also notice that M_ψ is bounded on $\mathcal{H}^2(\mathbb{D})$, so $\psi \in H^\infty$. Since ψ is bounded and analytic on $\mathbb{D}$, it follows by standard complex analysis that ψ possesses nontangential boundary limits almost everywhere on $\partial\mathbb{D}$. In particular, given the Denjoy–Wolff point $\alpha \in \overline{\mathbb{D}}$ of ϕ, the sequence of iterates $\phi^n(w)$ converges to α for each $w \in \mathbb{D}$, and by continuity of ψ in $\mathbb{D}$ or existence of boundary limits, the limit

$$\lim_{n\to\infty} \psi(\phi^n(w)) = \psi(\alpha)$$

exists as a finite complex number. Thus, ψ is constant on $\mathbb{D}$.

If ϕ is an elliptic automorphism, then $\phi(z) = e^{i\theta}z$ and iterates $\phi^n(z) = e^{in\theta}z$ rotate indefinitely inside $\mathbb{D}$. The invariance implies

$$\psi(z) = \psi(e^{i\theta}z).$$

Writing $\psi(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n e^{in\theta} z^n,$$

and so

$$a_n(1 - e^{in\theta}) = 0 \quad \forall n.$$

If $\frac{\theta}{2\pi}$ is irrational, then $a_n = 0$ for all $n \geq 1$, so ψ is constant. If $\frac{\theta}{2\pi} = \frac{p}{q}$ rational in lowest terms, then

$$e^{in\theta} = 1 \quad \Leftrightarrow \quad n \in q\mathbb{Z},$$

so the only possibly nonzero coefficients correspond to $n = kq$, and we get

$$\psi(z) = \sum_{k=0}^{\infty} a_{kq} z^{kq}.$$

This characterizes ψ as a function with q-fold rotational symmetry. This completes the proof. □

Theorem 2.2. *Suppose that the operators $C_\phi D$ and DC_ϕ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then, $C_\phi D = DC_\phi$ on $\mathcal{H}^2(\mathbb{D})$ if and only if ϕ is an identity map on $\mathbb{D}$.*

Proof. For any $f \in \mathcal{H}^2(\mathbb{D})$ and any $z \in \mathbb{D}$, we have $(C_\phi D)(f)(z) = (DC_\phi)(f)(z)$. For these to be equal for all f and z, we have

$$f'(\phi(z)) = f'(\phi(z))\phi'(z).$$

Since f' can assume arbitrary values at $\phi(z)$, this implies $\phi'(z) = 1 \quad \forall z \in \mathbb{D}$. Integrating, we get $\phi(z) = z + c$, for some constant $c \in \mathbb{C}$. Because ϕ maps the unit disk $\mathbb{D}$ into itself, and the map $z \mapsto z + c$ satisfies this property only if $c = 0$; otherwise, some points in $\mathbb{D}$ are mapped outside the disk. Therefore, ϕ is the identity map on $\mathbb{D}$. Conversely, if ϕ is the identity map, then $C_\phi = I$, and it trivially holds that $C_\phi D = D = DC_\phi$. □

Theorem 2.3. *Suppose that the operators DM_ψ and $M_\psi D$ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $DM_\psi = M_\psi D$ if and only if ψ is constant.*

Proof. For any $f \in \mathcal{H}^2(\mathbb{D})$ and any $z \in \mathbb{D}$, we have $(M_\psi D)(f)(z) = (DM_\psi)(f)(z)$, which means $\psi'(z)f(z) + \psi(z)f'(z) = \psi(z)f'(z)$. Thus $\psi'(z)f(z) = 0$. Because $f(z)$ can attain arbitrary values for each z, it follows that $\psi'(z) = 0$. Hence, ψ must be constant.

The converse is obvious since constant multiplication commutes with differentiation. □

Theorem 2.4. *Suppose that the operators $C_\phi D$ and $C_\phi M_\psi$ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $C_\phi D$ and $C_\phi M_\psi$ commute on $\mathcal{H}^2(\mathbb{D})$ if and only if either $\psi \equiv 0$, or ψ is a nonzero constant and ϕ the identity map on $\mathbb{D}$.*

Proof. For any $f \in \mathcal{H}^2(\mathbb{D})$ and any $z \in \mathbb{D}$, $C_\phi M_\psi \, C_\phi D(f)(z) = C_\phi D \, C_\phi M_\psi(f)(z)$. Thus for all f and all $z \in \mathbb{D}$, the following equation holds

$$\psi'(\phi(\phi(z)))\phi'(\phi(z))f(\phi(\phi(z))) = f'(\phi(\phi(z)))\big(\psi(\phi(z)) - \psi(\phi(\phi(z)))\phi'(\phi(z))\big). \tag{2.1}$$

Choosing $f(w) = 1$ gives:

$$\psi'(\phi(\phi(z)))\phi'(\phi(z)) = 0 \quad \text{for all } z \in \mathbb{D}$$

Since ϕ is non-constant, there exists z_0 such that $\phi'(\phi(z_0)) \neq 0$, which implies $\psi'(\phi(\phi(z_0))) = 0$. By analytic continuation, ψ is constant. Let $\psi(z) = c$. Choosing $f(w) = w$ in equation (2.1) gives

$$c(1 - \phi'(\phi(z))) = 0$$

If $c = 0$, then $\psi \equiv 0$, and we are done. If $c \neq 0$, then $\phi'(\phi(z)) = 1$ for all z, which implies $\phi'(w) = 1$ for all w in the range of ϕ. By analytic continuation, $\phi'(w) = 1$ for all $w \in \mathbb{D}$, so $\phi(z) = z + d$. Since $\phi : \mathbb{D} \to \mathbb{D}$, we must have $d = 0$, hence $\phi(z) = z$.

Conversely, if $\psi \equiv 0$ or if ψ is constant and $\phi(z) = z$, direct computation shows the operators commute. □

Theorem 2.5. *Suppose that the operators $C_\phi D$ and $M_\psi C_\phi$ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $C_\phi D$ and $M_\psi C_\phi$ commute on $\mathcal{H}^2(\mathbb{D})$ if and only if ψ is constant and either $\psi \equiv 0$, or $\phi(z) = z$ (the identity map).*

Proof. We can see that the operators commute if and only if

$$\left(\psi(z) - \psi(\phi(z))\phi'(\phi(z))\right) f'(\phi(\phi(z))) = \psi'(\phi(z)) f(\phi(\phi(z))) \tag{2.2}$$

for all f and all $z \in \mathbb{D}$. Choose $f(w) = 1$ (so $f'(w) = 0$), we get

$$\psi'(\phi(z)) = 0 \quad \text{for all } z \in \mathbb{D}$$

Since ϕ is non-constant, $\phi(\mathbb{D})$ contains an open subset of $\mathbb{D}$. By the identity theorem, $\psi' \equiv 0$ on $\mathbb{D}$, so ψ is constant. Let $\psi(z) = c$ for some constant c. Then equation (2.2) becomes

$$0 = \left(c - c\phi'(\phi(z))\right) f'(\phi(\phi(z))) = c\left(1 - \phi'(\phi(z))\right) f'(\phi(\phi(z)))$$

If $c = 0$, then $\psi \equiv 0$ and we are done. If $c \neq 0$, then we need $\phi'(\phi(z)) = 1$ for all $z \in \mathbb{D}$. This means $\phi'(w) = 1$ for all w in the range of ϕ. Since ϕ is non-constant, its range contains an open subset of $\mathbb{D}$. By analytic continuation, $\phi'(w) = 1$ for all $w \in \mathbb{D}$. Therefore, $\phi(z) = z + b$, $b \in \mathbb{C}$ is a constant. Since ϕ is a self-map of $\mathbb{D}$, we have $b = 0$, giving $\phi(z) = z$.

Conversely, if $\psi \equiv 0$ or if ψ is constant and $\phi(z) = z$, direct computation shows the operators commute. □

Theorem 2.6. *Suppose that the operators $C_\phi D$ and DM_ψ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $C_\phi D$ and DM_ψ commute if and only if ψ is constant and either $\psi \equiv 0$, or $\phi(z) = z$ (the identity map).*

Proof. The equality $(C_\phi D)(DM_\psi f) = (DM_\psi)(C_\phi D f)$ for all $f \in \mathcal{H}^2(\mathbb{D})$ is equivalent to

$$\begin{aligned}&\psi'(z) f'(\phi(z)) + \psi(z)\phi'(z) f''(\phi(z))\\ &= \psi''(\phi(z)) f(\phi(z)) + 2\psi'(\phi(z)) f'(\phi(z)) + \psi(\phi(z)) f''(\phi(z))\end{aligned}$$

which holds for for all $f \in \mathcal{H}^2(\mathbb{D})$ and all $z \in \mathbb{D}$. Therefore, we have

$$\begin{aligned}&\left(\psi(z)\phi'(z) - \psi(\phi(z))\right) f''(\phi(z))\\ &\qquad + \left(\psi'(z) - 2\psi'(\phi(z))\right) f'(\phi(z)) - \psi''(\phi(z)) f(\phi(z)) = 0\end{aligned} \tag{2.3}$$

Let $f(w) = 1$. (Then $f'(w) = f''(w) = 0$). Therefore for all $z \in \mathbb{D}$, we have $\psi''(\phi(z)) = 0$. Since $\phi(\mathbb{D})$ contains an open set, $\psi'' \equiv 0$, so $\psi(z) = az + b$. Again, let $f(w) = w$ (so $f'(w) = 1$, $f''(w) = 0$): $2\psi'(\phi(z)) - \psi'(z) = 0$ $2a - a = 0 \implies a = 0$ Therefore, $\psi(z) = b$ is constant. Further, let $f(w) = w^2$ (so $f'(w) = 2w$, $f''(w) = 2$). Substituting into equation (2.3) with $\psi \equiv b$, we have $2b(1 - \phi'(z)) = 0$ If $b = 0$, then $\psi \equiv 0$ and we're done. If $b \neq 0$, then $\phi'(z) = 1$.

This implies $\phi(z) = z + c$ for some constant c. For $\phi(z) = z + c$ to map $\mathbb{D}$ to $\mathbb{D}$ is possible only if $c = 0$. Therefore, $c = 0$ and $\phi(z) = z$.

Conversely, if $\psi \equiv 0$ or if ψ is constant and $\phi(z) = z$, direct computation shows the operators commute. □

Theorem 2.7. *Suppose that the operators $C_\phi D$ and DC_ϕ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $C_\phi D$ and DC_ϕ commute if and only if $\phi(z) = az + b$ where $a, b \in \mathbb{C}$ and $|a| + |b| < 1$.*

Proof. For commutation, we need $(C_\phi D)(DC_\phi f)$ and $(DC_\phi)(C_\phi Df)$ for all $f \in \mathcal{H}^2(\mathbb{D})$. Computing these, we get

$$(DC_\phi)(C_\phi Df)(z) = f''(\phi(\phi(z)))\phi'(\phi(z))\phi'(z) \tag{2.4}$$

$$(C_\phi D)(DC_\phi f)(z) = f''(\phi(\phi(z)))[\phi'(\phi(z))]^2 + f'(\phi(\phi(z)))\phi''(\phi(z)) \tag{2.5}$$

Equating (2.4) and (2.5), and rearranging the resulting equation, we get

$$f'(\phi(\phi(z)))\phi''(\phi(z)) + f''(\phi(\phi(z)))((\phi'(\phi(z)))^2 - \phi'(\phi(z))\phi'(z)) = 0$$

Taking $f(w) = w$ (so $f'(w) = 1$, $f''(w) = 0$): $\phi''(\phi(z)) = 0$ for all z. Since ϕ is non-constant, the image $\phi(\mathbb{D})$ contains an open subset $U \subseteq \mathbb{D}$. By the identity theorem for analytic functions, if $\phi''(w) = 0$ for all $w \in U$, then $\phi''(w) = 0$ for all $w \in \mathbb{D}$. Thus $\phi'' \equiv 0$. Integrating twice: $\phi'(z) = a$ (constant) and $\phi(z) = az + b$ for some $a, b \in \mathbb{C}$. For the constraint $\phi : \mathbb{D} \to \mathbb{D}$, we need $|a| + |b| \leq 1$.

Conversely, if $\phi(z) = az + b$ with $|a| + |b| < 1$, then direct computation shows:

$$(C_\phi D)(DC_\phi f)(z) = a^2 f''(a^2 z + ab + b)$$

$$(DC_\phi)(C_\phi Df)(z) = a^2 f''(a^2 z + ab + b)$$

Therefore, the operators commute if and only if $\phi(z) = az + b$ with $|a| + |b| < 1$. □

Theorem 2.8. *Suppose that the operators $C_\phi D$ and $M_\psi D$ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $C_\phi D$ and $M_\psi D$ commute if and only if ψ is constant and either $\psi \equiv 0$, or ϕ is the identity map.*

Proof. Using commutation of $C_\phi D$ and $M_\psi D$, we can easily have $(\psi(z)\phi'(z) - \psi(\phi(z)))f''(\phi(z)) - \psi'(\phi(z))f'(\phi(z)) = 0$ Now, to ensure this to holds for all f, we need the coefficients of $f'(\phi(z))$ and $f''(\phi(z))$ to be zero independently, that is, $\psi'(\phi(z)) = 0$ and $\psi(\phi(z)) - \psi(z)\phi'(z) = 0$ for all $z \in \mathbb{D}$. From first among these equations, using ϕ is non-constant and analytic, its range contains an open set. Therefore ψ' vanishes on an open set, so by the identity theorem, $\psi' \equiv 0$, and so $\psi(z) = c$ (constant). Substituting $\psi(z) = c$ into the second equation, we have $c(1 - \phi'(z)) = 0$ for all $z \in \mathbb{D}$. This gives us two cases: $c = 0$, so $\psi \equiv 0$. Otherwise, $c \neq 0$, so $1 - \phi'(z) = 0$, which means for all $z \in \mathbb{D}$, $\phi'(z) = 1$. If $\phi'(z) = 1$, then $\phi(z) = z + b$ for some constant b. Since $\phi : \mathbb{D} \to \mathbb{D}$, so $b = 0$. Therefore, $\phi(z) = z$ (the identity function).

If $\psi \equiv 0$, or if $\psi(z) = c \neq 0$ and $\phi(z) = z$, then we can easily see that the commutation holds. □

Theorem 2.9. *Suppose that the operators $C_\phi M_\psi$ and $M_\psi D$ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $C_\phi M_\psi$ and $M_\psi D$ commute if and only if ψ is constant and either $\psi \equiv 0$, or $\phi(z) = z$ (the identity map).*

Proof. We seek to find conditions under which $C_\phi M_\psi$ and $M_\psi D$ commute. For commutation, set these expressions equal for all analytic f, we can easily have

$$\psi(z)\,\psi'(\phi(z))\phi'(z)f(\phi(z)) + \psi(z)\,\psi(\phi(z))\phi'(z)f'(\phi(z)) = (\psi(\phi(z)))^2 f'(\phi(z)).$$

Therefore, we have $\psi(z)\,\psi'(\phi(z))\phi'(z) = 0$ and $(\psi(\phi(z)))^2 = \psi(z)\,\psi(\phi(z))\phi'(z)$ From the first among these equations, if $\psi(z)$ is not identically zero and $\phi'(z)$ is not identically zero, then $\psi'(\phi(z)) = 0$, so ψ is constant. From second one, if ψ is constant (say $\psi(z) \equiv c$), then $\phi'(z) = 1$, that is $\phi(z) = z + c$. Since $\phi : \mathbb{D} \to \mathbb{D}$, so $c = 0$ and $\phi(z) = z$.

Conversely, the conditions ψ is constant and either $\psi \equiv 0$, or $\phi(z) = z$ (the identity map) easily confirm commutation.

□

Theorem 2.10. *Suppose that the operators $C_\phi M_\psi$ and DM_ψ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $C_\phi M_\psi$ and DM_ψ commute if and only if ψ is constant and either $\psi \equiv 0$, or $\phi(z) = z$ (the identity map).*

Proof. For all analytic f and all z, if $C_\phi M_\psi$ and DM_ψ commute, then we must have

$$-\psi'(z)\psi(\phi(z)) - \psi(z)\psi'(\phi(z))\phi'(z) + \psi(\phi(z))\psi'(\phi(z)) = 0 \qquad (2.6)$$

$$-\psi(z)\psi(\phi(z))\phi'(z) + (\psi(\phi(z)))^2 = 0 \qquad (2.7)$$

for all $z \in \mathbb{D}$. If $\psi \equiv 0$, then we are done. If ψ is not identically equal to zero then an application of the identiy theorem in (2.6) and (2.7) shows that ψ is a nonzero constant function, say $\psi(z) \equiv c \neq 0$. Then $\psi' = 0$, $\psi(\phi(z)) = c$, and $\psi(z) = c$, so from above equations $\phi'(z) = 1$ So $\phi(z) = z + a$, i.e., $\phi(z)$ is an affine map with slope 1. The self-map restriction, then forces $a = 0$.

Conversely, the conditions ψ is constant and either $\psi \equiv 0$, or $\phi(z) = z$ (the identity map) easily confirm commutation. □

Theorem 2.11. *Suppose that the operators $C_\phi M_\psi$ and $M_\psi C_\phi$ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $C_\phi M_\psi$ and $M_\psi C_\phi$ commute if and only if $\psi(z)\psi(\phi^2(z)) = (\psi(\phi(z)))^2$ for every $z \in \mathbb{D}$.*

Proof. An easy calculation yields the proof. We omit the details. □

Theorem 2.12. *Suppose that the operators $M_\psi C_\phi$ and $M_\psi D$ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $M_\psi C_\phi$ and $M_\psi D$ commute if and only if either $\psi \equiv 0$, or ψ is constant and $\phi(z) = z$.*

Proof. For any $f \in \mathcal{H}^2(\mathbb{D})$ and for any $z \in \mathbb{D}$, the equality $(M_\psi C_\phi)(M_\psi D)(f)(z)$ $= (M_\psi D)(M_\psi C_\phi)(f)(z)$ easily yields $\psi(z)\psi'(z)f(\phi(z)) + (\psi(z))^2 f'(\phi(z))\phi'(z) = \psi(z)\psi(\phi(z))f'(\phi(z))$, which on rearranging, we have

$$\psi(z)\psi'(z)f(\phi(z)) = \psi(z)f'(\phi(z))(\psi(\phi(z)) - \psi(z)\phi'(z)).$$

Since this must hold for all $f \in \mathcal{H}^2(\mathbb{D})$, we test with specific functions. Take $f(w) = 1$. Then $f'(w) = 0$, and so we have $\psi(z)\psi'(z) = 0$ for all $z \in \mathbb{D}$. This means: for each z, either $\psi(z) = 0$ or $\psi'(z) = 0$. Again take $f(w) = w$ in the main equation. Then $f'(w) = 1$, and $\psi(z)\psi'(z)\phi(z) = \psi(z)(\psi(\phi(z)) - \psi(z)\phi'(z))$. If $\psi \equiv 0$, then we are done. Assuming $\psi(z) \neq 0$, we have $\psi'(z)\phi(z) = \psi(\phi(z)) - \psi(z)\phi'(z)$.

From first condition for all $z \in \mathbb{D}$, we have $\psi(z)\psi'(z) = 0$. If $Z = \{z \in \mathbb{D} : \psi(z) = 0\}$ and $S = \{z \in \mathbb{D} : \psi'(z) = 0\}$, then it implies that $Z \cup S = \mathbb{D}$. If $Z = \mathbb{D}$, then $\psi \equiv 0$ and we are done. Otherwise, $\psi \not\equiv 0$. Since ψ is analytic and $\psi \not\equiv 0$, the zero set Z is either empty or consists of isolated points. If $Z = \emptyset$, then $\psi(z) \neq 0$ for all z, so from first condition, we have $\psi'(z) = 0$ for all z. This means ψ is constant. If Z consists of isolated points, then using $Z \cup S = \mathbb{D}$ and Z is discrete, we must have that S contains all points except those in Z. In particular, S has an accumulation point.

But if $\psi'(z) = 0$ at an accumulation point of zeros of ψ', then by the Identity theorem, $\psi' \equiv 0$ on $\mathbb{D}$. This means ψ is constant. However, if ψ is constant and non-zero, then $Z = \emptyset$, contradicting our assumption that Z consists of isolated points. Therefore, it is impossible, and so either $\psi \equiv 0$ or ψ is constant. Suppose $\psi(z) = c$ for some constant $c \neq 0$. From second condition, then we have $c(1 - \phi'(z)) = 0$ Since $c \neq 0$, we have $\phi'(z) = 1 \quad$ for all $z \in \mathbb{D}$ This means $\phi(z) = z + b$ for some constant b. Since $\phi : \mathbb{D} \to \mathbb{D}$, the map $z \mapsto z + b$ must send $\mathbb{D}$ into itself. This is only possible if $b = 0$. Therefore: $\phi(z) = z$.
Converse is obvious.

□

Theorem 2.13. *Suppose that the operators DC_ϕ and $M_\psi D$ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then DC_ϕ and $M_\psi D$ commute if and only if for every $z \in \mathbb{D}$, we have $(\psi(z) - \psi(\phi(z)))(\phi'(z))^2 = 0$ and $\psi(z)\phi''(z) = \psi'(\phi(z))\phi'(z)$.*

Proof. An easy calculation yields the proof. We omit the details. □

Theorem 2.14. *Suppose that the operators DM_ψ and $M_\psi D$ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then DM_ψ and $M_\psi D$ commute if and only if for all $z \in \mathbb{D}$, $\psi''(z) = 0$. Equivalently, $\psi(z) = az + b$, $a, b \in \mathbb{C}$.*

Proof. For $f \in \mathcal{H}^2(\mathbb{D})$ and $z \in \mathbb{D}$, we can easily have that

$$(M_\psi D)(DM_\psi)(f)(z) = \psi(z)\psi''(z)f(z) + 2\psi(z)\psi'(z)f'(z) + (\psi(z))^2 f''(z)$$

and

$$(DM_\psi)(M_\psi D)(f)(z) = 2\psi(z)\psi'(z)f'(z) + (\psi(z))^2 f''(z).$$

Using commutation condition, we have

$$2\psi(z)\psi'(z)f'(z)+(\psi(z))^2 f''(z) = \psi(z)\psi''(z)f(z)+2\psi(z)\psi'(z)f'(z)+(\psi(z))^2 f''(z).$$

On canceling the terms $2\psi(z)\psi'(z)f'(z)$ and $(\psi(z))^2 f''(z)$ from both sides, we have $\psi(z)\psi''(z)f(z) = 0$. Since this must be true for all $f \in \mathcal{H}^2(\mathbb{D})$, by taking $f(z) = 1$ we have $\psi(z)\psi''(z) = 0$ for all $z \in \mathbb{D}$. We claim that $\psi''(z) = 0$ for all $z \in \mathbb{D}$. Set $Z := \{z \in \mathbb{D} : \psi(z) = 0\}$, and $U := \mathbb{D} \setminus Z = \{z \in \mathbb{D} : \psi(z) \neq 0\}$. On U, we have $\psi(z) \neq 0$, so by the given equation, $\psi''(z) = 0$ for all $z \in U$. Now, ψ'' is analytic on $\mathbb{D}$, and identically zero on the open set U. If U is non-empty (i.e., ψ is not identically zero), then by the identity theorem for holomorphic functions,

$$\psi''(z) = 0 \quad \text{for all } z \in \mathbb{D}.$$

If $U = \varnothing$, then $\psi(z) = 0$ everywhere on $\mathbb{D}$, then trivially $\psi''(z) = 0$ everywhere. Therefore, in all cases, $\psi''(z) = 0$ for all $z \in \mathbb{D}$. So ψ is linear, that is, $\psi(z) = az+b$

for constants $a, b \in \mathbb{C}$. Conversely, if $\psi(z) = az + b$ with $a, b \in \mathbb{C}$. Then

$$(DM_\psi)(f)(z) = \psi'(z)f(z) + \psi(z)f'(z) = af(z) + (az+b)f'(z)$$
$$(M_\psi D)(f)(z) = \psi(z)f'(z) = (az+b)f'(z)$$

Computing compositions, we have

$$(DM_\psi)(M_\psi D)(f)(z) = 2a(az+b)f'(z) + (az+b)^2 f''(z) \tag{2.8}$$

and

$$(M_\psi D)(DM_\psi)(f)(z) = 2a(az+b)f'(z) + (az+b)^2 f''(z) \tag{2.9}$$

The equality of expressions (2.8) and (2.9) assert that the given operators commute. This completes the proof. □

Theorem 2.15. *Suppose that the operators $M_\psi C_\phi$ and DM_ψ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then the operators $M_\psi C_\phi$ and DM_ψ commute if and only if*

$$\psi'(\phi(z)) = 2\psi'(z) \quad \textit{and} \quad \psi(\phi(z)) = \psi(z)\,\phi'(z)$$

for all $z \in \mathbb{D}$.

Proof. An easy calculation yields the proof. We omit the details. □

Theorem 2.16. *Assume that the operators $C_\phi M_\psi$ and DC_ϕ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then, the operators $C_\phi M_\psi$ and DC_ϕ commute if and only if*

$$\phi(z) = z \quad \textit{and} \quad \psi \equiv c \quad \textit{for some constant } c \in \mathbb{C}.$$

Proof. Suppose that $C_\phi M_\psi$ and DC_ϕ commute. Equating $(C_\phi M_\psi)(DC_\phi)f(z) = (C_\phi M_\psi)((f'\circ\phi)\cdot\phi')(z) = C_\phi\big(\psi\cdot(f'\circ\phi)\cdot\phi'\big)(z) = (\psi\circ\phi)(z)(f'\circ\phi\circ\phi)(z)(\phi'\circ\phi)(z)$. and

$$(DC_\phi)(C_\phi M_\psi)f(z) = \psi'(\phi(\phi(z)))\phi'(\phi(z))\phi'(z)(f\circ\phi\circ\phi)(z) + (\psi\circ\phi\circ\phi)(z)f'(\phi(\phi(z)))\phi'(\phi(z))\phi'(z).$$

and splitting the equation into terms involving f and f' gives two conditions that must hold for all f, we have

$$\phi'(z)\cdot\psi'(\phi(\phi(z)))\cdot\phi'(\phi(z)) = 0,$$

and

$$(\psi\circ\phi\circ\phi)(z)\cdot\phi'(\phi(z))\cdot\phi'(z) = (\psi\circ\phi)(z)\cdot\phi'(\phi(z)).$$

Since DC_ϕ is bounded on $\mathcal{H}^2$, ϕ cannot be a constant function. For the product to be zero for all z, and since ϕ' is not identically zero, we have that

$$\psi'(\phi(\phi(z))) = 0 \quad \text{for all } z \in \mathbb{D}.$$

Because $\phi(\phi(\mathbb{D})) \subseteq \mathbb{D}$ contains an accumulation point, analytic continuation implies

$$\psi'(w) = 0 \quad \text{for all } w \in \mathbb{D}.$$

Hence, ψ is constant: $\psi(z) \equiv c$. With $\psi \equiv c$, the second equation reduces to $c = c\cdot\phi'(z)$. If $c = 0$, then $C_\phi M_\psi$ is the zero operator, which trivially commutes with every operator. Otherwise ($c \neq 0$), $\phi'(z) = 1$ for all $z \in \mathbb{D}$. The only holomorphic self-map with constant derivative 1 are of the form $\phi(z) = z$. This completes the proof. □

Theorem 2.17. *Suppose that the operators $M_\psi C_\phi$ and DC_ϕ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $M_\psi C_\phi$ and DC_ϕ commute if and only if*

$$\phi(z) = z \quad \text{and} \quad \psi \equiv c \quad \text{for some constant } c \in \mathbb{C}.$$

Proof. An easy calculation yields the proof. We omit the details. □

Theorem 2.18. *Suppose that the operators DC_ϕ and DM_ψ are bounded on $\mathcal{H}^2(\mathbb{D})$. Then DC_ϕ and DM_ψ commute if and only if for all $z \in \mathbb{D}$, ϕ and ψ satisfy the equations: $\psi'(\phi(z)) = \psi'(z)$ and $\psi(\phi(z)) = \psi(z)$.*

Proof. An easy calculation yields the proof. We omit the details. □

We end this paper by investigating the commutativity of weighted composition operators including J-symmetric weighted composition operators, pairs consisting of weighted composition and composition differentiation operators, as well as pairs consisting of weighted composition operators and the adjoints of weighted composition operators.

The proof of our next result is straightforward and therefore omitted.

Theorem 2.19. *Let $\psi_1, \psi_2 \in \mathcal{H}(\mathbb{D})$ be analytic functions, and let ϕ_1, ϕ_2 be analytic self-maps of $\mathbb{D}$. Suppose that the weighted composition operators W_{ψ_1,ϕ_1}, $W_{\psi_2,\phi_2} : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$ are bounded. Then, the operators W_{ψ_1,ϕ_1} and W_{ψ_2,ϕ_2} commute if and only if the following two conditions hold for all $z \in \mathbb{D}$:*

(a) $\psi_1(z)\,\psi_2(\phi_1(z)) = \psi_2(z)\,\psi_1(\phi_2(z))$,
(b) $(\phi_1 \circ \phi_2)(z) = (\phi_2 \circ \phi_1)(z)$.

Corollary 2.1. *Let $\psi \in \mathcal{H}(\mathbb{D})$ be a univalent function and ϕ_1, ϕ_2 be analytic self-maps of $\mathbb{D}$ such that the weighted composition operators W_{ψ,ϕ_1} and W_{ψ,ϕ_2} are bounded on $\mathcal{H}^2(\mathbb{D})$. Then the operators W_{ψ,ϕ_1} and W_{ψ,ϕ_2} commute if and only if $\phi_1 = \phi_2$.*

Proof. Assume that the operators W_{ψ,ϕ_1} and W_{ψ,ϕ_2} commute. Then by (a) of Theorem 2.19, we have $\psi(z)\psi(\phi_1(z)) = \psi(z)\psi(\phi_2(z)) \quad$ for all $z \in \mathbb{D}$. Since ψ is not identically zero, the identity theorem guarantees that $\psi(\phi_1(z)) = \psi(\phi_2(z))$. Thus by the univalency of ψ, we have

$$\phi_1(z) = \phi_2(z) \quad \text{for all } z \in \mathbb{D}.$$

The converse direction is straightforward. We omit the details. □

As applications of Theorem 2.19 and Corollary 2.1, we present examples of commuting and non-commuting weighted composition operators.

Example 2.1. Let

$$\psi_1(z) = \psi_2(z) = \frac{az+b}{cz+d}, \quad \text{with } ad - bc \neq 0,$$

and define

$$\phi_1(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \quad \phi_2(z) = \frac{\beta - z}{1 - \overline{\beta} z},$$

where $|\alpha|, |\beta| < 1$. By Corollary 2.1, the weighted composition operators W_{ψ_1,ϕ_1} and W_{ψ_2,ϕ_2} commute if and only if $\alpha = \beta$.

Example 2.2. Let $\psi_1(z) = z$, $\psi_2(z) = z^2$, $\phi_1(z) = z/2$, $\phi_2(z) = z/4$. Then the weighted composition operators W_{ψ_1,ϕ_1} and W_{ψ_2,ϕ_2} are commuting by Theorem 2.19.

Example 2.3. Consider

$$\psi_1(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \quad \psi_2(z) = \frac{\beta - z}{1 - \overline{\beta} z},$$

with $|\alpha| = |\beta| < 1$ but $\alpha \neq \beta$, and set

$$\phi_1(z) = \phi_2(z) = \phi(z) = \frac{a - z}{1 - \overline{a} z}, \quad 0 < |a| < 1.$$

Then $W_{\psi_1,\phi_1} W_{\psi_2,\phi_2} \neq W_{\psi_2,\phi_2} W_{\psi_1,\phi_1}$.

Proof: Assume, for the sake of contradiction, that $W_{\psi_1,\phi_1} W_{\psi_2,\phi_2} = W_{\psi_2,\phi_2} W_{\psi_1,\phi_1}$. Then, by (a) of Theorem 2.19, we have

$$\psi_1(z)\psi_2(\phi_1(z)) = \psi_2(z)\psi_1(\phi_2(z)) \quad \text{for all } z \in \mathbb{D}.$$

Evaluating at $z = 0$, we get

$$\psi_1(0)\psi_2(\phi_1(0)) = \psi_2(0)\psi_1(\phi_2(0)). \tag{2.10}$$

Substituting $\psi_1(0) = \alpha$, $\psi_2(0) = \beta$, and $\phi_1(0) = a$ in (2.10), we have $\alpha \cdot \psi_2(a) = \beta \cdot \psi_1(a)$, which expands to $\alpha(1 - \overline{\alpha} a)(\beta - a) = \beta(1 - \overline{\beta} a)(\alpha - a)$. Upon expanding and rearranging terms, this equality holds if and only if $|\alpha|^2 = 1$. This contradicts the assumption $|\alpha| < 1$. Therefore, our initial assumption is false, and the operators W_{ψ_1,ϕ_1} and W_{ψ_2,ϕ_2} do not commute.

Theorem 2.20. *Let $\psi_1(z) = az + b$, $\psi_2(z) = cz + d$ be linear maps on $\mathbb{D}$ and $\phi(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$ such that $0 < |\alpha| < 1$ be an automorphism on $\mathbb{D}$. Then $W_{\psi_1,\phi}$ and $W_{\psi_2,\phi}$ commute if and only if $ad - bc = 0$.*

Proof. Suppose that $W_{\psi_1,\phi}$ and $W_{\psi_2,\phi}$ commute. Then by (a) of Theorem 2.19, we have

$$(az + b)\left(c\frac{\alpha - z}{1 - \overline{\alpha} z} + d\right) = (cz + d)\left(a\frac{\alpha - z}{1 - \overline{\alpha} z} + b\right) \quad \text{for all} \quad z \in \mathbb{D}.$$

In particular, by letting $z = 0$ we get $(bc - ad)\alpha = 0$. Since $\alpha \neq 0$, we have that the required condition holds. Conversely, suppose that $ad - bc = 0$. Then

$$\begin{aligned}\psi_1(z)\psi_2(\phi(z)) &= (az + b)\left(c\frac{\alpha - z}{1 - \overline{\alpha} z} + d\right) \\ &= \frac{ac\alpha z - acz^2 + adz - \overline{\alpha} adz + bc\alpha - bcz + bd - \overline{\alpha} bdz}{1 - \overline{\alpha} z}\end{aligned}$$

and

$$\begin{aligned}\psi_2(z)\psi_1(\phi(z)) &= (cz + d)\left(a\frac{\alpha - z}{1 - \overline{\alpha} z} + b\right) \\ &= \frac{ac\alpha z - acz^2 + bcz - \overline{\alpha} bcz^2 + ad\alpha - adz + bd - \overline{\alpha} bdz}{1 - \overline{\alpha} z}\end{aligned}$$

Clearly, $\psi_1(z)\psi_2(\phi(z)) = \psi_2(z)\psi_1(\phi(z)) \quad$ for all $\quad z \in \mathbb{D}$, and so $M_{\psi_1,\phi} M_{\psi_2,\phi} K_w(z) = M_{\psi_2,\phi} M_{\psi_1,\phi} K_w(z)$. Since, span$\{K_w : w \in \mathbb{D}\}$ is dense in $\mathcal{H}^2(\mathbb{D})$, it follows that $W_{\psi_1,\phi} W_{\psi_2,\phi} = W_{\psi_2,\phi} W_{\psi_1,\phi}$. This completes the proof. $\square$

Proposition 2.1. *Let W_{ψ_1,ϕ_1} and W_{ψ_2,ϕ_2} be two J-symmetric operators on $\mathcal{H}^2(\mathbb{D})$ where $Jf(z) = \overline{f(\overline{z})}$. Then the following are equivalent.*

(a) *W_{ψ_1,ϕ_1} and W_{ψ_2,ϕ_2} commute.*
(b) *ϕ_1 and ϕ_2 commute.*

Proof. Given W_{ψ_1,ϕ_1} and W_{ψ_2,ϕ_2} are J-symmetric. By Theorem 3.3 in [9],

$$\psi_1(z) = \frac{\alpha}{1 - a_0 z}, \qquad \phi_1(z) = a_0 + \frac{a_1 z}{1 - a_0 z}$$

$$\psi_2(z) = \frac{\beta}{1 - A_0 z}, \qquad \phi_2(z) = A_0 + \frac{A_1 z}{1 - A_0 z}$$

$(a) \implies (b)$ follows from Theorem 2.19.
$(b) \implies (a)$ Suppose that (b) holds. Then

$$\phi_1(\phi_2(z)) = \phi_2(\phi_1(z)) \quad \text{for all} \quad z \in \mathbb{D}.$$

Now,

$$\begin{aligned} \phi_1(\phi_2(z)) &= a_0 + \frac{a_1\left(A_0 + \frac{A_1 z}{1-A_0 z}\right)}{1 - a_0\left(A_0 + \frac{A_1 z}{1-A_0 z}\right)} \\ &= a_0 + \frac{a_1\left(A_0 - A_0^2 z + A_1 z\right)}{1 - A_0 z - a_0\left(A_0 - A_0^2 z + A_1 z\right)} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \phi_2(\phi_1(z)) &= A_0 + \frac{A_1\left(a_0 + \frac{a_1 z}{1-a_0 z}\right)}{1 - A_0\left(a_0 + \frac{a_1 z}{1-a_0 z}\right)} \\ &= A_0 + \frac{A_1\left(a_0 - a_0^2 z + a_1 z\right)}{1 - a_0 z - A_0\left(a_0 - a_0^2 z + a_1 z\right)} \end{aligned} \tag{2.12}$$

From (2.11) and (2.12) we have that

$$a_0 + \frac{a_1\left(A_0 - A_0^2 z + A_1 z\right)}{1 - A_0 z - a_0\left(A_0 - A_0^2 z + A_1 z\right)} = A_0 + \frac{A_1\left(a_0 - a_0^2 z + a_1 z\right)}{1 - a_0 z - A_0\left(a_0 - a_0^2 z + a_1 z\right)}$$

for all $z \in \mathbb{D}$. By letting $z = 0$, we get

$$a_0 - a_0^2 A_0 + a_1 A_0 = A_0 - A_0^2 a_0 + A_1 a_0$$

Therefore, a straightforward calculation shows that

$$\psi_1(z)\psi_2(\phi_1(z)) = \frac{\alpha\beta}{1 - A_0 a_0 - z(a_0 - A_0 a_0^2 + a_1 A_0)} = \psi_2(z)\psi_1(\phi_2(z)).$$

By virtue of Theorem 2.19, it follows that $W_{\psi_1,\phi_1}W_{\psi_2,\phi_2} = W_{\psi_2,\phi_2}W_{\psi_1,\phi_1}$. This completes the proof. □

Corollary 2.2. *In the above proposition if $\phi_1(0) = \phi_2(0) \neq 0$, Then $W_{\psi_1,\phi_1}W_{\psi_2,\phi_2} = W_{\psi_2,\phi_2}W_{\psi_1,\phi_1}$ if, and only if $\phi_1 = \phi_2$.*

Let $\phi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map of $\mathbb{D}$, then there exists an associated linear fractional map σ, again a self-map of $\mathbb{D}$, given by

$$\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}.$$

This map σ is commonly referred to as Cowen Auxiliary function (or Krein adjoint of ϕ) and is uniquely determined by ϕ. For more details, we refer to [2, 3]. These induce a weighted composition operator $W_{\psi,\phi} : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$, where $\psi = \alpha K_{\sigma(0)}$ for some nonzero constant α, with K_w denoting the reproducing kernel at w.

The next result provides the necessary and sufficient conditions for two such weighted composition operators to commute.

Theorem 2.21. *Let $\phi_1(z) = \frac{pz+q}{rz+s}$ and $\phi_2(z) = \frac{Pz+Q}{Rz+S}$ be two linear fractional self-maps of $\mathbb{D}$, and W_{ψ_1,ϕ_1} , W_{ψ_2,ϕ_2} be the weighted composition operators induced by these maps. Then $W_{\psi_1,\phi_1}W_{\psi_2,\phi_2} = W_{\psi_2,\phi_2}W_{\psi_1,\phi_1}$ on $\mathcal{H}^2(\mathbb{D})$ if and only if*

(a) $Qp + Sq = qP + sQ$
(b) $Rq = rQ$
(c) $Sr + Rp = Rs + rP$

Proof. First assume that $W_{\psi_1,\phi_1}W_{\psi_2,\phi_2} = W_{\psi_2,\phi_2}W_{\psi_1,\phi_1}$ on $\mathcal{H}^2(\mathbb{D})$. Then by Theorem 2.19, we have $\phi_1 \circ \phi_2 = \phi_2 \circ \phi_1$ and $\psi_1(z)\psi_2(\phi_1(z)) = \psi_2(z)\psi_1(\phi_2(z))$. We can easily compute the following compositions:

$$\phi_1(\phi_2(z)) = \frac{(Pp + Rq)z + (Qp + Sq)}{(Pr + Rs)z + (Qr + Ss)}, \tag{2.13}$$

and

$$\phi_2(\phi_1(z)) = \frac{(Pp + Qr)z + (Pq + Qs)}{(Rp + Sr)z + (Rq + Ss)}. \tag{2.14}$$

Equating (2.13) and (2.14) for all $z \in \mathbb{D}$, we obtain

$$\frac{(Pp + Rq)z + (Qp + Sq)}{(Pr + Rs)z + (Qr + Ss)} = \frac{(Pp + Qr)z + (Pq + Qs)}{(Rp + Sr)z + (Rq + Ss)}.$$

Substituting $z = 0$ gives

$$\frac{Qp + Sq}{Qr + Ss} = \frac{Pq + Qs}{Rq + Ss}. \tag{2.15}$$

Using the Cowen auxiliary functions σ_1, σ_2 corresponding to ϕ_1, ϕ_2 and $\psi_k = \alpha_k K_{\sigma_k(0)}, k = 1, 2$, we have

$$\begin{aligned}\psi_1(z)\psi_2(\phi_1(z)) &= \alpha_1 K_{\sigma_1(0)}(z) \cdot \alpha_2 K_{\sigma_2(0)}(\phi_1(z)) \\ &= \frac{\alpha_1}{1 - \overline{\sigma_1(0)}z} \cdot \frac{\alpha_2}{1 - \overline{\sigma_2(0)}\phi_1(z)}.\end{aligned}$$

Using $\phi_1(z) = \frac{pz+q}{rz+s}$ and $\sigma_1(0) = -\frac{\overline{r}}{\overline{s}}$, and similarly for ϕ_2 and $\sigma_2(0)$, we obtain

$$\psi_1(z)\psi_2(\phi_1(z)) = \frac{\alpha_1\alpha_2 sS}{(Sr + Rp)z + (Ss + Rq)},$$

and

$$\psi_2(z)\psi_1(\phi_2(z)) = \frac{\alpha_1\alpha_2 sS}{(Rs + Pr)z + (Ss + Qr)}.$$

Equating these expressions yields

$$(Sr + Rp)z + (Ss + Rq) = (Rs + Pr)z + (Ss + Qr).$$

By comparing coefficients of z and the constant terms, we obtain

$$\begin{aligned} Sr + Rp &= Rs + Pr, \\ Rq &= Qr. \end{aligned} \tag{2.16}$$

Using (2.16) in (2.15), we deduce that $Qp + Sq = Pq + Qs$. Thus, conditions (a), (b), and (c) hold.

Conversely, suppose (a), (b), and (c) hold. Then

$$\begin{aligned} (\phi_1 \circ \phi_2)(z) &= \frac{(Pp + Rq)z + (Qp + Sq)}{(Pr + Rs)z + (Qr + Ss)} = \frac{(Pp + Qr)z + (Pq + Qs)}{(Rp + Sr)z + (Rq + Ss)} \\ &= (\phi_2 \circ \phi_1)(z), \end{aligned}$$

and

$$\begin{aligned} \psi_1(z)\psi_2(\phi_1(z)) &= \frac{\alpha\beta Ss}{(Sr + Rp)z + (Ss + Rq)} = \frac{\alpha\beta Ss}{(Rs + Pr)z + (Ss + Qr)} \\ &= \psi_2(z)\psi_1(\phi_2(z)). \end{aligned}$$

Therefore, by Theorem 2.19, it follows that $W_{\psi_1,\phi_1}W_{\psi_2,\phi_2} = W_{\psi_2,\phi_2}W_{\psi_1,\phi_1}$. This completes the proof. □

The following theorem provides the necessary and sufficient conditions for a composition-differentiation operator D_{ϕ_1} and a weighted composition operator W_{ψ,ϕ_2} to commute on the Hardy space $\mathcal{H}^2(\mathbb{D})$.

Theorem 2.22. *Suppose that the operators $C_{\phi_1}D$ and W_{ψ,ϕ_2} are bounded on $\mathcal{H}^2(\mathbb{D})$. Then $C_{\phi_1}DW_{\psi,\phi_2} = W_{\psi,\phi_2}C_{\phi_1}D$ if and only if*

(a) *ψ is a constant function on $\mathbb{D}$,*
(b) *ϕ_2 is the identity map on $\mathbb{D}$.*

Proof. First suppose that $C_{\phi_1}DW_{\psi,\phi_2} = W_{\psi,\phi_2}C_{\phi_1}D$. Then for all $w, z \in \mathbb{D}$, $C_{\phi_1}DW_{\psi,\phi_2}K_w(z) = W_{\psi,\phi_2}C_{\phi_1}DK_w(z)$. Easy calculation yields the following:

$$\begin{aligned} C_{\phi_1}DW_{\psi,\phi_2}K_w(z) &= C_{\phi_1}D\left(\frac{\psi(z)}{1 - \overline{w}\phi_2(z)}\right) \\ &= \frac{\psi'(\phi_1(z))\left(1 - \overline{w}\phi_2(\phi_1(z))\right) + \overline{w}\psi(\phi_1(z))\phi_2'(\phi_1(z))}{\left(1 - \overline{w}\phi_2(\phi_1(z))\right)^2} \end{aligned}$$

and

$$W_{\psi,\phi_2}C_{\phi_1}DK_w(z) = W_{\psi,\phi_2}\left(\frac{\overline{w}}{(1 - \overline{w}\phi_1(z))^2}\right) = \frac{\psi(z)\,\overline{w}}{(1 - \overline{w}\phi_1(\phi_2(z)))^2}.$$

Equating above two equations, we have

$$\frac{\psi'(\phi_1(z))\left(1 - \overline{w}\phi_2(\phi_1(z))\right) + \overline{w}\psi(\phi_1(z))\phi_2'(\phi_1(z))}{\left(1 - \overline{w}\phi_2(\phi_1(z))\right)^2} = \frac{\psi(z)\,\overline{w}}{(1 - \overline{w}\phi_1(\phi_2(z)))^2}. \tag{2.17}$$

Set $w = 0$ in (2.17) to get $\psi'(\phi_1(z)) = 0$ for all $z \in \mathbb{D}$. Since ϕ_1 is non-constant analytic, the identity theorem implies

$$\psi' \equiv 0 \quad \Longrightarrow \quad \psi \equiv c, \quad \text{for some } c \in \mathbb{C} \setminus \{0\}.$$

Thus, ψ is constant, proving (a). Substitute $\psi \equiv c$ into (2.17), we get

$$\frac{c\,\overline{w}\,\phi_2'(\phi_1(z))}{(1-\overline{w}\phi_2(\phi_1(z)))^2} = \frac{c\,\overline{w}}{(1-\overline{w}\phi_1(\phi_2(z)))^2}.$$

Canceling $c\overline{w} \neq 0$ (for $w \neq 0$) gives

$$\frac{\phi_2'(\phi_1(z))}{(1-\overline{w}\phi_2(\phi_1(z)))^2} = \frac{1}{(1-\overline{w}\phi_1(\phi_2(z)))^2}, \quad \text{for all } w, z \in \mathbb{D}.$$

Setting $w = 0$ yields $\phi_2'(\phi_1(z)) = 1, \quad \forall z \in \mathbb{D}$. By the identity theorem and non-constancy of ϕ_1, it follows $\phi_2'(z) = 1, \quad \forall z \in \mathbb{D}$. Therefore, $\phi_2(z) = z + c_1$, for some constant c_1. Because ϕ_2 is a self-map of $\mathbb{D}$, the only possibility is $c_1 = 0$. Hence, $\phi_2(z) = z$, i.e., ϕ_2 is the identity. This proves (b).

The converse is straightforward and thus omitted. □

Theorem 2.23. *Let $\psi_1, \psi_2 \in \mathcal{H}(\mathbb{D})$ be nonzero analytic functions, i.e., $\psi_1 \not\equiv 0$ and $\psi_2 \not\equiv 0$, with $\psi_2(0) \neq 0$ (or $\psi_1(0) \neq 0$), and let $\phi_1, \phi_2 : \mathbb{D} \to \mathbb{D}$ be non-constant analytic self-maps of the unit disk $\mathbb{D}$ satisfying $\phi_1(0) = \phi_2(0) = 0$. Suppose the weighted composition operators $W_{\psi_1,\phi_1}, \quad W_{\psi_2,\phi_2} : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$ are bounded. Then $W_{\psi_1,\phi_1} W^*_{\psi_2,\phi_2} = W_{\psi_2,\phi_2} W^*_{\psi_1,\phi_1}$, where $W^*_{\psi,\phi} K_\beta = \overline{\psi(\beta)} K_{\phi(\beta)}$ if and only if*

(a) $\psi_1(z) = \alpha \psi_2(z)$;
(b) $\phi_1(z) = \beta \phi_2(z)$,

where α and β are non-zero real numbers.

Proof. Assume that $W_{\psi_1,\phi_1} W^*_{\psi_2,\phi_2} = W_{\psi_2,\phi_2} W^*_{\psi_1,\phi_1}$. Applying both sides to the reproducing kernel K_w and evaluating at $z \in \mathbb{D}$, we get

$$\frac{\overline{\psi_2(w)}\psi_1(z)}{1-\overline{\phi_2(w)}\phi_1(z)} = \frac{\overline{\psi_1(w)}\psi_2(z)}{1-\overline{\phi_1(w)}\phi_2(z)}.$$

Setting $w = 0$ gives

$$\psi_1(z) = \alpha \psi_2(z), \quad \text{where } \alpha = \frac{\overline{\psi_1(0)}}{\overline{\psi_2(0)}} \neq 0.$$

Substituting back and simplifying yields

$$\overline{\phi_2(w)}\phi_1(z) = \overline{\phi_1(w)}\phi_2(z) \quad \text{for all } w, z \in \mathbb{D}.$$

Fixing $w = w_0$ with $\phi_2(w_0) \neq 0$ implies

$$\phi_1(z) = \beta \phi_2(z), \quad \text{where } \beta = \frac{\overline{\phi_1(w_0)}}{\overline{\phi_2(w_0)}} \neq 0.$$

Conversely, if these conditions hold, we easily verify

$$W_{\psi_1,\phi_1} W^*_{\psi_2,\phi_2} K_w(z) = W_{\psi_2,\phi_2} W^*_{\psi_1,\phi_1} K_w(z).$$

By density of kernel functions, the equality of operators follows. □

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