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DETERMINATION OF THE UNKNOWN FUNCTION IN A BURGERS-TYPE PARABOLIC EQUATION

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Abstract. This paper studies the inverse problem of identifying an unknown coefficient on the right-hand side of a Burgers-type parabolic equation. Direct problems for the Burgers equation have been studied by J. Whitham, L. K. Martinson, and others. An additional condition for finding the unknown coefficient, which depends on the spatial variables, is given in integral form. Theorems on the existence, uniqueness, and stability of generalized solutions are proved.

1. Introduction

Inverse problems concerning the determination of unknown terms on the right-hand side of a parabolic equation under an additional integral condition have been studied in [1–10,12] and the references therein. In the works [1,2,3,4,12], the unknown functions depend on the time variable, whereas in [1,2,3,9,10] they depend on the spatial variables. The work [4] is devoted to inverse problems in non-cylindrical domains. In papers [1,9,10,12] issues of solvability (existence and uniqueness of the solution) have been investigated, while in [2,4], the stability of the solutions to the posed problems has also been examined. For approximate solution methods of inverse problems, the questions of stability play a crucial role in particular in [14].

The Burgers equation is a special case of the Navier-Stokes equation in the one-dimensional case and reflects both the effects of nonlinear propagation and the effects of diffusion.

The Burgers equation arises when considering a wide class of processes in hydromechanics, nonlinear acoustics and plasma physics.

The Burgers equation with a source describes the dynamics of a physical system located in an external field and is a natural generalization of the homogeneous equation corresponding to autonomous notions.

In papers [15,16] direct problems for the Burgers equation are considered.

In this paper, we consider an inverse problem of identifying an unknown coefficient on the right-hand side of a Burgers-type parabolic equation. An additional

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condition for finding the unknown coefficient, which depends on the spatial variables, is given in integral form. We prove theorems on the existence, uniqueness, and stability of generalized solutions.

2. Problem statement

Let R^n be a real n- dimensional Euclidean space, $D \subset R^n$ be a bounded domain with a sufficiently smooth boundary $\partial D, x = (x_1, \ldots, x_n)$ be an arbitrary point in the domain D , $Q = D \times (0, T]$, $S = \partial D \times [0, T]$, where $0 < T = const$. Functional spaces $C^l(\cdot), C^{l+\alpha}(\cdot), C^{l,l/2}(\cdot), C^{l+\alpha,(l+\alpha)/2}(\cdot)$, $l = 0, 1, 2;\ \alpha \in (0, 1)$ and the norms in these spaces are defined as in [13, chapter 1]:

$$\|w\|_B^{(l,k)} = \sum_{i=0}^{l} \sup_B \left|D_x^i w\right| + \sum_{j=1}^{k} \sup_B \left|D_t^j w\right|,$$

$$\|\upsilon\|_T^{(k)} = \sum_{j=0}^{k} \sup_{[0,T]} \left|\frac{d^j \upsilon(t)}{dt^j}\right|, \quad w_t = \frac{\partial w}{\partial t}, w_{x_i} = \frac{\partial w}{\partial x_i}, \quad i = \overline{1, n},$$

$$\Delta w = \sum_{i=1}^{n} \frac{\partial^2 w}{\partial x_i^2}, \quad \int_D w(x,t)dx = \int_D \ldots \int w(x,t)dx_1..dx_n,$$

$D_x^l w(x,t)$- all possible derivatives of a function $w(x,t)$ with respect to x_i of order $l, D_t^k w(x,t)$- derivative of function $w(x,t)$ with respect to t of order k.

We consider the following inverse problem in determining a pair of functions $\{f(x), u(x,t)\}$:

$$u_t - \Delta u + \sum_{i=1}^{n} \left(\frac{u^2}{2}\right)_{x_i} = f(x)g(x,t), \quad (x,t) \in Q, \tag{2.1}$$

$$u(x,0) = \varphi(x), \qquad x \in \overline{D} = D \cup \partial D, \tag{2.2}$$

$$u(x,t) = \psi(x,t), \quad (x,t) \in S, \tag{2.3}$$

$$\int_0^T u(x,t)dt = h(x), \quad x \in \overline{D} \tag{2.4}$$

where $g(x,t), \varphi(x), \psi(x,t), h(x)$ are the given functions.

Inverse problems for the Burgers equation in one- and two-dimensional cases (with respect to spatial variables) with a local additional condition are considered in [7,8], and with a non-local integral condition in [5,6].

Problem (2.1)-(2.4) belongs to the class of Hadamard ill-posed problems.

Therefore, this problem should be treated using the general concepts of the theory of ill-posed problems.

We take the following assumptions for the data of problem (2.1)-(2.4).

1^0. $g(x,t) \in C^{\alpha,\alpha/2}(\overline{Q}),\ g^* = \|g\|_Q^{(0)}, \beta_1 T^r \leq \int_0^T |g(x,t)|\, dt \leq \beta_2 T^r$, $g^*, \beta_1, \beta_2, r = const > 0,\ r \in \left(0, \frac{1}{2}\right)$;

2^0. $\varphi(x) \in C^{2+\alpha}(\overline{D}),\ \ \psi(x,t) \in C^{2+\alpha,1+\alpha/2}(S)$;

3^0. The zeroth and first order compatibility conditions are satisfied as follows:

$$\varphi(x)=\psi(x,0), x\in\partial D,\ \left[\psi_t(x,0)-\Delta\varphi(x)+\sum_{i=1}^{n}\left(\frac{\psi^2(x,0)}{2}\right)_{x_i}\right]\int_0^T g(x,t)dt=$$
$$=\left[\psi(x,T)-\varphi(x)-\Delta h(x)+\int_0^T\sum_{i=1}^{n}\left(\frac{\psi^2(x,t)}{2}\right)_{x_i}dt\right]g(x,0),\quad x\in\partial D;$$

4^0. $h(x)\in C^{2+\alpha}(\overline{D})$, $h_*T\le|\Delta h(x)|\le h^*T$, $h_*,h^*=const>0$.

Definition 2.1. The pair of functions $\{f(x),u(x,t)\}$ is called the solution of problem (2.1)-(2.4) if:

1) $f(x)\in C^{\alpha}(\overline{D})$;
2) $u(x,t)\in C^{2+\alpha,1+\alpha/2}(\overline{Q})$;
3) the relations (2.1)-(2.4) hold for these functions.

Problem (2.1)-(2.4) is reduced to an equivalent problem.

Lemma 2.1. *Let the initial data of problems (2.1)-(2.4) satisfy conditions* 1^0-4^0*, and* $\int_0^T\psi(x,t)dt=h(x),\quad x\in\partial D$.

If the pair $\{f(x),u(x,t)\}$ *is a solution to problem (2.1)-(2.4) in the sense of Definition 2.1, then this pair is also a solution to problem (2.1),(2.2),(2.3),*

$$f(x)=\left[u(x,T)-\varphi(x)-\Delta h(x)+\int_0^T\sum_{i=1}^{n}\left(\frac{u^2}{2}\right)_{x_i}dt\right]\Bigg/\int_0^T g(x,t)dt,\quad x\in\overline{D},\tag{2.5}$$

and vice versa, the solution to problem (2.1),(2.2),(2.3),(2.5) in the sense of Definition 2.1 is a solution to problem (2.1)-(2.4).

Proof. Assume $\{f(x),u(x,t)\}$ is a solution to the problem (2.1)-(2.4) in the sense of Definition 2.1. Integrate equation (2.1) over the interval $(0,T)$ with respect to t. Taking into account the conditions of Lemma 2.1, we obtain

$$\int_0^T u_t dt-\int_0^T\Delta u dt+\int_0^T\sum_{i=1}^{n}\left(\frac{u^2}{2}\right)_{x_i}dt=\int_0^T f(x)g(x,t)dt,\tag{2.6}$$

From here we get formula (2.5).

Now assume that the pair $\{f(x),u(x,t)\}$ is a solution to problem (2.1),(2.2), (2.3), (2.5) in the sense of Definition 2.1. We now show that condition (2.4) is satisfied. In equation (2.6) instead of $f(x)$ we substitute its value from (2.5), and denote $\theta(x)=\int_0^T u(x,t)dt-h(x)$

$$\Delta\theta=0,\ x\in D,\quad \theta(x)=0,\ x\in\partial D.$$

the associated Laplace problem has the unique zero solution, that is, $\theta(x)\equiv 0$, $x\in\overline{D}$:

$$\int_0^T u(x,t)dt=h(x).$$

Lemma 2.1 is proved. □

A solution to problem (2.1)-(2.4) does not always exist, and if it does, it may not be unique and unstable. Let us give an example of a violation of the correctness conditions namely, an example of the instability of the solution. Let us assume that $n = 1$, $D = (0, 1)$ in problem (2.1)-(2.4) and the input have the following form:

$g_s(x,t) = e^{-s(x+t)}(1+s) + e^{-2s(x+t)}$, $(x,t) \in D \times (0,T], \varphi_s(x) = e^{-sx}$, $x \in \overline{D}$, $\psi_{0s}(t) = e^{-st}$, $\psi_{1s}(t) = e^{-s(1+t)}$, $t \in [0,T]$,

$h_s(x) = -\frac{e^{-sx}}{s}\left(e^{-sT} - 1\right)$, $x \in \overline{D}$, s is a natural number. It is easy to verify that the functions $\left\{f_s(x) = -s,\ u_s(x,t) = e^{-s(x+t)}\right\}$ are the exact solution to problem (2.1)-(2.4) for the above input data and for any $s = 1, 2, \ldots$. It is clear that by choosing natural numbers m, s $(m \neq s)$, the differences $|g_m(x,t) - g_s(x,t)|$, $|\varphi_m(x) - \varphi_s(x)|$, $|\psi_{0m}(t) - \psi_{0s}(t)|$, $|\psi_{1m}(t) - \psi_{1s}(t)|$, $|h_m(x) - h_s(x)|$, that is, the perturbations of the input data can be made arbitrarily small in the sup-norm, however $|f_m(x) - f_s(x)| = |m - s|$, which shows the solution to problem (2.1)-(2.4) is unstable.

3. Existence of a generalized solution

Let $f(x) \in C^{\alpha}(\overline{D})$ be a given function and satisfy conditions $1^0 - 4^0$. Then there exists a solution $u(x,t) \in C^{2+\alpha, 1+\alpha/2}(\overline{Q})$ of the problem (2.1)-(2.3) that can be represented as [13, p 468]

$$u(x,t) = \Phi(x,t) + \int_0^t \int_D G(x,t;\xi,\tau) \left[f(\xi) g(\xi,\tau) - \sum_{i=1}^{n} \left(\frac{u^2}{2}\right)_{\xi_i} \right] d\xi d\tau, \qquad (3.1)$$

where $d\xi = d\xi_1 \ldots d\xi_n$,

$$\Phi(x,t) = F(x,t) + \int_0^t \int_D G(x,t;\xi,\tau) \left[\Delta F(\xi,\tau) - F_\tau(\xi,\tau)\right] d\xi d\tau, \qquad (3.2)$$

$F(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$, $F(x,0) = \varphi(x)$, $x \in \overline{D}$, $F(x,t) = \psi(x,t)$, $(x,t) \in S$, $G(x,t;\xi,\tau)$ is the Green function of the problem

$$u_t - \Delta u = f(x)g(x,t) - \sum_{i=1}^{n} \left(\frac{u^2}{2}\right)_{x_i} + \Delta F(x,t) - F_t(x,t), \quad (x,t) \in Q,$$

$$u(x,0) = 0, \quad x \in \overline{D}, \ u(x,t) = 0, \quad (x,t) \in S.$$

The estimates are true for the Green function [13, p.469]

$$\int_D \left| D_x^l G(x,t;\xi,\tau) \right| d\xi \leq m_1 (t-\tau)^{-l/2}, \ l = 0, 1, \qquad (3.3)$$

where $m_1 = const > 0$ depends only on the input data of problem (2.1)-(2.4). In what follows, constants depending only on the input data will be denoted by m_i, $i = 2, 3, \ldots$.

Definition 3.1. We will call the functions $\{f(x), u(x,t)\}$ a generalized solution to problem (2.5),(3.1) if:

1) $f(x) \in C(\overline{D})$;
2) $u(x,t) \in C^{1,0}(\overline{Q})$;
3) the relations (2.5),(3.1) are satisfied for these functions.

Theorem 3.1. *Let the unit data of problem (2.5),(3.1) satisfy conditions* 1^0-4^0*, respectively. Then, for sufficiently small* T^* *there exists a unique solution to problem (2.5),(3.1) in* $(x,t) \in \overline{Q_*} = \overline{D} \times [0, T^*]$.

Proof. Let us denote $B = C^{1,0}(\overline{Q})$ and $E = C(\overline{D})$.

$B^{'} = \{u|\ u(x,t) \in B,\ \left|D_x^l u(x,t)\right| \le u_0, l = 0,1, u_0 = const > 0,\ (x,t) \in \overline{Q}\}$ and $E^{'} = \left\{f|f(x) \in E,\ |f(x)| \le f_0, f_0 = const > 0,\ x \in \overline{D}\right\}$ are complete closed subspaces of B and E, respectively.

Using the method of compressed mapping, we prove that for each $f(x) \in E^{'}$ the problem of determining $u(x,t)$ from equation (3.1) has unique solution.

We write equation (3.1) in operator form

$$M_1[u] = u,\ M_1 : B \to B$$

$$M_1[u] = \Phi(x,t) + \int_0^t \int_D G(x,t,\xi,\tau) f(\xi) g(\xi,\tau) d\xi d\tau -$$
$$- \int_0^t \int_D G(x,t,\xi,\tau) \sum_{i=1}^n \left(\frac{u^2}{2}\right)_{\xi_i} d\xi d\tau.$$

We will show that for sufficiently small $T_1 (0 < T_1 \le T)$ and $f \in E^{'}$, one can choose a u_0 such that if $u(x,t) \in B^{'}$, then $M[B^{'}] \subset B^{'}$.

Considering (3.2),(3.3) and conditions $1^0 - 4^0$ for the input data, the definitions of $M_1[u]$, we obtain

$$|M_1[u]| \le |\Phi(x,t)| + \int_0^t \int_D G(x,t,\xi,\tau) |f(\xi)| |g(\xi,\tau)| d\xi d\tau +$$
$$+ \int_0^t \int_D G(x,t,\xi,\tau) \sum_{i=1}^n \left| \left(\frac{u^2}{2}\right)_{\xi_i} \right| d\xi d\tau,$$

$$|\Phi(x,t)| \le |F(x,t)| + \int_0^t \int_D G(x,t,\xi,\tau) \left[|\Delta F| + |F_i|\right] d\xi d\tau \le m_2, \quad (x,t) \in \overline{Q},$$

$$\int_0^t \int_D G(x,t,\xi,\tau) |f(\xi)| |g(\xi,\tau)| d\xi d\tau \le m_1 g^* f_0 T, \quad (x,t) \in \overline{Q},$$

$$\int_0^t \int_D G(x,t,\xi,\tau) \sum_{i=1}^n \left| \left(\frac{u^2}{2}\right)_{\xi_i} \right| d\xi d\tau \le m_1 n u_0^2 T, \quad (x,t) \in \overline{Q}.$$

Hence,

$$|M_1[u]| \leq m_3 + m_4 u_0^2 T.$$

Let $T_1 (0 < T_1 \leq T)$ be a number such that $4m_5\ T < 1 (m_5 = m_3 m_4)$. Then, for any u_0 in the interval

$$\left(a_1 \equiv \frac{1 - \sqrt{1 - 4m_5 T}}{2m_3 T}, \frac{1 + \sqrt{1 - 4m_5 T}}{2m_3 T} \equiv b_1\right)$$

$|M_1[u]| \leq u_0$.

Using the above method, we obtain a similar estimate for $(M_1[u]_{x_k}), k = \overline{1, n}$,

$$|M_1[u]_{x_k}| \leq m_6 + m_7 T^{1/2} u_0^2, \quad k = \overline{1, n}, \quad (x, t) \in \overline{Q}$$

Let $T_2 (0 < T_2 \leq T)$ be a number such that $4m_7 m_6 T^{1/2} < 1$. Then, for any u_0 in the interval

$$\left(a_2 \equiv \frac{1 - \sqrt{1 - 4m_6 m_7 T^{1/2}}}{2m_7 T^{1/2}}, \frac{1 + \sqrt{1 - 4m_6 m_7 T^{1/2}}}{2m_7 T^{1/2}} \equiv b_2\right)$$

$|M_1[u]|_{x_k} \leq u_0, k = \overline{1, n}$.

Thus, if $u_0 \in (a^*, b^*), a^* = \max(a_1, a_2), b^* = \min(b_1, b_n)$ then for $(x, t) \in \overline{Q}_3 = = \overline{D} \times [0, T_3], T_3 = \min(T_1, T_2)$ it follows $M_1[B'] \subset B'$.

We will show that for any $u_1(x, t), u_2(x, t) \in B'$ the following inequality is true:

$$\|M_1[u_1] - M_1[u_2]\|_Q^{(1,0)} \leq m_8 \|u_1 - u_2\|_Q^{(1,0)}, \ m_8 < 1.$$

By definition of $M_1[u]$ we have

$$|M_1[u_1] - M_1[u_2]| \leq \int_0^t \int_D G(x, t, \xi, \tau) \sum_{i=1}^n \left| \left(\frac{u_1^2}{2}\right)_{\xi_i} - \left(\frac{u_2^2}{2}\right)_{\xi_i} \right| d\xi d\tau \leq$$

$$\leq \int_0^t \int_D G(x, t, \xi, \tau) \sum_{i=1}^n \left(|u_1 - u_2| \, |u_{1\xi_i}| + |u_{1\xi_i} - u_{2\xi_i}| \, |u_2|\right) d\xi d\tau.$$

Considering the estimates (3.3) the definition of the set B', from the last inequality we have

$$|M_1[u_1] - M_1[u_2]| \leq \frac{2}{2 + \alpha} m_1 n u_0 T \|u_1 - u_2\|_Q^{(1,0)}, \quad (x, t) \in \overline{Q}.$$

From this we can write

$$\max_Q |M_1[u_1] - M_1[u_2]| \leq m_9 T \|u_1 - u_2\|_Q^{(1,0)}. \tag{3.4}$$

In the same way, we obtain

$$\left|(M_1[u_1] - M_1[u_2])_{x_k}\right| \leq$$

$$\leq \int_0^t \int_D |G_{x_k}(x, t, \xi, \tau)| \sum_{i=1}^n \left| \left(\frac{u_1^2}{2}\right)_{\xi_i} - \left(\frac{u_2^2}{2}\right)_{\xi_i} \right| d\xi d\tau, \quad k = \overline{1, n},$$

or

$$\max_Q \left|(M_1[u_1] - M_1[u_2])_{x_k}\right| \leq m_{10} T^{1/2} \|u_1 - u_2\|_Q^{(1,0)}, \quad k = \overline{1, n}. \tag{3.5}$$

Combining inequalities (3.4) and (3.5), we obtain

$$\|M_1[u_1] - M_1[u_2]\|_Q^{(1,0)} \le m_{11} T \|u_1 - u_2\|_Q^{(1,0)}. \tag{3.6}$$

Let $T_4 (0 < T_4 \le T)$ be a number such that $m_{11} T_4 < 1$.

Then from (3.6), it follows that $M_1[u]$ is a contraction operator under $(x,t) \in Q_4 = \overline{D} \times [0, T_4]$ [11, p.69].This means that for each given $f(x) \in E'$, the problem of determining $u(x,t)$ from (3.1) has a unique solution from $C^{(1,0)}(\overline{Q}_3)$.

We will now prove the existence of the function $f(x) \in C(\overline{D})$. We write equation (2.5) in operator form

$$M_2[f] = f, \quad M_2 : L \to L.$$

$$M_2[f] = \left[u(x,T) - \varphi(x) - \Delta h(x) + \int_0^T \sum_{i=1}^n \left(\frac{u^2}{2} \right)_{x_i} dt \right] \Bigg/ \int_0^T g(x,t) dt. \tag{3.7}$$

We will show that for a sufficiently small $T_5 (0 < T_5 \le T)$, one can choose such that if $f(x) \in E'$, then $M_2[E'] \subset E'$.

Considering (3.1),(3.3),(3.7) and conditions 1^0-4^0 for the input data, the definitions of $M_2[f]$, we obtain

$$|M_2[f]| \le \left[|u(x,T) - \varphi(x)| + |\Delta h(x)| + \int_0^T \sum_{i=1}^n \left| \left(\frac{u^2}{2} \right)_{x_i} \right| dt \right] \Bigg/ \int_0^T |g(x,t)| \, dt$$

$$|u(x,T) - \varphi(x)| \le \int_0^T \int_D G(x,t,\xi,\tau) \, |f(\xi)| \, |g(\xi,\tau)| \, d\xi d\tau +$$

$$+ \int_0^T \int_D G(x,t,\xi,\tau) \sum_{i=1}^n \left| \left(\frac{u_1^2}{2} \right)_{\xi_i} \right| d\xi d\tau \le m_1 f_0 g^* T + m_1 n u_0^2 T,$$

$$\int_0^T \sum_{i=1}^n \left| \left(\frac{u^2}{2} \right)_{x_i} \right| dt \le 2 n u_0^2 T$$

or

$$|M_2[f]| \le \left[m_1 f_0 g^* + m_1 n u_0^2 + h^* + 2 n u_0^2 \right] T^{1-r} / \beta_1.$$

Let T_5 $(0 < T_5 \le T)$ be a number such that $\beta_1 - m_1 g^* T^{1-r} > 0$.

Then, for any $f_0 \ge \left[m_1 n u_0^2 + h^* + 2 n u_0^2 \right] T^{1-r} / (\beta_1 - m_1 g^* T^{1-r})$. $|M_2[f]| \le f_0$. Thus, $M_2[E'] \subset E'$.

For any $f_1(x), f_2(x) \in E'$, we have

$$|M_2[f_1] - M_2[f_2]| \le [|u_1(x,T) - u_2(x,T)| +$$

$$+ \int_0^T \sum_{i=1}^n \left(|u_1 - u_2| \, |u_{1x_i}| + |u_{1x_i} - u_{2x_i}| \, |u_2| \right) dt] / \int_0^T |g(x,t)| dt. \tag{3.8}$$

For $|u_1(x,T) - u_2(x,T)|$ from equation (3.1), we obtain

$$|u_1(x,T) - u_2(x,T)| \leq \int_0^T \int_D G(x,t,\xi,\tau)\, |f_1(\xi) - f_2(\xi)|\, |g(\xi,\tau)|\, d\xi d\tau +$$

$$+ \int_0^T \int_D G(x,t,\xi,\tau) \sum_{i=1}^{n} \left(|u_1 - u_2|\, |u_{1x_i}| + |u_{1x_i} - u_{2x_i}|\, |u_2| \right) d\xi d\tau.$$

From the estimate (3.5) and the definition of the set B' we have

$$|u_1(x,T) - u_2(x,T)| \leq$$

$$\leq m_{12} T \, \|f_1 - f_2\|_D^{(0)} + m_{13} T \, \|u_1 - u_2\|_Q^{(1,0)}, \quad x \in \overline{D}. \tag{3.9}$$

We can also obtain an estimate for $|u_{1x_i} - u_{2x_i}|, k = \overline{1,n}$.

$$|u_{1x_i} - u_{2x_i}| \leq$$

$$\leq m_{14} T^{1/2} \, \|f_1 - f_2\|_D^{(0)} + m_{15} T^{1/2} \, \|u_1 - u_2\|_Q^{(1,0)}, \ x \in \overline{D} \tag{3.10}$$

Combining inequalities (3.9) and (3.10), we get

$$\|u_1 - u_2\|_Q^{(1,0)} \leq m_{16} T^{1/2} \, \|f_1 - f_2\|_D^{(0)} + m_{17} T^{1/2} \, \|u_1 - u_2\|_{Q_1}^{(1,0)}.$$

Let $T_6 (0 < T_6 \leq T)$ be a number such that $m_{17} T_6^{1/2} < 1$.

Then we have

$$\|u_1 - u_2\|_{Q_5}^{(1,0)} \leq m_{18} T^{1/2} \, \|f_1 - f_2\|_D^{(0)}, \tag{3.11}$$

Using inequality (3.11) in the right-hand side of (3.8), we obtain

$$|M_1[f_1] - M_1[f_2]| \leq m_{19} T^{1/2} \, \|f_1 - f_2\|_D^{(0)} / \beta_1 T^r, \quad x \in \overline{D},$$

or

$$\|M_1[f_1] - M_1[f_2]\|_D^{(0)} \leq m_{20} T^{(1-2r)/2} \, \|f_1 - f_2\|_D^{(0)}$$

Let $0 < T_6 \leq T$ be a number such that $m_{20} T_6^{(1-2r)/2} < 1$. Thus $M_2[f]$ is a contraction operator under $(x,t) \in \overline{D} \times [0, T_6]$.

Thus, under condition $(x,t) \in Q_* = \overline{D} \times [0, T^*], T^* = \min(T_3, T_6)$, problem (2.5), (3.1) concerning the determination of $\{f(x), u(x,t)\}$ has a unique solution from $C(\overline{D}) \times C^{1,0}(\overline{Q}_*)$.

The theorem is proved. □

4. Stability of the generalized solution

The estimation of the stability of solutions to inverse problems plays a central role in the study of their well-posedness.

Define the following set:

$$K = \left\{ (f,u) | f(x) \in C(\overline{D}), \ u(x,t) \in C^{1,0}(\overline{Q}), \quad |f(x)|, \right.$$

$$\left. |u(x,t)|, \quad |u_{x_i}(x,t)| \leq c_1, \ i = \overline{1,n}, \ (x,t) \in \overline{Q}, \quad 0 < c_1 = const \right\}.$$

Assume that we are given the two input sets

$$\{g_i(x,t), \varphi_i(x), \psi_i(x,t), h_i(x), \ i = 1,2\}.$$

We denote problem A with data $\{g_i(x,t), \varphi_i(x), \psi_i(x,t), h_i(x),\}$. By $A_i, i = 1,2$. Let $\{f_1(x), u_1(x,t)\}$ and $\{f_2(x), u_2(x,t)\}$ be solutions of problems A_1 and A_2 respectively.

Theorem 4.1. *Let the following conditions hold:*

1) the function $\{g_i(x,t), \varphi_i(x), \psi_i(x,t), h_i(x),\ i = 1,2\}$ satisfy conditions $1^0 - 4^0$, respectively;

2) solutions of problems A_1 and A_2 exist in the sense of Definition 3.1 and they belong to the set K.

Then there exists a $0 < \tilde{T} \le T$, such that for $(x,t) \in \overline{D} \times [0,\tilde{T}] = \overline{Q}_{\sim}$, the solution of problem A satisfies the stability estimate

$$\|u_1 - u_2\|_Q^{(1,0)} + \|f_1 - f_2\|_D^{(0)} \le$$
$$\le c_2 \left[\|g_1 - g_2\|_Q^{(0)} + \|\varphi_1 - \varphi_2\|_D^{(2)} + \|\psi_1 - \psi_2\|_S^{(1,0)} + \|h_1 - h_2\|_D^{(2)} \right], \qquad (4.1)$$

where $c_2 > 0$ depends on data of problem A_1,A_2, and the set K.

Proof. In what follows constants that depend on the data of problems A_1,A_2, and the set K, will be denoted by $c_i,\ i = 1,2,\dots$

Denote

$$z(x,t) = u_1(x,t) - u_2(x,t),\quad \lambda(x) = f_1(x) - f_2(x),\ \delta_1(x,t) = g_1(x,t) - g_2(x,t)$$

$$\begin{aligned} \delta_2(x) &= \varphi_1(x) - \varphi_2(x),\ \delta_3(x,t) = \psi_1(x,t) - \psi_2(x,t), \\ \delta_4(x) &= h_1(x) - h_2(x),\ \delta_5(x,t) = F_1(x,t) - F_2(x,t). \end{aligned}$$

Subtracting the relations of A_1 from the corresponding relations of A_2, we obtain the problem of determining a pair $\{\lambda(x), z(x,t)\}$

$$z(x,t) = \delta_5(x,t) + \int_0^t \int_D G(x,t;\xi,\tau) \{\lambda(\xi) g_1(\xi,\tau) - \delta_1(\xi,\tau) f_2(\xi) -$$
$$-\frac{1}{2}\sum_{i=1}^n [z(\xi,\tau)u_1(\xi,\tau)_{\xi_i} + z_{\xi_i}(\xi,\tau)u_2(\xi,\tau)] + \Delta\delta_5(\xi,\tau) - \delta_{5\tau}(\xi,\tau)\} d\xi d\tau, \quad (4.2)$$

$$\lambda(x) = \{z(x,T) - \delta_2(x) - \Delta\delta_4(x) + \frac{1}{2}\int_0^T \sum_{j=1}^n \left[z(x,t)u_1(x,t)_{x_j} + \right.$$
$$\left. + z_{x_j}(x,t)u_2(x,t) \right] dt\} \Big/ \int_0^T g_1(x,t)dt -$$
$$-\left\{ \left[u_2(x,T) - \delta_2(x) - \Delta h_2(x) + \frac{1}{2}\int_0^T \sum_{j=1}^n \left(u_2^2(x,t)_{x_j} \right) dt \right] \int_0^T \delta_1(x,t)dt \right\} \Big/$$
$$\Big/ \left[\int_0^T g_1(x,t)dt \cdot \int_0^T g_2(x,t)dt \right]. \qquad (4.3)$$

We will estimate the function $z(x,t)$ from (4.2)

$$|z(x,t)| \le |\delta_5(x,t)| + \int_0^t \int_D G(x,t;\xi,\tau) \{ |\lambda(\xi)| \, |g_1(\xi,\tau)| + |\delta_1(\xi,\tau)| \, |f_2(\xi)| + $$
$$+ \frac{1}{2} \sum_{j=1}^{n} \left[|z(\xi,\tau)| \left| u_1(\xi,\tau)_{\xi_j} \right| + \right.$$
$$\left. + \left| z_{\xi_j}(\xi,\tau) \right| |u_2(\xi,\tau)| \right] + |\Delta\delta_5(\xi,\tau)| + |\delta_{5\tau}(\xi,\tau)| \} \, d\xi d\tau.$$

Assume

$$\theta = \|z\|_Q^{(0)} + \sum_{j=1}^{n} \left\| z_{x_j} \right\|_Q^{(0)} + \|\lambda\|_D^{(0)}.$$

Under the conditions of Theorem 4.1 and the definition of the set K, and taking into account estimate (3.3) we will obtain

$$|z(x,t)| \le c_3 \left[\|\delta_5\|_D^{(2.,)} + \|\delta_1\|_Q^{(2,1)} \right] + c_4 \theta t, \quad (x,t) \in \overline{Q}, \tag{4.4}$$

Under the conditions of Theorem 4.1 from (4.3), we can write.

$$z_{x_i}(x,t) = \delta_{5x_i}(x,t) + \int_0^t \int_D G_{x_i}(x,t,\xi,\tau) \{ \lambda(\xi) g_1(\xi,\tau) - \delta_1(\xi,\tau) f_2(\xi) - $$
$$- \frac{1}{2} \sum_{j=1}^{n} \left[z(\xi,\tau) u_2(\xi,\tau)_{\xi_j} + z_{\xi_j}(\xi,\tau) u_2(\xi,\tau) \right] + $$
$$+ \Delta\delta_5(\xi,\tau) - \delta_5 \mathcal{T}(\xi,\tau) \} \, d\xi d\tau, \quad i = \overline{1,n} \tag{4.5}$$

In the same way, from (4.5) for $z_{x_i}(x,t), i = \overline{1,m}$ we have

$$|z_{x_i}(x,t)| \le c_5 \left[\|\delta_5\|_Q^{(2,1)} + \|\delta_1\|_Q^{(0)} \right] + c_6 \theta t^{1/2}, \quad i = \overline{1,n}, \quad (x,t) \in \overline{Q}, \tag{4.6}$$

For $\lambda(x)$ from (4.3), we get

$$|\lambda(x)| \le \left[|z(x,T) - \delta_2(x)| + |\Delta\delta_4(x)| + \frac{1}{2} \int_0^T \sum_{j=1}^{n} \left(|z(x,t)| \left| u_1(x,t)_{x_j} \right| + \right. \right.$$
$$\left. \left. + \left| z_{x_j}(x,t) \right| |u_2(x,t)| \right) dt \right] / \int_0^T |g_1(x,t)| \, dt + $$
$$+ \left\{ \left[|u_2(x,T) - \varphi_2(x)| + |\Delta h_2(x)| + \frac{1}{2} \int_0^T \sum_{j=1}^{n} \left| (u_2^2(x,t)_{x_j} \right| dt \right] \int_0^T |\delta_1(x,t)| \, dt \right\} / $$
$$/ \left[\int_0^T |g_1(x,t)| \, dt \int_0^T |g_2(x,t)| \, dt \right], \quad x \in \overline{D}$$

or

$$|\lambda(x)| \le c_7 \left[\|\delta_5\|_D^{(2,1)} T + c_8 \|\delta_1\|_Q^{(0)} T + c_9 \|\delta_4\|_D^{(2)} T + c_{10} \theta T^{(3-2r)/2} \right] / \beta_1 T^r + $$

$$+\left\{[c_{11}T + c_{12}T + c_1 nT]\,\|\delta_1\|_Q^{(0)}\,T^{1+r}\right\}/\,(\beta_1 T^r)^2$$

$$|\lambda(x)| \le c_{13}\left[\|\delta_1\|_Q^{(0)} + \|\delta_4\|_D^{(2)} + \|\delta_5\|_Q^{(2,1)}\right] + c_{14}\theta T^{(3-4r)/2},\quad x\in\overline{D}, \tag{4.7}$$

Inequalities (4.4),(4.6),(4.7) are satisfied at one values $(x,t)\in\overline{Q}$. Therefore they must be satisfied also for maximum values of the left parts

$$\|z\|_Q^{(0)} \le c_3\left[\|\delta_1\|_Q^{(0)} + \|\delta_5\|_Q^{(2,1)}\right] + c_4\theta T,\quad (x,t)\in\overline{Q},$$

$$\|z_{x_i}\|_Q^{(0)} \le c_5\left[\|\delta_1\|_D^{(0)} + \|\delta_5\|_Q^{(2,1)}\right] + c_6\theta T^{1/2},\quad (x,t)\in\overline{Q}$$

$$\|\lambda\|_D^{(0)} \le c_{13}\left[\|\delta_1\|_Q^{(0)} + \|\delta_4\|_D^{(2)} + \|\delta_5\|_T^{(2,1)}\right] + c_{14}\theta T^{(3-4r)/2},\quad x\in\overline{D}.$$

Combining the last three inequalities, we get

$$\theta \le c_{15}\left[\|\delta_1\|_Q^{(0)} + \|\delta_4\|_D^{(2)} + \|\delta_5\|_Q^{(2,1)}\right] + c_{16}\theta T^{1/2},\quad (x,t)\in\overline{Q}. \tag{4.8}$$

Let $\tilde{T}(0<\tilde{T}\le T)$ be such a number, that $c_{16}\tilde{T}^{1/2}<1$. Then from (4.8) we have

$$\theta \le c_{17}\left[\|\delta_1\|_Q^{(0)} + \|\delta_4\|_D^{(2)} + \|\delta_5\|_Q^{(2,1)}\right].$$

So, we get that the estimate (4.1) is correct.

It should be noted that if a solution to problem (2.5),(3.1) exists, in the sense of Definition 3.1, then it is unique on the set K. This follows under condition $g_1(x,t)=g_2(x,t)$, $\varphi_1(x)=\varphi_2(x)$, $\psi_1(x,t)=\psi_2(x,t)$, $h_1(x)=h_2(x)$ from inequality (4.1).

Theorem 4.1 is proved. □

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