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KHOVANOV HOMOLOGY, UNCOUPLING OPERATION AND SOME CLASSIFICATION OF ORIENTED GRAPH-LINKS

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Abstract. In [13], the second author of the current paper constructed Khovanov homology of graph-knots (graph-links with one component) in the case of the ground field of characteristic two. To generalize this construction for the case of graph-links with many components one has to define additional structure (orientability) on graph-links. This was done by the first named author in [11] and [9]. In the present paper we summarize all the results together, define Khovanov homology for oriented graph-links and construct a new invariant. In the end of the paper we calculate invariants for graph-links with less than 5 vertices and get the full classification of such graph-links.

1. Introduction

The aim of the present paper is to define Khovanov homology for graph-links with many components. Recall that graph-links [6, 7] are a generalization of virtual links considered up to mutation (a piece of a virtual link diagram inside a box is cut, then it is turned by a half-twist and returned to the initial position): we consider graphs instead of chord diagrams of virtual diagrams and moves on graphs originated from the Reidemeister moves rewritten in the language of intersection graphs of chord graphs.

More precisely, a diagram-representative of a graph-link is a simple unoriented labelled graph, and graph-links themselves are equivalence classes of such graphs modulo some moves. If we want to associate a graph-link with a (connected) virtual link diagram, we have to consider any rotating circuit (an Euler tour rotating at any classical crossing) on the virtual diagram, the corresponding chord diagram with some labels, see below, and its intersection graph. It is known that the intersection graph determines a chord diagram up to mutations [3, 4]. Therefore, any link invariant which is preserved under mutations may be considered as a candidate for the role of an invariant of graph-links. It may turn out that some link invariants cannot be extended to graph-links, as some links that give the same graph-link may be non-transformable into each other by Reidemeister moves and mutations. It is not known whether such links exist. Therefore, it is a non-trivial problem to extend a given link invariant which is preserved under

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mutations to an invariant of graph-links. In the case of the Jones polynomial this problem was solved in [6, 7, 18].

Khovanov's discovery of a homological knot invariant in the 1990s [12] led to a number of new invariants, known as the categorifications of polynomial knot invariants. It is worth to note that in the case of classical knots Khovanov homology detects the unknot but in the case of virtual knots it does not do since Khovanov homology does not feel Z-move (or, another name, virtualization). Later odd Khovanov homology was defined in the work in [15] and was shown that it is preserved under mutations [1], so one can expect that this invariant can be extended to graph-links. Let us note that in the case of a field of characteristic 2 the odd Khovanov homology coincides with Khovanov homology and the integral Khovanov homology of links with many components is not preserved under mutations [1] (in the case of knots the problem is still unsolved). The construction of odd Khovanov homology modulo 2 used in [1] is transferred to the case of graph-knots. As a result Khovanov homology of graph-knots was defined in [13]. In the case of links Khovanov homology is defined for oriented links. Therefore, for the case of graph-links with many components we have to define any additional structure on them, which is analogous to the orientation in the case of links. It was done in [9, 11].

The paper is organized as follows. In the next section we give the main definitions concerning oriented graph-links. In Section 3 we define Khovanov homology for oriented graph-links by using the construction from [13]. Section 4 is dedicated to a new operation for graph-links with many components. By using the last operation and the other invariants in Section 5 we classify all graph-links with representative-diagrams on $\leqslant 4$ vertices.

2. Graph-links and oriented graph-links

2.1. Graph-links. Let G be a simple finite graph (a graph without loops and multiple edges), and let $V = V(G)$ be its vertex set.

Definition 2.1. A graph is called *labeled* if every chord is endowed with a label from the cartesian product $\{0, 1\} \times \{+, -\}$, where the first argument is the *framing* of the vertex, and the second one is the *sign* of the vertex. We denote $\mathrm{fr}(v)$ and $\mathrm{sgn}(v)$ the framing and the sign of the vertex v, respectively. All labeled graphs are considered up to label-preserving isomorphism.

We define the moves on labeled graphs below. This definition is essentially a reformulation of the definition of the usual Reidemeister moves on link diagrams: we look at how the intersection graphs of chord diagrams of rotating circuits are transformed when the moves are applied to the original link diagrams. The resulting graph surgeries turn out to be local, which allows us to apply them to arbitrary graphs, including those that do not correspond to any link diagrams, since there are simple graphs which are not intersection graphs of chord diagrams, see [2]. A simple graph H is called *realizable* if there is a chord diagram D such that H is the intersection graph of D. Otherwise, a graph is called *non-realizable.*

Before describing moves on labeled graphs we define two operations on simple unlabeled graphs.

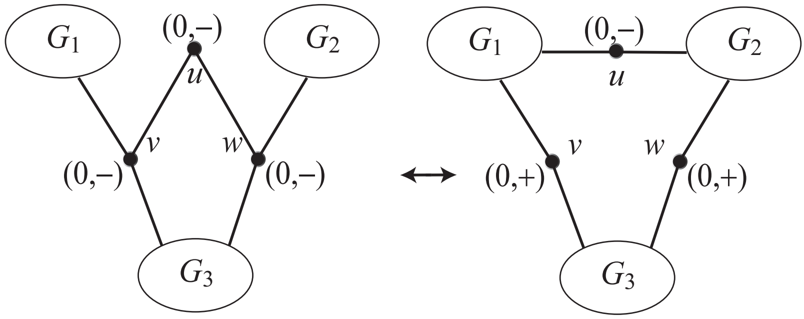

FIGURE 1. The third graph-move

Definition 2.2 (Local complementation)**.** Let G be a graph. The *local complementation* of G at $v \in V(G)$ is the operation which toggles adjacencies between $u, w \in N(v)$, $u \neq w$, and does not change the rest of G. Denote the graph obtained from G by the local complementation at a vertex v by $\mathrm{LC}(G; v)$.

Definition 2.3 (Pivot operation)**.** Let G be a graph with distinct adjacent vertices u and v. The *pivoting operation* of G at u and v or at the edge uv is the composition of three local complementations: the first one at u, then at v and at last at u. Denote the graph obtained from G by the pivoting operation at the vertices u and v by $\mathrm{piv}(G; u, v) = \mathrm{LC}(\mathrm{LC}(\mathrm{LC}(G; u); v); u)$.

Now we are ready to define *graph-moves*, i.e. moves on labeled graphs.

Definition 2.4. $\Omega_g 1$. *The first graph-move* is an addition/removal of an isolated vertex with the framing 0.

$\Omega_g 2$. *The second graph-move* is an addition/removal of two non-adjacent (respectively, adjacent) vertices labeled $(0, +)$ and $(0, -)$ (respectively, $(1, +)$ and $(1, -)$) and having the same adjacencies with the other vertices.

$\Omega_g 3$. *The third graph-move* is defined as follows. Let u, v, w be three vertices of G all having the label $(0, -)$ so that u is adjacent only to v and w in G, and v and w are not adjacent to each other. Then we only change the adjacencies of u with the vertices v, w and $t \in \big(N(v) \setminus N(w)\big) \cup \big(N(w) \setminus N(v)\big)$ (for the other pairs of vertices we do not change their adjacencies). In addition, we switch $\mathrm{sgn}(v)$ and $\mathrm{sgn}(w)$, see Fig. 1 (we depict only a part of G where the changes occur). The inverse operation is also called the third Reidemeister graph-move.

$\Omega_g 4$. *The fourth graph-move* for G is defined as follows. We take two adjacent vertices u and v with $\mathrm{fr}(u) = \mathrm{fr}(v) = 0$, respectively. Replace G with $\mathrm{piv}(G; u, v)$ and switch $\mathrm{sgn}(u)$ and $\mathrm{sgn}(v)$.

$\Omega_g 4'$. In this *fourth graph-move* we take a vertex v with $\mathrm{fr}(v) = 1$. Replace G with $\mathrm{LC}(G; v)$ and switch $\mathrm{sgn}(v)$ and $\mathrm{fr}(u)$ for each $u \in N(v)$.

Remark 2.1. The third graph-move $\Omega_g 3$ does not exhaust all the possibilities for representing the third Reidemeister move on labels graphs. It can be shown that all the other versions of the third Reidemeister move, see Figs. 2 and 3 for some possibilities, are combinations of the second, third and fourth graph-moves, see [14] for details.

The fourth graph-moves $\Omega_g 4$ and $\Omega_g 4'$ in the realizable case correspond to a rotating circuit change on a virtual diagram.

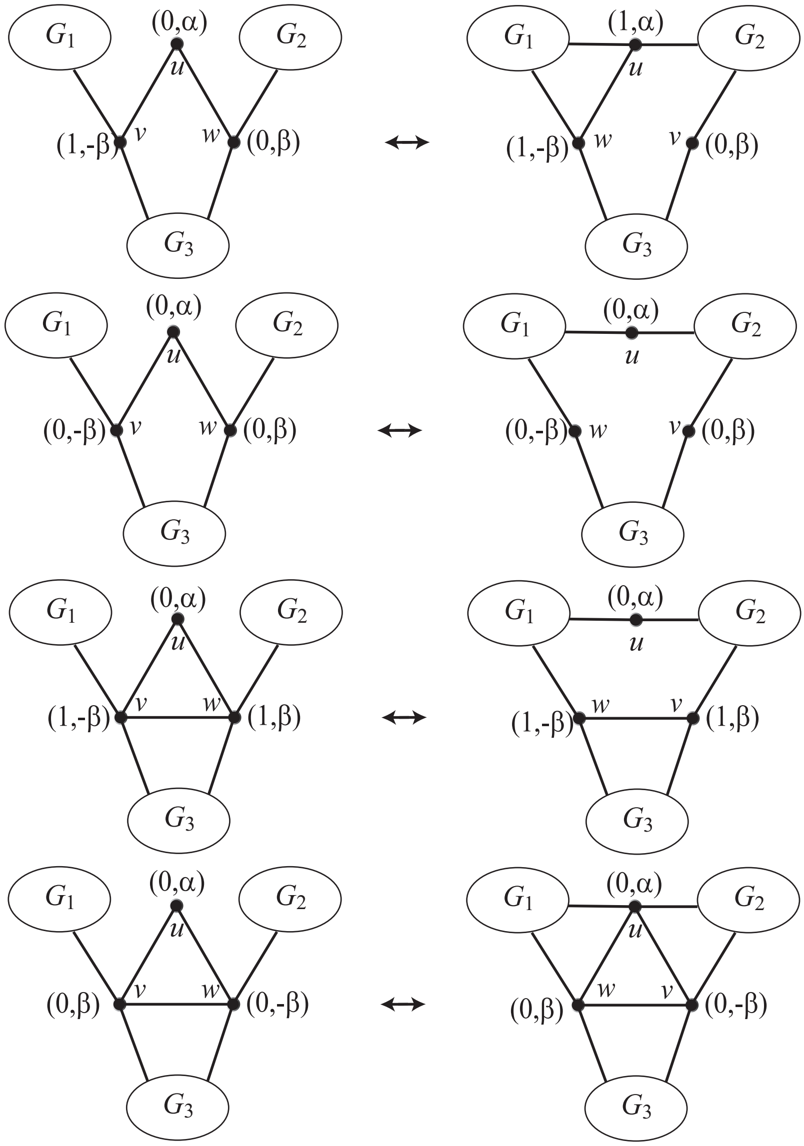

FIGURE 2. The other versions of third graph-move

Definition 2.5. A *graph-link* is an equivalence class of labeled simple graphs modulo $\Omega_g1 - \Omega_g4'$ graph-moves. Any graph from a graph-link $\{G\}$ is called a *representative* for $\{G\}$.

We call an equivalence class of labeled simple graphs modulo Ω_g4 and Ω_g4' moves a *diagram* of the corresponding graph-link. In this case we denote $\{G\}_4$ the equivalence class of G.

Remark 2.2. The free loop is also considered as a labeled graph. The graph-link generated by the free loop is called *trivial*. When we apply a first or second graph-move to the free loop, we delete the free loop and get the graph with one or two vertices. Conversely, applying the decreasing first (second) graph-move to

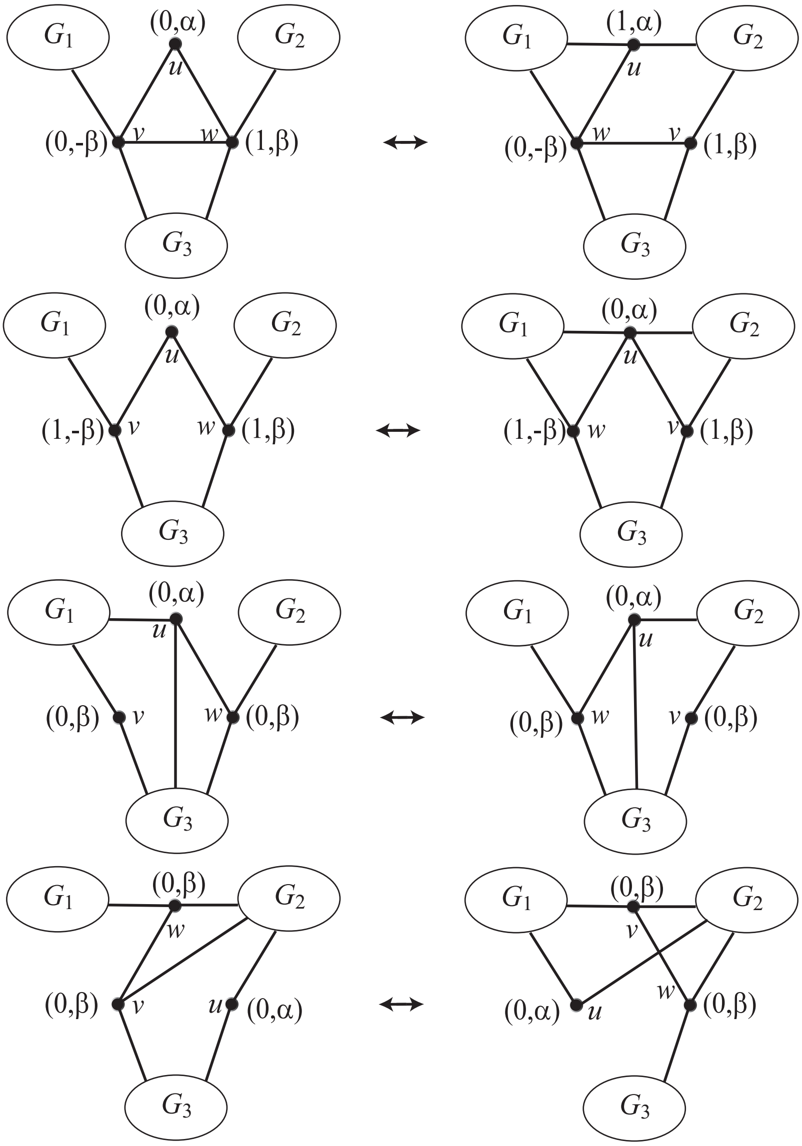

FIGURE 3. The other versions of third graph-move

a labeled graph with one vertex (two vertices) we get the free loop rather than empty graph.

Definition 2.6. The *adjacency matrix* $A(G) = (a_{ij})$ of a labeled graph G is the matrix over $\mathbb{Z}_2$ defined as follows: $a_{ii} = \mathrm{fr}(v_i)$, and $a_{ij} = 1$, $i \neq j$, if and only if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise.

Proposition 2.1 ([6])**.** *The number* $\operatorname{corank}(A(G) + E)$ *is an invariant of the graph-link* $\{G\}$, *i.e.* $\operatorname{corank}(A(G_1) + E) = \operatorname{corank}(A(G_2) + E)$ *for any* $G_1, G_2 \in \{G\}$, *and* $\operatorname{corank} A$ *is the corank or nullity of the matrix* A, *i.e. the difference between the size of the matrix and its rank* rk (*the rank is calculated over* $\mathbb{Z}_2$).

Definition 2.7. Define the *number of components* of a graph-link $\{G\}$ to be equal to $\operatorname{corank}(A(G)+E)+1$, where E is the identity matrix. A *graph-knot* is a graph-link with one component.

A graph-link is called *non-realizable*, if each of its representatives is non-realizable graph.

Note that in the realizable case the number of components of a graph-link coincides with the number of components of the corresponding links.

2.2. Oriented graph-links. To define the notion of orientation on graph-links we first partition the vertices of a labeled graph into classes, see [9, 11]. In the realizable case, the classes correspond to crossings where one component or two different components intersect. Recall that this partition relies on the Circuit-Nullity Formula, see [16, 17].

Let $B(G) = A(G) + E$ and $B_i(G) = A(G) + E + E_{ii}$, where E_{ii} is the matrix with the only one non-zero element equal to 1 in the ith column and ith row.

Definition 2.8. We say that a vertex $v_i \in V(G)$ *lies on one component* of G or is a *pure vertex* of G if

$$\operatorname{corank} B_i(G) \neq \operatorname{corank} B(G \setminus \{v_i\}).$$

Otherwise, we say that v_i *belongs to different components* of G or is a *mixed vertex* of G. Denote $MV(G)$ the set of mixed vertices of G.

Let v_i and v_j be two mixed vertices of G. We say that *two components meet* at these vertices or these vertices have the *same component type* if either $v_i = v_j$, or v_i is a pure vertex of the labeled graph $G \setminus \{v_j\}$, i.e. either

$$\operatorname{corank} B_i(G \setminus \{v_j\}) \neq \operatorname{corank} B(G \setminus \{v_j, v_i\})$$

if $i < j$, or

$$\operatorname{corank} B_{i-1}(G \setminus \{v_j\}) \neq \operatorname{corank} B(G \setminus \{v_j, v_i\})$$

if $i > j$. Otherwise, we say that *different components meet* at these vertices or these vertices have a *different component type*.

Remark 2.3. If two vertices have the same component type, in common case we cannot say what these components are.

From [11] we have that if the labeled graph G_2 is obtained from a labeled graph G_1 by a graph-move Ω then

(1) the relation "to have the same component type" preserves for the corresponding vertices of the graphs G_1 and G_2, not participating in Ω or participating in $\Omega = \Omega_g 3$, $\Omega_g 4$ or $\Omega_g 4'$;
(2) the mixed vertices participating in $\Omega = \Omega_g 2$ have the same component type.

Let G be a labeled graph with k components. There exists a set of $k-1$ mixed vertices such that the labeled graph obtained from G by deleting these $k-1$ vertices has only one component. We call this set a *binding set* of G, i.e. a binding set is a set U of $k-1$ vertices such that $\operatorname{corank} B(G \setminus U) = 0$. For any sequence $\Lambda = (\alpha_1, \dots, \alpha_{k-1}) \in \{+,-\}^{k-1}$ and any enumerated binding set $U = \{u_1, \dots, u_{k-1}\}$ there exists a labeled graph $G(U, \Lambda)$ in the diagram $\mathcal{G}$ of G such that the vertices of $G(U, \Lambda)$ corresponding to $u_1, \dots, u_{k-1}$ have the prescribed signs $\alpha_1, \dots, \alpha_{k-1}$. Note that a graph $G(U, \Lambda)$ is not unique, and by deleting the vertices corresponding to $u_1, \dots, u_{k-1}$ from any graph in $\mathcal{G}$ we get a labeled graph with one component. We call the pair (U, Λ) a *signed binding set* of G.

From [11] it follows that for any pure vertex v_i we can define the writhe number at it by setting

$$\mathrm{wr}_i = \mathrm{sgn}(v_i)(-1)^{\mathrm{corank}\, B_i(G) - \mathrm{corank}\, B(G)},$$

and this formula is invariant under a deletion of m vertices if the resulting graph has the number of components being equal to the number of components of G minus m. To define the writhe number at a mixed vertex of G we need to construct a graph $G_{(U,\Lambda)} = G(U, \Lambda) \setminus U$ with one component.

Definition 2.9. The *writhe number* $\mathrm{wr}_j(G, U, \Lambda)$ at a vertex $v_j \in V(G)$ with respect to a signed binding set (U, Λ) is the writhe number at the corresponding vertex of $G_{(U,\Lambda)}$, i.e.

$$\mathrm{wr}_j(G, U, \Lambda) = \mathrm{sgn}(v_j)(-1)^{\mathrm{corank}\, B_j(G_{(U,\Lambda)})},$$

if $v_j \neq u_i$ for $i = 1, \dots, k-1$, and $\mathrm{wr}_j(G, U, \Lambda) = \alpha_i$ otherwise.

Definition 2.10. We say that two signed binding sets (U_1, Λ_1) and (U_2, Λ_2), where $U_1 = \{v_{i_1}, \dots, v_{i_{k-1}}\}$, are *equivalent* if $\mathrm{wr}_{i_p}(G, U_1, \Lambda_1) = \mathrm{wr}_{i_p}(G, U_2, \Lambda_2)$ for all $p = 1, \dots, k-1$.

The relation from Def. 2.10 on signed binding sets is an equivalence relation, see [9].

Definition 2.11. We call the equivalence class $\mathcal{O} = [(U, \Lambda)]$ of a signed binding set (U, Λ) of a labeled graph G an *orientation* of G. We say that a labeled graph G is *oriented* if any orientation $\mathcal{O}$ is fixed (we denote the oriented labeled graph $(G, \mathcal{O})$).

Let G' be a labeled graph obtained from a labeled graph G by a graph-move Ω, and $\mathcal{O}$ is an orientation of G. We can choose $(U, \Lambda) \in \mathcal{O}$ such that in the case $\Omega = \Omega_g 2$ none of the vertices of U takes part in the move. Then the pair (U, Λ) is a signed binding set of G'.

Definition 2.12. We say that the oriented labeled graph $(G', \mathcal{O}')$, where $\mathcal{O}' = [(U', \Lambda')]$, is obtained from the oriented labeled graph $(G, \mathcal{O})$, where $\mathcal{O} = [(U, \Lambda)]$, by the *oriented graph move* Ω.

An *oriented diagram* of a graph-link is an equivalence class of oriented labeled graphs modulo oriented $\Omega_g 4$ and $\Omega_g 4'$ graph-moves

An *oriented graph-link* is an equivalence class of oriented labeled graphs modulo oriented $\Omega_g 1 - \Omega_g 4'$ graph-moves.

Let G be a labeled graph and $\mathcal{V}_1, \dots, \mathcal{V}_r$ be the equivalence classes of vertices of G modulo the relation "to have the same component type". Let us consider any orientation $\mathcal{O} = [(U, \Lambda)]$ on G. For each $\mathcal{V}_i = \{v_{i_1}, \dots, v_{i_{s_i}}\}$, $i = 1, \dots, r$, we define the *linking number* of this class:

$$\mathrm{lk}(\mathcal{V}_i) = \frac{1}{2} \sum_{j=1}^{s_i} \mathrm{wr}_{i_j}(G, U, \Lambda).$$

Definition 2.13. The *oriented linking multiset* of an oriented labeled graph $(G, \mathcal{O})$ is

$$\mathrm{lk}(G, \mathcal{O}) = \{\mathrm{lk}(\mathcal{V}_1), \dots, \mathrm{lk}(\mathcal{V}_r)\}.$$

The *linking multiset* of a labeled graph G is

$$\mathrm{lk}(G) = \{|\mathrm{lk}(\mathcal{V}_1)|, \dots, |\mathrm{lk}(\mathcal{V}_r)|\}.$$

Theorem 2.1 (see [9])**.** *The* (*oriented*) *linking multiset is an invariant of* (*oriented*) *graph-links up to addition/removal of* 0*.*

It is not difficult to prove the following result.

Proposition 2.2. *Let* $(G_i, \mathcal{O}_i)$*,* $i = 1, 2$*, be two oriented labeled graphs. Then*

$$\mathrm{lk}(G_1 \sqcup G_2, \mathcal{O}_1 \sqcup \mathcal{O}_2) = \mathrm{lk}(G_1, \mathcal{O}_1) \sqcup \mathrm{lk}(G_2, \mathcal{O}_2).$$

Let us recall the definition of the Kauffman bracket polynomial and the Jones polynomial which is an invariant of oriented graph-links.

Definition 2.14 ([6, 7])**.** We call a subset of $V(G)$ a *state* of G.
The *Kauffman bracket polynomial* of G is

$$\langle G\rangle(a) = \sum_s a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{\mathrm{corank}\, A(G(s))},$$

where the sum is taken over all states s of G, $\alpha(s)$ is equal to the number of the vertices labeled $(a, -)$ from s and the vertices labeled $(b, +)$ from $V(G) \setminus s$, $\beta(s) = |V(G)| - \alpha(s)$.

Theorem 2.2 ([6, 7])**.** *The Kauffman bracket polynomial of a labeled graph is invariant under* $\Omega_g 2 - \Omega_g 4'$ *and gets multiplied by* $(-a^{\pm 3})$ *under* $\Omega_g 1$*.*

The *writhe number* of an oriented labeled graph $(G, \mathcal{O})$, where $\mathcal{O} = [(U, \Lambda)]$, is

$$\mathrm{wr}(G, \mathcal{O}) = \sum_{i=1}^{n} \mathrm{wr}_i(G, U, \Lambda).$$

Definition 2.15. Let $\mathfrak{G}$ be an oriented graph-link. Define the *Jones polynomial* as

$$X(G, \mathcal{O})(q) = (-a)^{-3\mathrm{wr}(G,\mathcal{O})}\langle G\rangle(a)\Big|_{a=q^{-1/4}},$$

where $(G, \mathcal{O})$ is any representative of $\mathfrak{G}$.

The following two propositions are not difficult to prove.

Proposition 2.3. *Let* $\mathcal{O}_i$*,* $i = 1, 2$*, be two orientations of a labeled graph* G*. Then*

$$X(G, \mathcal{O}_2)(q) = q^{3/2(\|\mathrm{lk}(G,\mathcal{O}_2)\| - \|\mathrm{lk}(G,\mathcal{O}_1)\|)} X(G, \mathcal{O}_1)(q),$$

where $\|\mathrm{lk}(G, \mathcal{O}_i)\|$ *is the sum of elements of* $\mathrm{lk}(G, \mathcal{O}_i)$*.*

Proposition 2.4. *Let* $(G_i, \mathcal{O}_i)$*,* $i = 1, 2$*, be two oriented labeled graphs. Then*

$$X(G_1 \sqcup G_2, \mathcal{O}_1 \sqcup \mathcal{O}_2)(q) = X(G_1, \mathcal{O}_1)(q) \cdot X(G_2, \mathcal{O}_2)(q).$$

3. Khovanov homology

Let G be a simple labeled graph with n vertices, and $A(G) = (a_{ij})$ its adjacency matrix. Given a state $s \subset V(G)$, let us consider the vector space

$$\mathcal{V}(G,s) = \mathbb{Z}_2\langle x_1, \dots, x_n \mid r_1^{(G,s)}, \dots, r_n^{(G,s)} \rangle,$$

where the relations $r_1^{(G,s)}, \dots, r_n^{(G,s)}$ are given by the formula

$$r_i^{(G,s)} = \begin{cases} x_i + \sum\limits_{\{j \mid v_j \in s\}} a_{ij}x_j, & \text{if } v_i \notin s, \\ \sum\limits_{\{j \mid v_j \in s\}} a_{ij}x_j, & \text{if } v_i \in s. \end{cases} \tag{3.1}$$

Note that if G is realized by a chord diagram and, therefore, by a virtual link diagram then the relations $r_1^{(G,s)}, \dots, r_n^{(G,s)}$ generate the cycle space of the graph of circles (the touch-graph) for a resolution of the link diagram, i.e. at each crossing we perform a smoothing, see [1, 10, 17]. It is not difficult to show that the dimension of $\mathcal{V}(G,s)$ is equal to $\operatorname{corank} A(G(s))$, where $G(s)$ is the induced subgraph of G on s, see [10, 17].

The next step is to construct the state cube. We have a natural bijection between states $s \subset V(G)$ and vertices of the hypercube $\{0,1\}^n$: the state s corresponds to the vertex $\sigma(s)$, where the i-th coordinate is equal to 1 if and only if either $v_i \in s$ and $\operatorname{sgn}(v_i) = +$ or $v_i \notin s$ and $\operatorname{sgn}(v_i) = -$. Every edge of the hypercube is of the type $s \to s \oplus i$, where $s \oplus i = s \triangle \{v_i\}$. We orient the edge so that $\|\sigma(s \oplus i)\| = \|\sigma(s)\| + 1$, where $\|\sigma(s)\|$ is the sum of coordinates of $\sigma(s)$.

We assign to every oriented edge $s \to s \oplus i$ the map

$$\partial^s_{s\oplus i} \colon \bigwedge\nolimits^* \mathcal{V}(G,s) \to \bigwedge\nolimits^* \mathcal{V}(G, s \oplus i)$$

of exterior algebras defined by the formula

$$\partial^s_{s\oplus i}(u) = \begin{cases} x_i \wedge u & \text{if } x_i = 0 \in \mathcal{V}(G,s), \\ u & \text{if } x_i \neq 0 \in \mathcal{V}(G,s). \end{cases} \tag{3.2}$$

Consider the chain complex

$$\mathcal{C}(G) = \bigoplus_{s \subset V(G)} \bigwedge\nolimits^* \mathcal{V}(G,s)$$

with differential

$$\partial(u) = \sum_{\{s, s' \subset V(G) \mid s \to s'\}} \partial^s_{s'}(u).$$

Definition 3.1. The homology $\operatorname{Kh}(G)$ of the complex $(\mathcal{C}(G), \partial)$ are called the *reduced* (*odd*) *Khovanov homology* of the labeled simple graph G.

Let $(G, \mathcal{O})$ be an oriented labeled graph with n vertices and $u \in \bigwedge^r \mathcal{V}(G,s)$.

Definition 3.2. The *homological grading* of u is equal to

$$M(u) = \|\sigma(s)\| + \frac{\operatorname{wr}(G,\mathcal{O}) - n}{2}.$$

The *quantum grading* of u is equal to

$$Q(u) = \dim \mathcal{V}(G,s) - 2r + \|\sigma(s)\| + \frac{3\operatorname{wr}(G,\mathcal{O}) - n}{2}.$$

From [9, 11, 13] it follows that the differential increases the grading M and leaves the grading Q unchanged. Let $\mathrm{Kh}_{m,q}(G,\mathcal{O})$ be the homogenous part of $\mathrm{Kh}(G)$ having the gradings $M=m$ and $Q=q$. As a result we have the following theorems.

Theorem 3.1. *The groups* $\mathrm{Kh}_{m,q}(G,\mathcal{O})$, $m,q\in\mathbb{Z}$, *are invariants of* $(G,\mathcal{O})$.

Proof. For an element $u\in\bigwedge^r\mathcal{V}(G,s)\subset\mathcal{C}(G)$ of the Khovanov complex, consider the unshifted gradings

$$M_0(u)=\|\sigma(s)\|,\qquad Q_0(u)=\dim\mathcal{V}(G,s)-2r+\|\sigma(s)\|.$$

The differential increases the grading M_0 by 1 and does not change Q_0. Hence, the homology group $\mathrm{Kh}(G)$ is bigraded.

Reidemeister moves shift the gradings M_0,Q_0 of the Khovanov complex as shown in the following table [8, 13].

	M_0	Q_0
Ω_g1^+	0	-1
Ω_g1^-	1	2
Ω_g2	1	1
Ω_g3	0	0
Ω_g4	0	0
Ω_g4'	0	0

Here Ω_g1^ϵ, $\epsilon=\pm$, denotes the adding an isolated vertex with the label $(0,\epsilon)$ and Ω_g2 denotes the increasing second Reidemeister move.

On the other hand, the moves $\Omega_g2,\Omega_g3,\Omega_g4,\Omega_g4'$ do not change the writhe number $\mathrm{wr}(G,\mathcal{O})$, and the moves Ω_g1^ϵ change it by ϵ. Hence, the corrected gradings M and Q are invariant under Reidemeister moves. □

By the definition of Khovanov complex we have

Theorem 3.2. *Let* $(G_i,\mathcal{O}_i)$, $i=1,2$, *be two oriented labeled graphs. Then*

$$\mathrm{Kh}_{m,q}(G_1\sqcup G_2,\mathcal{O}_1\sqcup\mathcal{O}_2)=\bigoplus_{m_1+m_2=m,\ q_1+q_2=q}\mathrm{Kh}_{m_1,q_1}(G_1,\mathcal{O}_1)\otimes\mathrm{Kh}_{m_2,q_2}(G_2,\mathcal{O}_2).$$

Theorem 3.3. *The bigraded Khovanov homology* $\mathrm{Kh}_{m,q}(G,\mathcal{O})$ *categorifies the Jones polynomial in the sense that*

$$\sum_{m,q\in\mathrm{Z}}(-1)^m\dim_{\mathbb{Z}_2}\mathrm{Kh}_{m,q}(G,\mathcal{O})\cdot t^q=X(G,\mathcal{O})(it^{-1/2}).$$

Definition 3.3. The Laurent polynomial

$$Kh(G,\mathcal{O})(x,y)=\sum_{m,\,q\in\mathbb{Z}}\dim_{\mathbb{Z}_2}\mathrm{Kh}_{m,q}(G,\mathcal{O})x^my^q$$

is called a *Poincaré polynomial.*

From the definitions we immediately obtain the following facts.

Proposition 3.1. *Let* $\mathcal{O}_i$, $i=1,2$, *be two orientations of a labeled graph* G. *Then*

$$Kh(G,\mathcal{O}_2)(x,y)=x^{\|\mathrm{lk}(G,\mathcal{O}_2)\|-\|\mathrm{lk}(G,\mathcal{O}_1)\|}y^{3(\|\mathrm{lk}(G,\mathcal{O}_2)\|-\|\mathrm{lk}(G,\mathcal{O}_1)\|)}Kh(G,\mathcal{O}_1)(x,y).$$

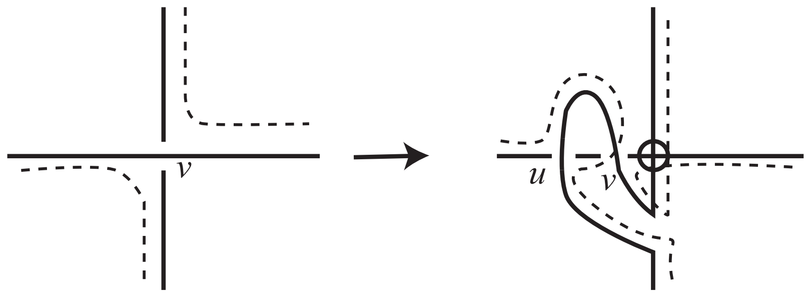

FIGURE 4. Uncoupling operation in the realizable case

Proposition 3.2. *Let* $(G_i, \mathcal{O}_i)$, $i = 1, 2$, *be two oriented labeled graphs. Then*

$$Kh(G_1 \sqcup G_2, \mathcal{O}_1 \sqcup \mathcal{O}_2)(x, y) = Kh(G_1, \mathcal{O}_1)(x, y) \cdot Kh(G_2, \mathcal{O}_2)(x, y).$$

4. Uncoupling operation

Let G be a labeled graph with more than 1 component and v be its mixed vertex. We define the labeled graph $G_{uc(v)}$ to be the graph obtained from G by changing the framing of v and adding a new vertex $\tilde{v}$ which is adjacent only to v and has the label $(0, \operatorname{sgn}(v))$. We say that the graph $G_{uc(v)}$ is obtained from G by *uncoupling operation at the mixed vertex* v (see Fig 4 for the realizable case).

Lemma 4.1. *The graphs* G *and* $G_{uc(v)}$ *have the same number of components and* $MV(G_{uc(v)}) = MV(G) \cup \{\tilde{v}\}$.

Proof. Let us enumerate the vertices of G and $G_{uc(v)}$ such that the number of v in G is 1, the number of $\tilde{v}$ in $G_{uc(v)}$ is 1 and the numbers of the other vertices of $G_{uc(v)}$ are 1 more the numbers of the corresponding vertices of G.

We have

$$\operatorname{corank} B(G_{uc(v)}) = \operatorname{corank} \begin{pmatrix} 1 & 1 & \mathbf{0}^\top \\ 1 & \operatorname{fr}(v) & \mathbf{a}^\top \\ \mathbf{0} & \mathbf{a} & B(G \setminus \{v\}) \end{pmatrix}$$
$$= \operatorname{corank} \begin{pmatrix} 1 & 0 & \mathbf{0}^\top \\ 0 & \operatorname{fr}(v) + 1 & \mathbf{a}^\top \\ \mathbf{0} & \mathbf{a} & B(G \setminus \{v\}) \end{pmatrix} = \operatorname{corank} B(G),$$

$$\operatorname{corank} B_1(G_{uc(v)}) = \operatorname{corank} \begin{pmatrix} 0 & 1 & \mathbf{0}^\top \\ 1 & \operatorname{fr}(v) & \mathbf{a}^\top \\ \mathbf{0} & \mathbf{a} & B(G \setminus \{v\}) \end{pmatrix}$$
$$= \operatorname{corank} \begin{pmatrix} 0 & 1 & \mathbf{0}^\top \\ 1 & 0 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{0} & B(G \setminus \{v\}) \end{pmatrix} = \operatorname{corank} B(G \setminus \{v\})$$
$$= \operatorname{corank} B_1(G) = \operatorname{corank} B(G_{uc(v)} \setminus \{\tilde{v}\}),$$

$$\operatorname{corank} B_2(G_{uc(v)}) = \operatorname{corank} \begin{pmatrix} 1 & 1 & \mathbf{0}^\top \\ 1 & \operatorname{fr}(v)+1 & \mathbf{a}^\top \\ \mathbf{0} & \mathbf{a} & B(G\setminus\{v\}) \end{pmatrix}$$

$$= \operatorname{corank} \begin{pmatrix} 1 & 0 & \mathbf{0}^\top \\ 0 & \operatorname{fr}(v) & \mathbf{a}^\top \\ \mathbf{0} & \mathbf{a} & B(G\setminus\{v\}) \end{pmatrix} = \operatorname{corank} B_1(G)$$

$$= \operatorname{corank} B(G\setminus\{v\}) = \operatorname{corank} \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & B(G\setminus\{v\}) \end{pmatrix} = \operatorname{corank} B(G_{uc(v)}\setminus\{v\})$$

and for any vertex w with the number $i \geqslant 2$

$$\operatorname{corank} B_i(G_{uc(v)}) = \operatorname{corank} \begin{pmatrix} 1 & 1 & \mathbf{0}^\top \\ 1 & \operatorname{fr}(v) & \mathbf{a}^\top \\ \mathbf{0} & \mathbf{a} & B_{i-1}(G\setminus\{v\}) \end{pmatrix}$$

$$= \operatorname{corank} \begin{pmatrix} 1 & 0 & \mathbf{0}^\top \\ 0 & \operatorname{fr}(v)+1 & \mathbf{a}^\top \\ \mathbf{0} & \mathbf{a} & B_{i-1}(G\setminus\{v\}) \end{pmatrix} = \operatorname{corank} B_{i-1}(G),$$

$$\operatorname{corank} B(G_{uc(v)}\setminus\{w\}) = \operatorname{corank} \begin{pmatrix} 1 & 1 & \mathbf{0}^\top \\ 1 & \operatorname{fr}(v) & \mathbf{a}^\top \\ \mathbf{0} & \mathbf{a} & B(G\setminus\{v,w\}) \end{pmatrix}$$

$$= \operatorname{corank} \begin{pmatrix} 1 & 0 & \mathbf{0}^\top \\ 0 & \operatorname{fr}(v)+1 & \mathbf{a}^\top \\ \mathbf{0} & \mathbf{a} & B(G\setminus\{v,w\}) \end{pmatrix} = \operatorname{corank} B(G\setminus\{w\}).$$

□

Denote G_{uc} the graph obtained from G by uncoupling operations at all the mixed vertices of G. From the definition it follows that G_{uc} does not depend on a sequence of the mixed vertices when we apply the uncoupling operation.

Before proving the main theorem of this section we justify the following lemma.

Lemma 4.2. *The number of mixed vertices taking part in* $\Omega_g 3$ *is equal to either* 0, *or* 2, *or* 3. *Moreover, if this number is equal to* 2 *then these two mixed vertices have the same component type, and if it equals* 3 *then each pair of these vertices have different component type.*

Proof. Let G be a labeled graph and vertices v_1, v_2, v_3 take part in $\Omega_g 3$. From [11, Lemma 3.4] it suffices to consider the right part of Fig. 1. We have

$$B(G) = \begin{pmatrix} 1 & 0 & 0 & \mathbf{a}^\top \\ 0 & 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & 1 & (\mathbf{a}+\mathbf{b})^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{a}+\mathbf{b} & C \end{pmatrix}.$$

To check or disprove the equality $\operatorname{corank} B_i(G) = \operatorname{corank} B(G\setminus\{v_i\})$, $i = 1, 2, 3$, we will use [11, Lemma 3.1].

Since $\mathbf{a}^\top\Lambda + \mathbf{b}^\top\Lambda = (\mathbf{a}+\mathbf{b})^\top\Lambda$, where Λ is a column-vector consisting of 0 and 1, we get that either none of the first three columns of $B(G)$ or the sum of two of them is a linear combination of columns of the matrix

$$\begin{pmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \\ (\mathbf{a}+\mathbf{b})^\top \\ C \end{pmatrix}.$$

We immediately get the validity of the first part of the lemma.

The validity of the second part of the lemma follows from [11, Lemma 3.6]. Indeed, if two vertices from v_1, v_2, v_3 are mixed, then the sum of columns is a linear combination of columns of the matrix

$$\begin{pmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \\ (\mathbf{a}+\mathbf{b})^\top \\ C \end{pmatrix}$$

and each column separately from these two columns is not a linear combination of the columns of $B(G)$ except these two columns. If v_1, v_2, v_3 are mixed, then we have three linear combinations for three sums of two columns, and, therefore, each column of the first three columns of $B(G)$ is a linear combination of the columns of $B(G)$ except the two columns. $\square$

Theorem 4.1. *If* $\{G\} = \{G'\}$ *then* $\{G_{uc}\} = \{G'_{uc}\}$.

Proof. Let G' be obtained from G by a one graph-move. We will show that G'_{uc} is obtained from G_{uc} by graph-moves. The validity of this assertion is obvious if there is no mixed vertex in the graph-move, so it is obvious for the first graph-move, see [11, Lemma 3.2].

Let G' be obtained from G by Ω_g2 by adding vertices u and v. Then u and v are mixed vertices in G'. Let $\tilde{u}$ and $\tilde{v}$ be the additional vertices of G'_{uc}, being adjacent only to u and v, respectively. Then we apply Ω_g4' to G'_{uc} at u and v if $\mathrm{fr}(u) = \mathrm{fr}(v) = 0$ in G' or Ω_g4 to G'_{uc} at u and v if $\mathrm{fr}(u) = \mathrm{fr}(v) = 1$ in G' and then two times Ω_g2 for the pairs $(u, \tilde{u})$ and $(v, \tilde{v})$. As a result we get G_{uc}.

Let G' be obtained from G by Ω_g3, see Fig. 1. By Lemma 4.1 and [11, Lemma 3.4] we have four cases: (1) u, $v \in MV(G)$ and u, $w \in MV(G')$; (2) u, $w \in MV(G)$ and u, $v \in MV(G')$; (3) v, $w \in MV(G)$ and v, $w \in MV(G')$; (4) u, v, $w \in MV(G)$ and u, v, $w \in MV(G')$. In each case we apply Ω_g4' to G_{uc} and G'_{uc} at the mixed vertices of G and G' from u, v, w and then Ω_g2 to the pairs of $(u, \tilde{u})$, $(v, \tilde{v})$ and $(w, \tilde{w})$ if they exist.

Let G' be obtained from G by Ω_g4 at vertices u and v where $\mathrm{sgn}(u) = \alpha$ and $\mathrm{sgn}(v) = \beta$. By [11, Lemma 3.5] we have three cases: (1) $u \in MV(G)$ and $u \in MV(G')$; (2) $v \in MV(G)$ and $v \in MV(G')$; (3) u, $v \in MV(G)$ and u, $v \in MV(G')$. In the first two cases we apply Ω_g4' to G_{uc} and G'_{uc} at the one mixed vertex of G and G' from u, v and then Ω_g2 to the pair of $(u, \tilde{u})$, $(v, \tilde{v})$ if it exists. In the third case we apply to G_{uc} the following sequence of graph-moves: Ω_g4' at u; Ω_g2 to the pair $(u, \tilde{u})$; Ω_g2 which adds two adjacent vertices w_1 and w_2 with framing 1 and signs β and $-\beta$, respectively, being adjacent only to v and each vertex from $N(v) \setminus \{\tilde{v}\}$ in the new graph; Ω_g4' at w_1; Ω_g4' at v; Ω_g2 to the pair $(v, \tilde{v})$. As a result we get the labeled graph isomorphic to the graph obtained from G'_{uc} by applying Ω_g4' at u and Ω_g2 to the pair $(u, \tilde{u})$

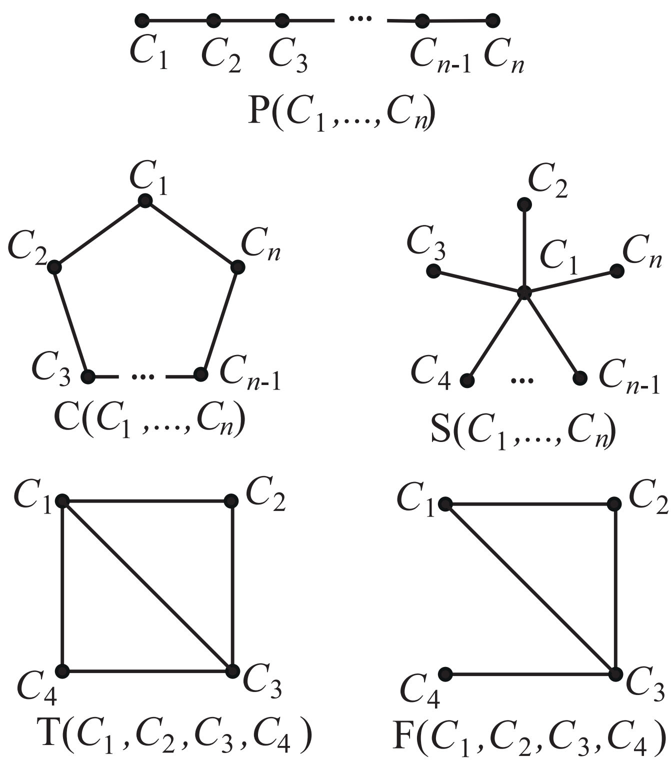

FIGURE 5. Graphs

Let G' be obtained from G by $\Omega_g 4'$ at vertex u with sign α. To get the labeled graph being isomorphic G'_{uc} from G_{uv} we have to perform the following sequence of graph-moves to G_{uv}: $\Omega_g 2$ which adds two adjacent vertices w_1 and w_2 with framing 1 and signs α and $-\alpha$, respectively, being adjacent only to u and each vertex from $N(u) \setminus \{\tilde{u}\}$; $\Omega_g 4'$ at w_1; $\Omega_g 4'$ at u; $\Omega_g 2$ to the pair $(u, \tilde{u})$. $\square$

5. Classification of graph-links with diagrams on 4 vertices

In [9] the authors classified all graph-links having representatives with less than 4 vertices, where for each diagram representing a graph-link with more than one component they considered all orientations on it and found the linking sets and the Jones polynomials. In this section we do the same for graph-links with representatives on 4 vertices. Let us note that this classification was started by Ilyutko and Maslennikova and continued in Maslennikova's coursework, but was not completed.

First, let us introduce some notations. In our tables we will rename the vertex labels by putting $A_0 = (0,+)$, $A_1 = (1,+)$, $B_0 = (0,-)$, $B_1 = (1,-)$ (in fact this vertex designation is natural, since the sings correspond to A- and B-smoothings). Let us denote $\mathrm{K}(C_1, \ldots, C_n)$ the complete graph with labels $C_1, \ldots, C_n \in \{A_0, A_1, B_0, B_1\}$, and the graphs $\mathrm{P}(C_1, \ldots, C_n)$, $\mathrm{C}(C_1, \ldots, C_n)$, $\mathrm{S}(C_1, \ldots, C_n)$, $\mathrm{T}(C_1, C_2, C_3, C_4)$ and $\mathrm{F}(C_1, C_2, C_3, C_4)$ are depicted in Fig. 5.

By Theorem 3.3, the Jones polynomial is expressed through a Poincaré polynomial. Therefore below we calculate only Poincaré polynomials. Now, we recall the results from [9] with specifying only the Poincaré polynomials for the first oriented linking multiset in the case of graph-links with many components (see Propositions 2.3, 3.1) and denoting $\mathrm{GrLk}(n, m)_i$ graph-links with n vertices and m components. Also we will not write the oriented linking multiset and Poincaré polynomials for disjoint unions of graph-links (see Propositions 2.2 and 3.2). We recall that the empty graph we denote $\bigcirc$.

Let us consider labeled graphs on four vertices. There are 996 of such graphs and they are partitioned into 228 diagrams. Among them, 102 diagrams cannot be reduced by a decreasing first or second Reidemeister move.

Let us first consider graph-knots. We have 34 diagrams out of 102 and 18 new values of Poincaré polynomials. We show that 9 diagrams are equivalent to diagrams on $\leqslant 3$ vertices, and there are 7 pairs of diagrams on 4 vertices. As a result we have 18 different graph-knots with representatives on 4 vertices but no on < 4 vertices.

Equivalences of $\{\mathrm{S}(B_1, A_0, A_0, A_1), \mathrm{S}(B_0, A_0, A_0, B_1), \mathrm{K}(A_1, A_1, A_0, A_1)\}_4$ and $\{\mathrm{S}(A_0, A_1, B_0, B_0), \mathrm{S}(A_1, B_1, B_0, B_0), \mathrm{K}(B_0, B_1, B_1, B_1)\}_4$ to diagrams on 3 vertices are shown in Fig. 6 (here we use third graph-moves from Figs. 2 and 3).

Applying a third graph-move and then a first graph-move to
$\{\mathrm{T}(A_1, A_0, A_1, A_0), \mathrm{F}(A_1, A_1, B_1, A_0), \mathrm{P}(A_0, B_1, B_0, A_0)\}_4$,
$\{\mathrm{T}(B_1, B_0, B_1, B_0), \mathrm{F}(B_1, B_1, A_1, B_0), \mathrm{P}(B_0, A_1, A_0, B_0)\}_4$,
$\{\mathrm{P}(A_0, B_0, B_0, A_0), \mathrm{C}(A_0, A_0, A_0, A_0)\}_4$
and $\{\mathrm{P}(B_0, A_0, A_0, B_0), \mathrm{C}(B_0, B_0, B_0, B_0)\}_4$ we get the diagrams on 3 vertices.

Applying a third graph-move and then a second-graph move to
$\{\mathrm{T}(A_0, A_0, A_1, A_1), \mathrm{C}(A_0, A_1, B_1, A_0), \mathrm{F}(A_1, A_0, B_1, A_1), \mathrm{T}(A_1, B_1, B_0, A_0),$
$\mathrm{P}(A_1, B_1, B_0, A_1), \mathrm{F}(A_1, A_0, B_0, B_1), \mathrm{F}(B_0, A_1, B_1, B_1), \mathrm{P}(B_1, B_0, B_0, A_1),$
$\mathrm{P}(A_1, B_1, B_1, B_1), \mathrm{F}(B_0, A_1, B_0, A_1), \mathrm{P}(B_1, B_0, B_1, B_1)\}_4$,
$\{\mathrm{T}(A_0, A_0, B_1, A_1), \mathrm{C}(A_0, A_1, B_1, B_0), \mathrm{F}(A_1, A_0, A_1, A_1), \mathrm{T}(A_1, B_1, B_0, B_0),$
$\mathrm{P}(A_1, B_1, A_0, A_1), \mathrm{F}(A_1, A_0, A_0, B_1), \mathrm{F}(B_0, B_1, B_1, B_1), \mathrm{P}(B_1, B_0, A_0, A_1),$
$\mathrm{P}(A_1, B_1, A_1, B_1), \mathrm{F}(B_0, B_1, B_0, A_1), \mathrm{P}(B_1, B_0, A_1, B_1)\}_4$
and $\{\mathrm{T}(A_0, A_1, B_1, B_0), \mathrm{C}(B_1, A_1, B_0, B_0), \mathrm{F}(A_0, B_1, A_1, A_1), \mathrm{T}(B_0, B_1, B_1, B_0),$
$\mathrm{P}(A_1, A_1, A_0, A_1), \mathrm{F}(A_0, B_1, A_0, B_1), \mathrm{F}(B_0, B_1, A_1, B_1), \mathrm{P}(B_1, A_0, A_0, A_1),$
$\mathrm{P}(A_1, A_1, A_1, B_1), \mathrm{F}(B_0, B_1, A_0, A_1), \mathrm{P}(B_1, A_0, A_1, B_1)\}_4$ we get the diagrams on 2 vertices.

The diagrams of $\mathrm{GrLk}(4, 1)_i$, $10 \leqslant i \leqslant 15$, form two Ω_4-equivalence classes which are connected by a third graph-move, see Figs. 2 and 3.

The diagrams of $\mathrm{GrLk}(4, 1)_{17}$ form two Ω_4-equivalence classes. To show that those classes are connected by graph-moves, we use the map χ from [5]. We get two looped graphs and their equivalence is shown in Fig. 7 (we use moves from [5] and we use marks 1, 2 and 3 for vertices the third move from [5, Fig. 9] is applied to).

Remark 5.1. We have obtained that graph-knots with a representative on $\leqslant 4$ vertices are classified by the Jones polynomial.

Let us first consider graph-links with two components. This condition distinguishes 39 diagrams out of 68. They give 27 new values of Poincaré polynomials, and 4 values which coincide with that of diagrams on 3 vertices. We show that

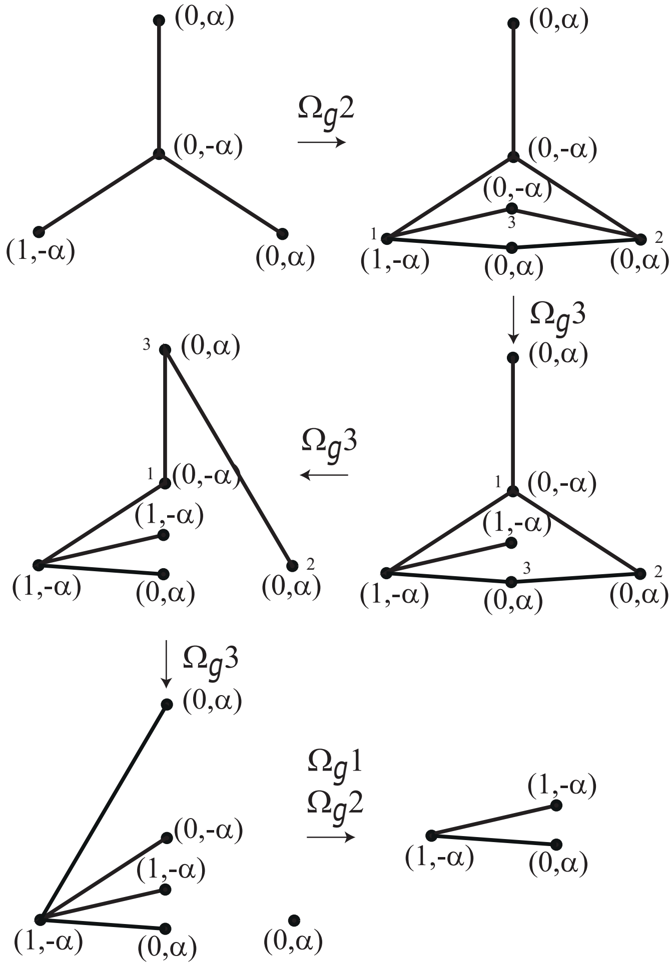

FIGURE 6. Equivalence of the labeled graphs (we use marks 1, 2 and 3 for vertices the third-graph move is applied to)

8 diagrams are equivalent to diagrams on $\leqslant 3$ vertices, and there are 31 new graph-links with 2 components.

It is not difficult to find third graph-moves from Figs. 2 and 3 after performing which to the diagrams $\{\mathrm{P}(A_0, B_0, B_1), \mathrm{P}(A_0, B_1, A_1), \mathrm{C}(A_0, A_1, A_1)\}_4 \sqcup \{\mathrm{P}(A_1), \mathrm{P}(B_1)\}_4$, $\{\mathrm{P}(A_1, A_0, B_0), \mathrm{P}(B_1, A_1, B_0), \mathrm{C}(B_0, B_1, B_1)\}_4 \sqcup \{\mathrm{P}(A_1), \mathrm{P}(B_1)\}_4$, $\{\mathrm{T}(A_1, A_0, A_1, A_1), \mathrm{C}(A_0, A_0, B_1, A_0), \mathrm{F}(A_1, A_0, B_1, A_0), \mathrm{P}(A_1, B_1, B_0, A_0), \mathrm{P}(B_1, B_0, B_0, A_0)\}_4$, $\{\mathrm{T}(B_1, A_0, B_1, A_1), \mathrm{C}(A_0, B_0, B_1, B_0), \mathrm{F}(A_1, A_0, A_1, B_0), \mathrm{P}(A_1, B_1, A_0, B_0), \mathrm{P}(B_1, B_0, A_0, B_0)\}_4$, $\{\mathrm{T}(A_1, A_0, A_1, B_1), \mathrm{C}(A_0, A_0, A_1, A_0), \mathrm{F}(A_1, B_0, B_1, A_0), \mathrm{P}(B_1, B_1, B_0, A_0), \mathrm{P}(A_1, B_0, B_0, A_0)\}_4$, $\{\mathrm{T}(A_1, A_0, B_1, B_1), \mathrm{C}(A_0, A_0, A_1, B_0), \mathrm{F}(A_1, B_0, B_1, B_0), \mathrm{F}(A_1, B_0, A_1, A_0), \mathrm{P}(B_1, B_1, B_0, B_0), \mathrm{P}(B_1, B_1, A_0, A_0), \mathrm{P}(A_1, B_0, B_0, B_0), \mathrm{P}(A_1, B_0, A_0, A_0)\}_4$, $\{\mathrm{T}(B_1, A_1, B_1, B_0), \mathrm{C}(B_1, B_0, B_0, B_0), \mathrm{F}(A_0, B_1, A_1, B_0), \mathrm{P}(A_1, A_1, A_0, B_0), \mathrm{P}(B_1, A_0, A_0, B_0)\}_4$, $\{\mathrm{T}(A_1, B_0, A_1, B_1), \mathrm{C}(B_0, A_0, A_1, A_0), \mathrm{F}(B_1, B_0, B_1, A_0), \mathrm{P}(B_1, A_1, B_0, A_0), \mathrm{P}(A_1, A_0, B_0, A_0)\}_4$ we can apply a first or second graph-move in every case.

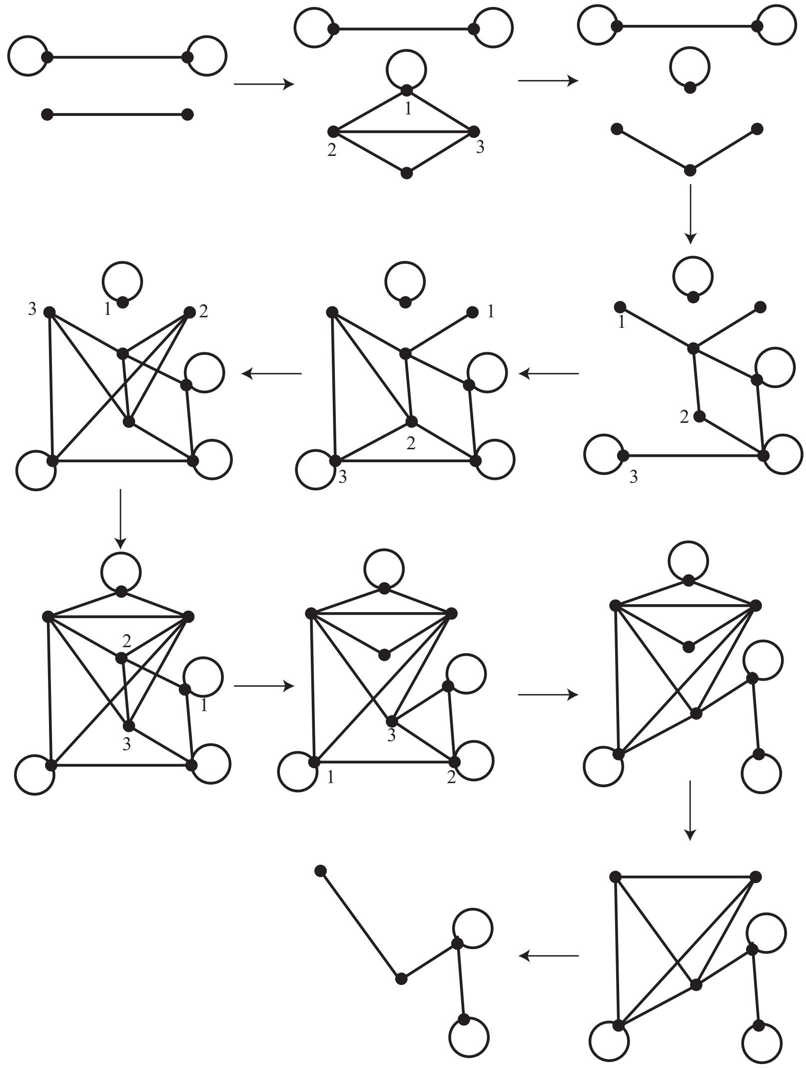

FIGURE 7. Equivalence of two labeled graphs

We have the following tables for graph-links with 2 components (recall that we specify only the Poincaré polynomials for the first oriented linking multiset).

From Tables 2 and 6 we see that the diagrams $\mathrm{GrLk}(3,2)_1$ and $\mathrm{GrLk}(4,2)_{15}$ have the same invariants. The same is true for the pairs $(\mathrm{GrLk}(3,2)_2, \mathrm{GrLk}(4,2)_{12})$, $(\mathrm{GrLk}(3,2)_4, \mathrm{GrLk}(4,2)_{16})$ and $(\mathrm{GrLk}(3,2)_5, \mathrm{GrLk}(4,2)_{11})$. To show that the each pair gives two non-equivalent diagrams we consider the uncoupling operation. It is easy to see that each diagram $\mathrm{GrLk}(3,2)_{1_{uc}}$, $\mathrm{GrLk}(3,2)_{2_{uc}}$, $\mathrm{GrLk}(3,2)_{4_{uc}}$, and $\mathrm{GrLk}(3,2)_{5_{uc}}$ is equivalent to $\mathrm{GrLk}(2,2)_1$, and the diagrams $\mathrm{GrLk}(4,2)_{15_{uc}}$, $\mathrm{GrLk}(4,2)_{12_{uc}}$, $\mathrm{GrLk}(4,2)_{16_{uc}}$ and $\mathrm{GrLk}(4,2)_{11_{uc}}$ are equivalent to $\mathrm{GrLk}(4,2)_{24}$, $\mathrm{GrLk}(4,2)_{26}$, $\mathrm{GrLk}(4,2)_{24}$ and $\mathrm{GrLk}(4,2)_{26}$, respectively.

Let us finally consider 29 remaining diagrams. We get 29 new values of Poincaré polynomials, therefore all of them give new graph-links.

As a result we get 78 graph-links with minimal representatives on 4 vertices and Tables 1– 8.

Graph-link	Representative graphs
$\mathrm{GrLk}(0,1)$	$\bigcirc$
$\mathrm{GrLk}(1,2)$	$\mathrm{P}(A_1)$, $\mathrm{P}(B_1)$
$\mathrm{GrLk}(2,1)_1$	$\mathrm{P}(A_1,A_1)$, $\mathrm{P}(A_0,B_1)$
$\mathrm{GrLk}(2,1)_2$	$\mathrm{P}(B_1,B_1)$, $\mathrm{P}(A_1,B_0)$
$\mathrm{GrLk}(2,2)_1$	$\mathrm{P}(A_0,A_0)$, $\mathrm{P}(B_0,B_0)$
$\mathrm{GrLk}(2,2)_2$	$\mathrm{P}(A_0,B_0)$
$\mathrm{GrLk}(2,3)$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(1,2)$
$\mathrm{GrLk}(3,1)_1$	$\mathrm{P}(A_0,B_0,A_0)$
$\mathrm{GrLk}(3,1)_2$	$\mathrm{P}(B_0,A_0,B_0)$
$\mathrm{GrLk}(3,1)_3$	$\mathrm{P}(A_0,B_0,A_1)$, $\mathrm{P}(A_0,B_1,B_1)$, $\mathrm{C}(A_1,A_1,B_0)$
$\mathrm{GrLk}(3,1)_4$	$\mathrm{P}(A_1,A_1,B_0)$, $\mathrm{P}(B_1,A_0,B_0)$, $\mathrm{C}(A_0,B_1,B_1)$
$\mathrm{GrLk}(3,2)_1$	$\mathrm{P}(A_1,A_0,A_1)$, $\mathrm{P}(B_1,A_1,A_1)$, $\mathrm{C}(B_0,B_1,A_0)$, $\mathrm{P}(B_1,A_0,B_1)$
$\mathrm{GrLk}(3,2)_2$	$\mathrm{P}(A_1,B_0,A_1)$, $\mathrm{P}(B_1,B_1,A_1)$, $\mathrm{C}(B_0,A_1,A_0)$, $\mathrm{P}(B_1,B_0,B_1)$
$\mathrm{GrLk}(3,2)_3$	$\mathrm{P}(A_0,B_1,A_0)$, $\mathrm{C}(A_1,A_1,A_1)$
$\mathrm{GrLk}(3,2)_4$	$\mathrm{P}(A_1,A_0,B_1)$, $\mathrm{P}(B_1,A_1,B_1)$, $\mathrm{P}(A_1,A_1,A_1)$, $\mathrm{C}(B_0,B_1,B_0)$, $\mathrm{C}(A_0,B_1,A_0)$
$\mathrm{GrLk}(3,2)_5$	$\mathrm{P}(A_1,B_0,B_1)$, $\mathrm{P}(B_1,B_1,B_1)$, $\mathrm{P}(A_1,B_1,A_1)$, $\mathrm{C}(B_0,A_1,B_0)$, $\mathrm{C}(A_0,A_1,A_0)$
$\mathrm{GrLk}(3,2)_6$	$\mathrm{P}(B_0,A_1,B_0)$, $\mathrm{C}(B_1,B_1,B_1)$
$\mathrm{GrLk}(3,2)_7$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(2,1)_1$
$\mathrm{GrLk}(3,2)_8$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(2,1)_2$
$\mathrm{GrLk}(3,3)_1$	$\mathrm{C}(A_0,A_0,A_0)$, $\mathrm{C}(B_0,B_0,A_0)$
$\mathrm{GrLk}(3,3)_2$	$\mathrm{C}(A_0,A_0,B_0)$, $\mathrm{C}(B_0,B_0,B_0)$
$\mathrm{GrLk}(3,3)_3$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(2,2)_1$
$\mathrm{GrLk}(3,3)_4$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(2,2)_2$
$\mathrm{GrLk}(3,4)$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(1,2)$

TABLE 1. Graph-links with $\leqslant 3$ vertices

Graph-link	linking multiset	Poincaré polynomial
$\mathrm{GrLk}(0,1)$		1
$\mathrm{GrLk}(1,2)$	$\{1/2\}$, $\{-1/2\}$	$y+xy^2$
$\mathrm{GrLk}(2,1)_1$		$y^2+xy^3+x^2y^5$
$\mathrm{GrLk}(2,1)_2$		$x^{-2}y^{-5}+x^{-1}y^{-3}+y^{-2}$
$\mathrm{GrLk}(2,2)_1$	$\{0\}$	$y^{-1}+y$
$\mathrm{GrLk}(2,2)_2$	$\{1\}$, $\{-1\}$	$y+x^2y^5$
$\mathrm{GrLk}(3,1)_1$		$y^2+x^2y^6+x^3y^8$
$\mathrm{GrLk}(3,1)_2$		$x^{-3}y^{-8}+x^{-2}y^{-6}+y^{-2}$
$\mathrm{GrLk}(3,1)_3$		$x^{-2}y^{-4}+x^{-1}y^{-2}+xy+y^{-1}+1$
$\mathrm{GrLk}(3,1)_4$		$1+x^{-1}y^{-1}+y+xy^2+x^2y^4$
$\mathrm{GrLk}(3,2)_1$	$\{0\}$	$2y+xy^2+x^2y^4$
$\mathrm{GrLk}(3,2)_2$	$\{0\}$	$x^{-2}y^{-4}+x^{-1}y^{-2}+2y^{-1}$
$\mathrm{GrLk}(3,2)_3$	$\{3/2\}$, $\{-3/2\}$	$y^3+xy^4+x^2y^6+x^3y^8$
$\mathrm{GrLk}(3,2)_4$	$\{1\}$, $\{-1\}$	$y^3+xy^4+x^2y^5+x^2y^6$
$\mathrm{GrLk}(3,2)_5$	$\{1\}$, $\{-1\}$	$1+y+xy^2+x^2y^3$
$\mathrm{GrLk}(3,2)_6$	$\{3/2\}$, $\{-3/2\}$	$y+xy^3+x^2y^5+x^3y^6$
$\mathrm{GrLk}(3,3)_1$	$\{-1/2,1/2,1/2\}$, $\{-1/2,-1/2,-1/2\}$	$2+y^2+x^2y^4$
$\mathrm{GrLk}(3,3)_2$	$\{1/2,1/2,1/2\}$, $\{-1/2,-1/2,1/2\}$	$y^2+x^2y^4+2x^2y^6$

TABLE 2. Invariants of graph-links with $\leqslant 3$ vertices

Graph-link	Representative graphs
$\mathrm{GrLk}(4,1)_1$	$\mathrm{S}(B_1,A_0,A_0,A_0)$, $\mathrm{K}(A_1,A_1,A_1,A_1)$
$\mathrm{GrLk}(4,1)_2$	$\mathrm{S}(B_0,A_0,A_0,A_1)$, $\mathrm{S}(B_1,A_0,A_0,B_1)$, $\mathrm{K}(A_1,A_1,B_0,A_1)$
$\mathrm{GrLk}(4,1)_3$	$\mathrm{S}(A_1,A_1,B_0,B_0)$, $\mathrm{S}(A_0,B_1,B_0,B_0)$, $\mathrm{K}(A_0,B_1,B_1,B_1)$
$\mathrm{GrLk}(4,1)_4$	$\mathrm{S}(A_1,B_0,B_0,B_0)$, $\mathrm{K}(B_1,B_1,B_1,B_1)$
$\mathrm{GrLk}(4,1)_5$	$\mathrm{T}(B_1,A_0,B_1,A_0)$, $\mathrm{F}(A_1,A_1,A_1,B_0)$, $\mathrm{P}(A_0,B_1,A_0,B_0)$
$\mathrm{GrLk}(4,1)_6$	$\mathrm{T}(A_1,B_0,A_1,B_0)$, $\mathrm{F}(B_1,B_1,B_1,A_0)$, $\mathrm{P}(B_0,A_1,B_0,A_0)$
$\mathrm{GrLk}(4,1)_7$	$\mathrm{P}(A_0,B_0,A_0,B_0)$, $\mathrm{C}(A_0,B_0,A_0,B_0)$
$\mathrm{GrLk}(4,1)_8$	$\mathrm{T}(B_0,A_0,B_1,A_0)$, $\mathrm{F}(A_1,A_1,A_1,B_1)$, $\mathrm{P}(A_0,B_1,A_0,B_1)$, $\mathrm{F}(A_1,A_1,A_0,A_1)$, $\mathrm{P}(A_0,B_1,A_1,A_1)$, $\mathrm{F}(A_0,B_1,B_0,A_0)$
$\mathrm{GrLk}(4,1)_9$	$\mathrm{T}(A_0,B_0,A_1,B_0)$, $\mathrm{F}(B_1,B_1,B_1,A_1)$, $\mathrm{P}(B_0,A_1,B_0,A_1)$, $\mathrm{F}(B_1,B_1,B_0,B_1)$, $\mathrm{P}(B_0,A_1,B_1,B_1)$, $\mathrm{F}(B_0,A_1,A_0,B_0)$
$\mathrm{GrLk}(4,1)_{10}$	$\mathrm{T}(A_0,A_0,A_1,A_0)$, $\mathrm{F}(A_1,A_1,B_1,A_1)$, $\mathrm{P}(A_0,B_1,B_0,A_1)$, $\mathrm{F}(A_1,A_1,B_0,B_1)$, $\mathrm{P}(A_0,B_1,B_1,B_1)$, $\mathrm{F}(B_0,A_1,B_0,A_0)$, $\mathrm{T}(B_0,A_0,B_1,A_1)$, $\mathrm{C}(A_0,B_1,B_1,B_0)$, $\mathrm{F}(A_1,A_0,A_1,B_1)$, $\mathrm{T}(A_1,A_1,B_0,B_0)$, $\mathrm{P}(A_1,B_1,A_0,B_1)$, $\mathrm{F}(A_1,A_0,A_0,A_1)$, $\mathrm{F}(A_0,B_1,B_1,B_1)$, $\mathrm{P}(B_1,B_0,A_0,B_1)$, $\mathrm{P}(A_1,B_1,A_1,A_1)$, $\mathrm{F}(A_0,B_1,B_0,A_1)$, $\mathrm{P}(B_1,B_0,A_1,A_1)$
$\mathrm{GrLk}(4,1)_{11}$	$\mathrm{T}(A_0,A_0,B_1,A_0)$, $\mathrm{F}(A_1,A_1,A_1,A_1)$, $\mathrm{P}(A_0,B_1,A_0,A_1)$, $\mathrm{F}(A_1,A_1,A_0,B_1)$, $\mathrm{P}(A_0,B_1,A_1,B_1)$, $\mathrm{F}(B_0,B_1,B_0,A_0)$, $\mathrm{T}(B_0,A_1,B_1,B_0)$, $\mathrm{C}(B_1,B_1,B_0,B_0)$, $\mathrm{F}(A_0,B_1,A_1,B_1)$, $\mathrm{P}(A_1,A_1,A_0,B_1)$, $\mathrm{F}(A_0,B_1,A_0,A_1)$, $\mathrm{P}(B_1,A_0,A_0,B_1)$, $\mathrm{P}(A_1,A_1,A_1,A_1)$,
$\mathrm{GrLk}(4,1)_{12}$	$\mathrm{T}(A_1,A_0,B_0,A_0)$, $\mathrm{F}(A_1,A_1,B_1,B_1)$, $\mathrm{P}(A_0,B_1,B_0,B_1)$, $\mathrm{F}(A_1,A_1,B_0,A_1)$, $\mathrm{P}(A_0,B_1,B_1,A_1)$, $\mathrm{F}(A_1,A_0,B_0,A_0)$, $\mathrm{T}(A_1,A_0,B_0,A_1)$, $\mathrm{C}(A_0,A_0,B_1,B_1)$, $\mathrm{F}(A_1,A_0,B_1,B_1)$, $\mathrm{P}(A_1,B_1,B_0,B_1)$, $\mathrm{F}(A_1,A_0,B_0,A_1)$, $\mathrm{P}(B_1,B_0,B_0,B_1)$, $\mathrm{P}(A_1,B_1,B_1,A_1)$
$\mathrm{GrLk}(4,1)_{13}$	$\mathrm{T}(A_0,A_0,A_1,B_1)$, $\mathrm{C}(A_0,A_1,A_1,A_0)$, $\mathrm{F}(A_1,B_0,B_1,A_1)$, $\mathrm{P}(B_1,B_1,B_0,A_1)$, $\mathrm{F}(A_1,B_0,B_0,B_1)$, $\mathrm{P}(A_1,B_0,B_0,A_1)$, $\mathrm{P}(B_1,B_1,B_1,B_1)$, $\mathrm{T}(A_1,B_0,B_0,B_0)$, $\mathrm{F}(B_1,B_1,B_1,B_1)$, $\mathrm{P}(B_0,A_1,B_0,B_1)$, $\mathrm{F}(B_1,B_1,B_0,A_1)$, $\mathrm{P}(B_0,A_1,B_1,A_1)$, $\mathrm{F}(A_1,A_0,A_0,B_0)$
$\mathrm{GrLk}(4,1)_{14}$	$\mathrm{T}(A_0,A_0,B_1,B_1)$, $\mathrm{C}(A_0,A_1,A_1,B_0)$, $\mathrm{F}(A_1,B_0,A_1,A_1)$, $\mathrm{T}(A_1,B_1,A_0,B_0)$, $\mathrm{P}(B_1,B_1,A_0,A_1)$, $\mathrm{F}(A_1,B_0,A_0,B_1)$, $\mathrm{F}(B_0,B_1,B_1,A_1)$, $\mathrm{P}(A_1,B_0,A_0,A_1)$, $\mathrm{P}(B_1,B_1,A_1,B_1)$, $\mathrm{F}(B_0,B_1,B_0,B_1)$, $\mathrm{P}(A_1,B_0,A_1,B_1)$, $\mathrm{T}(B_0,B_0,B_1,B_0)$, $\mathrm{F}(B_1,B_1,A_1,B_1)$, $\mathrm{P}(B_0,A_1,A_0,B_1)$, $\mathrm{F}(B_1,B_1,A_0,A_1)$, $\mathrm{P}(B_0,A_1,A_1,A_1)$, $\mathrm{F}(A_0,B_1,A_0,B_0)$
$\mathrm{GrLk}(4,1)_{15}$	$\mathrm{T}(A_0,B_0,B_1,B_0)$, $\mathrm{F}(B_1,B_1,A_1,A_1)$, $\mathrm{P}(B_0,A_1,A_0,A_1)$, $\mathrm{F}(B_1,B_1,A_0,B_1)$, $\mathrm{P}(B_0,A_1,A_1,B_1)$, $\mathrm{F}(B_0,B_1,A_0,B_0)$, $\mathrm{T}(A_0,B_0,B_1,B_1)$, $\mathrm{C}(B_0,A_1,A_1,B_0)$, $\mathrm{F}(B_1,B_0,A_1,A_1)$, $\mathrm{P}(B_1,A_1,A_0,A_1)$, $\mathrm{F}(B_1,B_0,A_0,B_1)$, $\mathrm{P}(A_1,A_0,A_0,A_1)$, $\mathrm{P}(B_1,A_1,A_1,B_1)$
$\mathrm{GrLk}(4,1)_{16}$	$\mathrm{GrLk}(2,1)_1 \sqcup \mathrm{GrLk}(2,1)_1$
$\mathrm{GrLk}(4,1)_{17}$	$\mathrm{GrLk}(2,1)_1 \sqcup \mathrm{GrLk}(2,1)_2$, $\mathrm{T}(B_0,A_0,B_1,B_1)$, $\mathrm{C}(A_0,B_1,A_1,B_0)$, $\mathrm{F}(A_1,B_0,A_1,B_1)$, $\mathrm{T}(A_1,A_1,A_0,B_0)$, $\mathrm{P}(B_1,B_1,A_0,B_1)$, $\mathrm{F}(A_1,B_0,A_0,A_1)$, $\mathrm{F}(A_0,B_1,B_1,A_1)$, $\mathrm{P}(A_1,B_0,A_0,B_1)$, $\mathrm{P}(B_1,B_1,A_1,A_1)$, $\mathrm{F}(A_0,B_1,B_0,B_1)$, $\mathrm{P}(A_1,B_0,A_1,A_1)$
$\mathrm{GrLk}(4,1)_{18}$	$\mathrm{GrLk}(2,1)_2 \sqcup \mathrm{GrLk}(2,1)_2$

Table 3: Graph-knots with 4 vertices

Graph-link	Poincaré polynomial
$\mathrm{GrLk}(4,1)_1$	$x^4y^{11}+x^3y^9+x^2y^7+xy^5+y^4$
$\mathrm{GrLk}(4,1)_2$	$x^{-4}y^{-9}+x^{-3}y^{-7}+x^{-2}y^{-6}+x^{-2}y^{-5}+x^{-1}y^{-4}+x^{-1}y^{-3}+y^{-2}$
$\mathrm{GrLk}(4,1)_3$	$x^4y^9+x^3y^7+x^2y^6+x^2y^5+xy^4+xy^3+y^2$
$\mathrm{GrLk}(4,1)_4$	$x^{-4}y^{-11}+x^{-3}y^{-9}+x^{-2}y^{-7}+x^{-1}y^{-5}+y^{-4}$
$\mathrm{GrLk}(4,1)_5$	$x^{-4}y^{-9}+x^{-3}y^{-8}+x^{-3}y^{-7}+2x^{-2}y^{-6}+x^{-1}y^{-4}+y^{-2}$
$\mathrm{GrLk}(4,1)_6$	$xy^4+2x^2y^6+x^3y^7+x^3y^8+x^4y^9+y^2$
$\mathrm{GrLk}(4,1)_7$	$x^{-2}y^{-4}+x^{-1}y^{-2}+xy^2+x^2y^4+1$
$\mathrm{GrLk}(4,1)_8$	$x^3y^7+2x^2y^5+xy^4+xy^3+2y^2$
$\mathrm{GrLk}(4,1)_9$	$x^{-3}y^{-7}+2x^{-2}y^{-5}+x^{-1}y^{-4}+x^{-1}y^{-3}+2y^{-2}$
$\mathrm{GrLk}(4,1)_{10}$	$x^3y^6+x^2y^5+x^2y^4+2xy^3+y^2+y$
$\mathrm{GrLk}(4,1)_{11}$	$x^3y^8+x^2y^7+x^2y^6+xy^5+y^4$
$\mathrm{GrLk}(4,1)_{12}$	$x^{-2}y^{-3}+xy^2+x^{-1}y^{-1}+2$
$\mathrm{GrLk}(4,1)_{13}$	$x^{-3}y^{-8}+x^{-2}y^{-7}+x^{-2}y^{-6}+x^{-1}y^{-5}+y^{-4}$
$\mathrm{GrLk}(4,1)_{14}$	$x^{-3}y^{-6}+x^{-2}y^{-5}+x^{-2}y^{-4}+2x^{-1}y^{-3}+y^{-2}+y^{-1}$
$\mathrm{GrLk}(4,1)_{15}$	$x^{-1}y^{-2}+x^2y^3+xy+2$

TABLE 4. Invariants of graph-knots with 4 vertices

Graph-link	Representative graphs
$\mathrm{GrLk}(4,2)_1$	$\mathrm{S}(B_0,A_0,A_0,A_0)$
$\mathrm{GrLk}(4,2)_2$	$\mathrm{S}(B_0,A_0,A_1,A_1)$, $\mathrm{S}(B_1,A_0,B_1,A_1)$, $\mathrm{S}(B_0,A_0,B_1,B_1)$, $\mathrm{K}(A_1,B_0,A_0,A_1)$
$\mathrm{GrLk}(4,2)_3$	$\mathrm{S}(B_1,A_0,A_1,A_1)$, $\mathrm{S}(B_0,A_0,B_1,A_1)$, $\mathrm{K}(A_1,A_0,A_0,A_1)$, $\mathrm{S}(B_1,A_0,B_1,B_1)$, $\mathrm{K}(A_1,B_0,B_0,A_1)$
$\mathrm{GrLk}(4,2)_4$	$\mathrm{S}(A_0,A_1,A_1,B_0)$, $\mathrm{S}(A_1,B_1,A_1,B_0)$, $\mathrm{S}(A_0,B_1,B_1,B_0)$, $\mathrm{K}(B_0,A_0,B_1,B_1)$
$\mathrm{GrLk}(4,2)_5$	$\mathrm{S}(A_1,A_1,A_1,B_0)$, $\mathrm{S}(A_0,B_1,A_1,B_0)$, $\mathrm{K}(A_0,A_0,B_1,B_1)$, $\mathrm{S}(A_1,B_1,B_1,B_0)$, $\mathrm{K}(B_0,B_0,B_1,B_1)$
$\mathrm{GrLk}(4,2)_6$	$\mathrm{S}(A_0,B_0,B_0,B_0)$
$\mathrm{GrLk}(4,2)_7$	$\mathrm{T}(A_0,A_0,A_0,A_0)$, $\mathrm{F}(B_0,A_0,B_0,A_0)$, $\mathrm{T}(B_0,A_0,B_0,A_0)$
$\mathrm{GrLk}(4,2)_8$	$\mathrm{T}(A_0,A_0,B_0,A_0)$, $\mathrm{F}(B_0,B_0,B_0,A_0)$, $\mathrm{F}(A_0,A_0,B_0,A_0)$
$\mathrm{GrLk}(4,2)_9$	$\mathrm{T}(A_0,A_0,A_0,A_1)$, $\mathrm{C}(A_0,A_1,B_1,A_1)$, $\mathrm{T}(A_1,B_1,B_0,A_1)$, $\mathrm{F}(B_0,A_0,B_1,B_1)$, $\mathrm{C}(A_0,B_1,B_1,B_1)$, $\mathrm{F}(B_0,A_0,B_0,A_1)$, $\mathrm{T}(B_0,A_0,B_0,A_1)$
$\mathrm{GrLk}(4,2)_{10}$	$\mathrm{T}(A_0,A_0,B_0,A_1)$, $\mathrm{C}(A_0,A_1,B_1,B_1)$, $\mathrm{T}(A_1,B_1,B_0,B_1)$, $\mathrm{T}(A_1,A_1,B_0,A_1)$, $\mathrm{F}(B_0,B_0,B_1,B_1)$, $\mathrm{F}(A_0,A_0,B_1,B_1)$, $\mathrm{F}(B_0,B_0,B_0,A_1)$, $\mathrm{F}(A_0,A_0,B_0,A_1)$
$\mathrm{GrLk}(4,2)_{11}$	$\mathrm{T}(A_0,A_0,A_0,B_1)$, $\mathrm{C}(A_0,A_1,A_1,A_1)$, $\mathrm{T}(A_1,B_1,A_0,A_1)$, $\mathrm{F}(B_0,A_0,B_1,A_1)$, $\mathrm{C}(A_0,B_1,A_1,B_1)$, $\mathrm{F}(B_0,A_0,B_0,B_1)$, $\mathrm{T}(B_0,A_0,B_0,B_1)$
$\mathrm{GrLk}(4,2)_{12}$	$\mathrm{T}(A_0,A_0,B_0,B_1)$, $\mathrm{C}(A_0,A_1,A_1,B_1)$, $\mathrm{T}(A_1,B_1,A_0,B_1)$, $\mathrm{T}(A_1,A_1,A_0,A_1)$, $\mathrm{F}(B_0,B_0,B_1,A_1)$, $\mathrm{F}(A_0,A_0,B_1,A_1)$, $\mathrm{F}(B_0,B_0,B_0,B_1)$, $\mathrm{F}(A_0,A_0,B_0,B_1)$
$\mathrm{GrLk}(4,2)_{13}$	$\mathrm{T}(B_1,A_0,B_1,B_1)$, $\mathrm{C}(A_0,B_0,A_1,B_0)$, $\mathrm{F}(A_1,B_0,A_1,B_0)$, $\mathrm{P}(B_1,B_1,A_0,B_0)$, $\mathrm{P}(A_1,B_0,A_0,B_0)$
$\mathrm{GrLk}(4,2)_{14}$	$\mathrm{T}(A_0,A_1,B_1,A_1)$, $\mathrm{C}(B_1,A_1,A_1,B_0)$, $\mathrm{F}(A_0,A_0,A_1,A_1)$, $\mathrm{T}(A_0,B_1,B_1,B_1)$, $\mathrm{T}(B_0,B_1,A_0,B_0)$, $\mathrm{F}(A_0,A_0,A_0,B_1)$, $\mathrm{F}(B_0,B_0,A_1,A_1)$, $\mathrm{F}(B_0,B_0,A_0,B_1)$
$\mathrm{GrLk}(4,2)_{15}$	$\mathrm{T}(B_0,A_1,B_1,A_1)$, $\mathrm{C}(B_1,B_1,A_1,B_0)$, $\mathrm{F}(A_0,A_0,A_1,B_1)$, $\mathrm{T}(B_0,B_1,B_1,B_1)$, $\mathrm{T}(B_0,A_1,A_0,B_0)$, $\mathrm{F}(A_0,A_0,A_0,A_1)$, $\mathrm{F}(B_0,B_0,A_1,B_1)$, $\mathrm{F}(B_0,B_0,A_0,A_1)$
$\mathrm{GrLk}(4,2)_{16}$	$\mathrm{T}(A_0,A_1,A_0,B_0)$, $\mathrm{C}(B_1,A_1,B_0,A_1)$, $\mathrm{T}(B_0,B_1,B_1,A_1)$, $\mathrm{F}(B_0,A_0,A_1,B_1)$, $\mathrm{C}(B_1,B_1,B_0,B_1)$, $\mathrm{F}(B_0,A_0,A_0,A_1)$, $\mathrm{T}(B_0,A_1,B_0,B_0)$
$\mathrm{GrLk}(4,2)_{17}$	$\mathrm{T}(A_1,A_1,A_1,B_0)$, $\mathrm{C}(B_1,A_0,B_0,A_0)$, $\mathrm{F}(A_0,B_1,B_1,A_0)$, $\mathrm{P}(A_1,A_1,B_0,A_0)$, $\mathrm{P}(B_1,A_0,B_0,A_0)$
$\mathrm{GrLk}(4,2)_{18}$	$\mathrm{T}(A_0,A_1,B_1,B_1)$, $\mathrm{C}(B_1,A_1,B_1,B_0)$, $\mathrm{C}(A_1,A_1,A_1,B_0)$, $\mathrm{F}(A_0,B_0,A_1,A_1)$, $\mathrm{T}(B_0,B_1,B_0,B_0)$, $\mathrm{T}(A_0,B_1,A_0,B_0)$, $\mathrm{F}(A_0,B_0,A_0,B_1)$
$\mathrm{GrLk}(4,2)_{19}$	$\mathrm{T}(A_0,B_0,A_0,B_0)$, $\mathrm{F}(B_0,A_0,A_0,B_0)$, $\mathrm{T}(B_0,B_0,B_0,B_0)$
$\mathrm{GrLk}(4,2)_{20}$	$\mathrm{T}(A_0,B_0,B_0,B_0)$, $\mathrm{F}(B_0,B_0,A_0,B_0)$, $\mathrm{F}(A_0,A_0,A_0,B_0)$
$\mathrm{GrLk}(4,2)_{21}$	$\mathrm{P}(A_0,B_1,A_1,B_0)$, $\mathrm{F}(A_1,A_1,A_0,B_0)$, $\mathrm{F}(B_1,B_1,B_0,A_0)$
$\mathrm{GrLk}(4,2)_{22}$	$\mathrm{P}(A_0,B_1,B_1,A_0)$, $\mathrm{F}(A_1,A_1,B_0,A_0)$
$\mathrm{GrLk}(4,2)_{23}$	$\mathrm{P}(B_0,A_1,A_1,B_0)$, $\mathrm{F}(B_1,B_1,A_0,B_0)$
$\mathrm{GrLk}(4,2)_{24}$	$\mathrm{GrLk}(2,1)_1 \sqcup \mathrm{GrLk}(2,2)_1$
$\mathrm{GrLk}(4,2)_{25}$	$\mathrm{GrLk}(2,1)_1 \sqcup \mathrm{GrLk}(2,2)_2$
$\mathrm{GrLk}(4,2)_{26}$	$\mathrm{GrLk}(2,1)_2 \sqcup \mathrm{GrLk}(2,2)_1$
$\mathrm{GrLk}(4,2)_{27}$	$\mathrm{GrLk}(2,1)_2 \sqcup \mathrm{GrLk}(2,2)_2$
$\mathrm{GrLk}(4,2)_{28}$	$\mathrm{GrLk}(3,1)_1 \sqcup \mathrm{GrLk}(1,2)$
$\mathrm{GrLk}(4,2)_{29}$	$\mathrm{GrLk}(3,1)_2 \sqcup \mathrm{GrLk}(1,2)$
$\mathrm{GrLk}(4,2)_{30}$	$\mathrm{GrLk}(3,1)_3 \sqcup \mathrm{GrLk}(1,2)$
$\mathrm{GrLk}(4,2)_{31}$	$\mathrm{GrLk}(3,1)_4 \sqcup \mathrm{GrLk}(1,2)$

Table 5: Graph-links with 4 vertices and 2 components

Graph-link	linking multiset	Poincaré polynomial
$\mathrm{GrLk}(4,2)_1$	$\{2\}$, $\{-2\}$	$x^2y^7 + x^3y^9 + x^4y^{11} + y^3$
$\mathrm{GrLk}(4,2)_2$	$\{1\}$, $\{-1\}$	$x^{-2}y^{-3} + x^{-1}y^{-1} + 2xy^2 + x^2y^3 + y + 2$
$\mathrm{GrLk}(4,2)_3$	$\{0\}$	$x^{-2}y^{-5} + x^{-2}y^{-4} + x^{-1}y^{-3} + x^{-1}y^{-2} + y^{-2} + 2y^{-1} + x$
$\mathrm{GrLk}(4,2)_4$	$\{1\}$, $\{-1\}$	$2xy^4 + x^2y^5 + 2x^2y^6 + x^3y^7 + x^4y^9 + y^3$
$\mathrm{GrLk}(4,2)_5$	$\{0\}$	$xy^2 + xy^3 + x^2y^4 + x^2y^5 + x^{-1} + y^2 + 2y$
$\mathrm{GrLk}(4,2)_6$	$\{2\}$, $\{-2\}$	$xy^3 + x^2y^5 + x^4y^9 + y$
$\mathrm{GrLk}(4,2)_7$	$\{0\}$	$x^2y^5 + x^3y^7 + 2y$
$\mathrm{GrLk}(4,2)_8$	$\{1\}$, $\{-1\}$	$x^2y^5 + x^2y^7 + x^3y^9 + y^3$
$\mathrm{GrLk}(4,2)_9$	$\{1\}$, $\{-1\}$	$xy^3 + x^2y^4 + x^2y^5 + x^3y^6 + 2y$
$\mathrm{GrLk}(4,2)_{10}$	$\{0\}$	$x^{-2}y^{-3} + x^{-1}y^{-1} + xy^2 + y^{-1} + y + 1$
$\mathrm{GrLk}(4,2)_{11}$	$\{1\}$, $\{-1\}$	$xy^2 + x^2y^3 + y + 1$
$\mathrm{GrLk}(4,2)_{12}$	$\{0\}$	$x^{-2}y^{-4} + x^{-1}y^{-2} + 2y^{-1}$
$\mathrm{GrLk}(4,2)_{13}$	$\{1/2\}$, $\{-1/2\}$	$x^{-2}y^{-4} + 2x^{-1}y^{-2} + xy^2 + x^2y^3 + y^{-1} + xy + 1$
$\mathrm{GrLk}(4,2)_{14}$	$\{0\}$	$x^{-1}y^{-2} + x^2y^3 + y^{-1} + xy + y + 1$
$\mathrm{GrLk}(4,2)_{15}$	$\{0\}$	$xy^2 + x^2y^4 + 2y$
$\mathrm{GrLk}(4,2)_{16}$	$\{1\}$, $\{-1\}$	$xy^4 + x^2y^5 + x^2y^6 + y^3$
$\mathrm{GrLk}(4,2)_{17}$	$\{1/2\}$, $\{-1/2\}$	$xy^3 + xy^4 + 2x^2y^5 + x^3y^7 + x^{-1} + y^2 + y$
$\mathrm{GrLk}(4,2)_{18}$	$\{1\}$, $\{-1\}$	$xy^3 + 2x^2y^5 + x^{-1} + y^2 + y$
$\mathrm{GrLk}(4,2)_{19}$	$\{0\}$	$x^{-3}y^{-7} + x^{-2}y^{-5} + 2y^{-1}$
$\mathrm{GrLk}(4,2)_{20}$	$\{1\}$, $\{-1\}$	$x^{-1}y^{-3} + x^2y^3 + y^{-1} + y$
$\mathrm{GrLk}(4,2)_{21}$	$\{2\}$, $\{-2\}$	$xy^4 + 2x^2y^6 + x^3y^8 + x^4y^9 + y^3$
$\mathrm{GrLk}(4,2)_{22}$	$\{0\}$	$x^{-2}y^{-3} + x^{-1}y^{-1} + 2xy^2 + x^2y^4 + 2y + 1$
$\mathrm{GrLk}(4,2)_{23}$	$\{0\}$	$x^{-2}y^{-4} + 2x^{-1}y^{-2} + x^2y^3 + 2y^{-1} + xy + 1$

Table 6. Invariants of graph-links with 4 vertices and 2 components

Graph-link	Representative graphs
$\mathrm{GrLk}(4,3)_1$	$\mathrm{S}(A_0,A_1,A_1,A_1)$, $\mathrm{S}(A_1,B_1,A_1,A_1)$, $\mathrm{S}(A_0,B_1,B_1,A_1)$, $\mathrm{K}(B_0,A_0,A_0,B_1)$, $\mathrm{S}(A_1,B_1,B_1,B_1)$, $\mathrm{K}(B_0,B_0,B_0,B_1)$
$\mathrm{GrLk}(4,3)_2$	$\mathrm{S}(A_1,A_1,A_1,A_1)$, $\mathrm{S}(A_0,B_1,A_1,A_1)$, $\mathrm{K}(A_0,A_0,A_0,B_1)$, $\mathrm{S}(A_1,B_1,B_1,A_1)$, $\mathrm{S}(A_0,B_1,B_1,B_1)$, $\mathrm{K}(B_0,B_0,A_0,B_1)$
$\mathrm{GrLk}(4,3)_3$	$\mathrm{S}(B_0,A_1,A_1,A_1)$, $\mathrm{S}(B_1,B_1,A_1,A_1)$, $\mathrm{S}(B_0,B_1,B_1,A_1)$, $\mathrm{K}(B_0,A_0,A_0,A_1)$, $\mathrm{S}(B_1,B_1,B_1,B_1)$, $\mathrm{K}(B_0,B_0,B_0,A_1)$
$\mathrm{GrLk}(4,3)_4$	$\mathrm{S}(B_1,A_1,A_1,A_1)$, $\mathrm{S}(B_0,B_1,A_1,A_1)$, $\mathrm{K}(A_0,A_0,A_0,A_1)$, $\mathrm{S}(B_1,B_1,B_1,A_1)$, $\mathrm{S}(B_0,B_1,B_1,B_1)$, $\mathrm{K}(B_0,B_0,A_0,A_1)$
$\mathrm{GrLk}(4,3)_5$	$\mathrm{T}(A_0,A_1,A_0,A_1)$, $\mathrm{C}(B_1,A_1,A_1,A_1)$, $\mathrm{T}(A_0,B_1,A_0,B_1)$, $\mathrm{T}(B_0,B_1,A_0,A_1)$, $\mathrm{C}(B_1,B_1,A_1,B_1)$, $\mathrm{T}(B_0,A_1,B_0,A_1)$, $\mathrm{T}(B_0,B_1,B_0,B_1)$
$\mathrm{GrLk}(4,3)_6$	$\mathrm{T}(A_0,A_1,B_0,A_1)$, $\mathrm{C}(B_1,A_1,A_1,B_1)$, $\mathrm{T}(A_0,B_1,B_0,B_1)$
$\mathrm{GrLk}(4,3)_7$	$\mathrm{T}(A_1,A_1,A_1,A_1)$, $\mathrm{C}(B_1,A_0,A_1,A_0)$, $\mathrm{F}(A_0,A_0,B_1,A_0)$, $\mathrm{T}(A_1,B_1,A_1,B_1)$, $\mathrm{F}(B_0,B_0,B_1,A_0)$
$\mathrm{GrLk}(4,3)_8$	$\mathrm{T}(B_1,A_1,B_1,A_1)$, $\mathrm{C}(B_1,B_0,A_1,B_0)$, $\mathrm{F}(A_0,A_0,A_1,B_0)$, $\mathrm{T}(B_1,B_1,B_1,B_1)$, $\mathrm{F}(B_0,B_0,A_1,B_0)$
$\mathrm{GrLk}(4,3)_9$	$\mathrm{T}(A_0,A_1,A_0,B_1)$, $\mathrm{C}(B_1,A_1,B_1,A_1)$, $\mathrm{C}(A_1,A_1,A_1,A_1)$, $\mathrm{T}(B_0,B_1,B_0,A_1)$, $\mathrm{C}(B_1,B_1,B_1,B_1)$
$\mathrm{GrLk}(4,3)_{10}$	$\mathrm{T}(A_1,A_1,A_1,B_1)$, $\mathrm{C}(B_1,A_0,B_1,A_0)$, $\mathrm{C}(A_1,A_0,A_1,A_0)$, $\mathrm{F}(A_0,B_0,B_1,A_0)$
$\mathrm{GrLk}(4,3)_{11}$	$\mathrm{T}(B_1,A_1,B_1,B_1)$, $\mathrm{C}(B_1,B_0,B_1,B_0)$, $\mathrm{C}(A_1,B_0,A_1,B_0)$, $\mathrm{F}(A_0,B_0,A_1,B_0)$
$\mathrm{GrLk}(4,3)_{12}$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(3,2)_1$
$\mathrm{GrLk}(4,3)_{13}$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(3,2)_2$
$\mathrm{GrLk}(4,3)_{14}$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(3,2)_3$
$\mathrm{GrLk}(4,3)_{15}$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(3,2)_4$
$\mathrm{GrLk}(4,3)_{16}$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(3,2)_5$
$\mathrm{GrLk}(4,3)_{17}$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(3,2)_6$
$\mathrm{GrLk}(4,3)_{18}$	$\mathrm{GrLk}(2,2)_1\sqcup\mathrm{GrLk}(2,2)_1$
$\mathrm{GrLk}(4,3)_{19}$	$\mathrm{GrLk}(2,2)_1\sqcup\mathrm{GrLk}(2,2)_2$
$\mathrm{GrLk}(4,3)_{20}$	$\mathrm{GrLk}(2,2)_2\sqcup\mathrm{GrLk}(2,2)_2$
$\mathrm{GrLk}(4,3)_{21}$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(2,1)_1$
$\mathrm{GrLk}(4,3)_{22}$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(2,1)_2$
$\mathrm{GrLk}(4,4)_1$	$\mathrm{K}(A_0,A_0,A_0,A_0)$, $\mathrm{K}(B_0,B_0,A_0,A_0)$, $\mathrm{K}(B_0,B_0,B_0,B_0)$
$\mathrm{GrLk}(4,4)_2$	$\mathrm{K}(A_0,A_0,A_0,B_0)$, $\mathrm{K}(B_0,B_0,B_0,A_0)$
$\mathrm{GrLk}(4,4)_3$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(3,3)_1$
$\mathrm{GrLk}(4,4)_4$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(3,3)_2$
$\mathrm{GrLk}(4,4)_5$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(2,2)_1$
$\mathrm{GrLk}(4,4)_6$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(2,2)_2$
$\mathrm{GrLk}(4,5)$	$\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(1,2)\sqcup\mathrm{GrLk}(1,2)$

Table 7: Graph-links with 4 vertices and $\geqslant 3$ components

Graph-link	linking multiset	Poincaré polynomial
$\mathrm{GrLk}(4,3)_1$	$\{-1/2, 1/2, 1/2\}$, $\{-1/2, -1/2, -1/2\}$	$2xy^3 + x^2y^4 + 2x^2y^5 + 3y^2$
$\mathrm{GrLk}(4,3)_2$	$\{1/2, 1/2, 1/2\}$, $\{-1/2, -1/2, 1/2\}$	$xy^5 + 3x^2y^6 + x^2y^7 + x^3y^7 + x^4y^9 + y^4$
$\mathrm{GrLk}(4,3)_3$	$\{-1/2, 1/2, 1/2\}$, $\{-1/2, -1/2, -1/2\}$	$x^{-2}y^{-3} + x^{-1}y^{-1} + x^2y^2 + y^{-1} + xy + 3$
$\mathrm{GrLk}(4,3)_4$	$\{1/2, 1/2, 1/2\}$, $\{-1/2, -1/2, 1/2\}$	$2xy^3 + 3x^2y^4 + y^2 + 2y$
$\mathrm{GrLk}(4,3)_5$	$\{0, 1\}$, $\{0, -1\}$	$2xy^3 + 2x^2y^4 + x^2y^5 + 2y^2 + y$
$\mathrm{GrLk}(4,3)_6$	$\{0, 0\}$	$x^{-2}y^{-3} + x^{-1}y^{-1} + x^2y^3 + xy + 4$
$\mathrm{GrLk}(4,3)_7$	$\{1/2, 1/2, 1\}$, $\{-1/2, 1/2, -1\}$, $\{-1/2, -1/2, 1\}$	$xy^5 + x^2y^6 + x^2y^7 + x^3y^7 + x^3y^9 + y^4$
$\mathrm{GrLk}(4,3)_8$	$\{-1/2, 1/2, 1\}$, $\{1/2, 1/2, -1\}$, $\{-1/2, -1/2, -1\}$	$xy^2 + xy^3 + x^2y^4 + x^3y^5 + y^2 + 1$
$\mathrm{GrLk}(4,3)_9$	$\{1, 1\}$, $\{-1, 1\}$, $\{-1, -1\}$	$xy^5 + x^2y^5 + 2x^2y^6 + x^2y^7 + x^3y^7 + x^4y^8 + y^4$
$\mathrm{GrLk}(4,3)_{10}$	$\{-1/2, 1/2, 1\}$, $\{1/2, 1/2, -1\}$, $\{-1/2, -1/2, -1\}$	$2xy^3 + x^2y^5 + x^3y^7 + 2y^2$
$\mathrm{GrLk}(4,3)_{11}$	$\{1/2, 1/2, 1\}$, $\{-1/2, -1/2, 1\}$, $\{-1/2, 1/2, -1\}$	$xy^4 + 2x^2y^6 + 2x^3y^7 + y^2$
$\mathrm{GrLk}(4,4)_1$	$\{-1/2, 1/2, 1/2, 1/2\}$, $\{-1/2, -1/2, -1/2, 1/2\}$	$x^2y^3 + 3x^2y^5 + y^3 + 3y$
$\mathrm{GrLk}(4,4)_2$	$\{1/2, 1/2, 1/2, 1/2\}$, $\{-1/2, -1/2, 1/2, 1/2\}$, $\{-1/2, -1/2, -1/2, -1/2\}$	$3x^2y^5 + 3x^2y^7 + x^4y^9 + y^3$

Table 8. Invariants of graph-links with 4 vertices and $\geqslant 3$ components

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