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ESTIMATING THE UNIFORM EQUICONVERGENCE RATE FOR THE SCHRÖDINGER OPERATOR WITH SUMMABLE POTENTIAL ON A COMPACT SET

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Abstract. One-dimensional Schrödinger operator is considered on the interval $G=(0,1)$. It is assumed that the potential is a complex-valued function of the class $L_r(G)$, $r\geq 1$. Uniform equiconvergence estimates are obtained depending on r and p, where $p\geq 1$ is the order of summability of the expanded function $f(x)$.

1. Introduction and statement of results

Consider the one-dimensional Schrödinger operator

$$Lu=-u''+q(x)\,u$$

with a complex-valued potential $q(x)\in L_r(G)$, $r\geq 1$, on the interval $G=(0,1)$. Root functions (eigen- and associated functions) of the operator L are understood in a generalized sense, i.e. irrespective of boundary conditions (see [1,2]).

In [1,2], a constructive necessary and sufficient condition is obtained for the uniform equiconvergence of expansion in eigen- and associated functions of a differential operator of arbitrary order (with smooth coefficients) with a trigonometric Fourier series on a compact $K\subset G$. In [5,6] (see also the references therein), for the expanded function $f(x)\in L_p(G)$, $1\leq p\leq\infty$, the equiconvergence rate is studied in L_s- metrics, $1\leq s\leq\infty$ (in case $s=\infty$, it is about a uniform equiconvergence) on a compact, for a differential operator with summable coefficients. The order of summability of the expanded function does not appear in some of the obtained equiconvergence rate estimates, i.e. the estimate stays the same for every value of p, $1\leq p\leq\infty$.

The rate of convergence in metrics L_p, $1\leq p<\infty$, under certain additional conditions regarding the decreasing order of biorthogonal coefficients and the norms of eigenfunctions and associated functions, has been studied in [9] and [10]. The rate of uniform convergence on a compact set for an nth-order differential operator with non-zero coefficients of the $(n-1)$-th derivative and two-point regular boundary conditions is investigated in [11–13]. Estimates of uniform convergence have been obtained in terms of the integral moduli of continuity of the expanded function and the coefficient of the $(n-1)$-th derivative.

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We will deal with the uniform equiconvergence rate of spectral (biorthogonal) expansion of the function $f(x) \in L_p(G)$, $p \in [1,\infty]$, in root functions of the operator L with trigonometric Fourier series on the compact $K \subset G$. More precisely, we will study the impact of summability order p of the expanded function and summability order r of the potential on the uniform equiconvergence rate.

Let $\{u_k(x)\}_{k=1}^{\infty}$ be an arbitrary system consisting of eigen- and associated functions (root functions) of the operator L and $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding system of eigenvalues. We require that, together with each associated function of order $s, s \geq 1$, the system $\{u_k(x)\}_{k=1}^{\infty}$ also contains the corresponding eigenfunctions and all associated functions of order less than s. This means that each element of the system $\{u_k(x)\}_{k=1}^{\infty}$ is identically non-zero, is absolutely continuous together with its first order derivative on $\overline{G}$, and satisfies the equation $Lu_k + \lambda_k u_k = \theta_k u_{k-1}$ almost everywhere in G, where θ_k equals either 0 (in this case $u_k(x)$ is an eigenfunction) or 1 (in this case $u_k(x)$ is an associated function of order $l \geq 1$, and $u_{k-1}(x)$ is an associated function of order $l-1$, $\lambda_k = \lambda_{k-1} = \ldots = \lambda_{k-l}$, $\theta_k = \theta_{k-1} = \theta_{k-l+1} = 1$ with $\theta_{k-l} = 0$).

Let the system $\{u_k(x)\}_{k=1}^{\infty}$ and the system of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ of the operator L satisfy the conditions A_p (Il'in conditions):

1) $\{u_k(x)\}_{k=1}^{\infty}$ is closed and minimal in $L_p(G)$ for some fixed p, $p \geq 1$.

2) Carleman's condition and the "sum of units" condition are satisfied:

$$|Im\,\mu_k| \leq const,\ k = 1,2,\ldots; \tag{1.1}$$

$$\sum_{\tau \leq \rho_k \leq \tau+1} 1 \ \leq const,\ \forall \tau \geq 0, \tag{1.2}$$

where $\mu_k^2 = \lambda_k$, $\rho_k = Re\,\mu_k \geq 0$;

3) for every compact $K \subset G$ there exists a constant $C_0(K)$ such that

$$\|u_k\|_{p,K} \|\upsilon_k\|_q \leq C_0(K), \quad k = 1,2,\ldots, \tag{1.3}$$

where $p^{-1}+q^{-1} = 1$, $\{\upsilon_k(x)\}_{k=1}^{\infty}$ is a biorthogonally conjugate system to $\{u_k(x)\}_{k=1}^{\infty}$ consisting of root functions of the formally conjugate operator $L^* = -\frac{d^2}{dx^2} + \overline{q(x)}$ (i.e. $L^*\upsilon_k = \overline{\lambda_k}\,\upsilon_k - \theta_k\,\upsilon_{k+1}$, $\theta_k = 0$ or $\theta_k = 1$); $\|\cdot\|_{p,K} = \|\cdot\|_{L_p(K)}$, $\|\cdot\|_p = \|\cdot\|_{L_p(G)}$.

Denote

$$\sigma_\nu(x,f) = \sum_{\rho_k \leq \nu} f_k u_k(x), \ \ \nu > 0,\ f_k = (f,\upsilon_k) = \int_0^1 f(x)\, \overline{\upsilon_k}(x)\ dx$$

$$\Delta_\nu(f,K) = \|\sigma_\nu(\cdot,f) - S_\nu\ (\cdot,\ f)\|_{C(K)},$$

where $S_\nu\ (x,f)$ is a partial sum of the trigonometric Fourier series of the function $f(x) \in L_p(G)$, i.e.

$$S_\nu(x,f) = \frac{a_0}{2} + \sum_{2\pi k \leq \nu} (a_k \cos 2\pi kx + b_k \sin 2\pi kx),$$

$$a_k = 2\int_0^1 f(x)\, \cos\, 2\pi kx\, dx,\ k = 0,1,2,\ldots;$$

$$b_k = 2\int_0^1 f(x)\, \sin\, 2\pi kx dx,\ k = 1,2,\ldots.$$

Let us introduce some notations to be used in the sequel:

$$\overset{\wedge}{f_k} = f_k \left\| \upsilon_k \right\|_q^{-1};\ \omega_p (f, \delta) = \sup_{0<h\leq\delta} \left\{ \int_0^{1-h} \left| f(x+h) - f(x) \right|^p dx \right\}^{\frac{1}{p}};$$

$$\Omega \left(f, \frac{\nu}{2}, \beta \right) = \nu^{-1} \sum_{1\leq\rho_k\leq\nu/2} \rho_k^{-\beta} \left| \overset{\wedge}{f_k} \right|, \beta > 0;$$

$$\Omega_1 \left(f, \frac{\nu}{2} \right) = \nu^{-1} \sum_{1\leq\rho_k\leq\nu/2} \rho_k^{-1} \ln \rho_k \left| \overset{\wedge}{f_k} \right|;$$

$$\varphi(f, \nu) = \max_{\rho_k\geq\nu/2} \left| \overset{\wedge}{f_k} \right| + \omega_1 \left(f, \nu^{-1} \right),$$

$$\varphi_1 (f, \nu) = \nu^{-1} \left\| f \right\|_p + \max_{\rho_k\geq\nu/2} \left| \overset{\wedge}{f_k} \right| + \omega_1 \left(f,\ \nu^{-1} \right),$$

$$\psi \left(f, \frac{\nu}{2}, r \right) = \begin{cases} \Omega \left(f,\ \frac{\nu}{2},\ 1 - \frac{1}{r} \right), & 1 \leq r < \infty, \\ \Omega_1 \left(f,\ \frac{\nu}{2} \right), & r = \infty. \end{cases}$$

Definition 1.1. If $\Delta_\nu\ (f,\ K) \to 0$ as $\nu \to +\infty$, then we will say that the expansions of the function $f(x)$ into biorthogonal series with respect to the system $\{u_k(x)\}_{k=1}^{\infty}$ and into the trigonometric Fourier series are uniformly equiconvergent on the compact $K \subset G$.

Definition 1.2. The system $\{\varphi_k(x)\}_{k=1}^{\infty} \subset L_q(G)$, $q \geq 2$, is said to be a Riesz system (or to satisfy the Riesz inequality) if there exists a constant $M = M(p)$ such that for every function $f(x) \in L_p(G)$, $1 < p \leq 2$, the inequality

$$\sum_{k=1}^{\infty} \left| (f, \varphi_k) \right|^q \leq M \left\| f \right\|_p^q$$

holds, where $p^{-1} + q^{-1} = 1$.

We have two theorems to prove in this work:

Theorem 1.1. *Let $q(x) \in L_r(G)$, $r \geq 1$, and the system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy the conditions A_p for some fixed $p \geq 1$. Then the expansions of an arbitrary function $f(x) \in L_p(G)$ into biorthogonal series with respect to the system $\{u_k(x)\}_{k=1}^{\infty}$ and into the trigonometric Fourier series are uniformly equiconvergent on every compact $K \subset G$ and the following estimates are true for $\nu \geq 2$:*

I. If $p > 1$, then

$$\Delta_\nu (f,\ K) \leq C(K) \begin{cases} \nu^{-1} \left\| f \right\|_p + \omega_1 \left(f,\ \nu^{-1} \right),\ r = \infty, \\ \nu^{\frac{1}{r} - \frac{1}{\alpha} - 1} \left\| f \right\|_p + \omega_1 \left(f,\ \nu^{-1} \right), & \alpha' \left(1 - \frac{1}{r} \right) \neq 1, \\ \nu^{-1} \ln^{\frac{1}{\alpha'}} \nu \left\| f \right\|_p + \omega_1 \left(f, \nu^{-1} \right),\ \alpha' \left(1 - \frac{1}{r} \right) = 1, \end{cases} \quad (1.4)$$

where $\alpha = \max\{2, q\}$, $\alpha' = \min\{2, p\}$, $\alpha^{-1} + \alpha'^{-1} = 1$;

II. If $p = 1$, then

$$\Delta_\nu (f, K) \le C(K) \begin{cases} \nu^{-1} \ln^2 \nu \ \|f\|_1 + \omega_1 \left(f, \nu^{-1}\right), & r = \infty, \\ \nu^{\frac{1}{r}-1} \ \|f\|_1 + \omega_1 \left(f, \nu^{-1}\right), & 1 \le r < \infty; \end{cases} \tag{1.5}$$

III. If $p \ge 1$, $q(x) \equiv 0$, then

$$\Delta_\nu (f, K) \le C(K) \left\{\nu^{-1} \ \|f\|_1 + \omega_1 \left(f, \nu^{-1}\right)\right\}, \tag{1.6}$$

where the constant $C(K)$ does not depend on ν and $f(x)$; and $\omega_1(f, \delta)$ is a modulus of continuity of the function $f(x)$ in $L_1(G)$.

The next theorem does not require that $\{\upsilon_k(x)\}_{k=1}^{\infty}$ must be a system of root functions of the formally conjugate operator $L^* = -\frac{d^2}{dx^2} + \overline{q(x)}$.

Theorem 1.2. *Let the conditions 1), 2) and the inequality (1.3) be satisfied, and in case $1 < p \le 2$ the system $\left\{\upsilon_k(x) \ \|\upsilon_k\|_q^{-1}\right\}_{k=1}^{\infty}$ satisfy the Riesz inequality in $L_p(G)$, $p^{-1} + q^{-1} = 1$. Then the expansions of an arbitrary function $f(x) \in L_p(G)$, $p \ge 1$, into biorthogonal series with respect to the system $\{u_k(x)\}_{k=1}^{\infty}$ and into the trigonometric Fourier series are uniformly equiconvergent on every compact $K \subset G$ and the following estimates are true for $\nu \ge 2$:*

a) If $1 < p \le 2$, then

$$\Delta_\nu (f, K) \le C(K) \begin{cases} \nu^{-1} \|f\|_p + \varphi(f, \nu), & r = \infty, \\ \nu^{\frac{1}{r}-\frac{1}{q}-1} \|f\|_p + \varphi(f, \nu), & p\left(1 - \frac{1}{r}\right) \ne 1, \\ \nu^{-1} \ln^{\frac{1}{p}} \nu \ \|f\|_p + \varphi(f, \nu), & p\left(1 - \frac{1}{r}\right) = 1; \end{cases} \tag{1.7}$$

b) If $p = 1$ or $p > 2$, then

$$\Delta_\nu (f, K) \le C(K) \begin{cases} \nu^{-1} \ln^2 \nu \|f\|_p + \varphi(f, \nu), & r = \infty, \\ \nu^{\frac{1}{r}-1} \ \|f\|_p + \varphi(f, \nu), & 1 \le r < \infty; \end{cases} \tag{1.8}$$

c) If $p \ge 1$, $q(x) \equiv 0$, then

$$\Delta_\nu (f, K) \le C(K) \left\{\nu^{-1} \ln^2 \nu \ \|f\|_p + \varphi(f, \nu)\right\}, \tag{1.9}$$

where $C(K)$ does not depend on ν and $f(x)$.

Note that in Theorem 1.2, the Riesz property for the system $\left\{\upsilon_k(x) \ \|\upsilon_k\|_q^{-1}\right\}_{k=1}^{\infty}$ is required only in case $f(x) \in L_p(G)$, $1 < p \le 2$. There is no such requirement in the rest of cases, i.e. in cases $p = 1$, $p > 2$.

2. Some auxiliary facts

To prove our theorems, we need the following lemmas.

Lemma 2.1. *Let the system* $\{\upsilon_k(x)\}_{k=1}^{\infty}$ *consist of root functions of the formally conjugate operator* $L^* = -\frac{d^2}{dx^2} + \overline{q(x)}$. *Then, under the condition 2), the Riesz inequality holds in* $L_p(G)$, $p^{-1}+q^{-1}=1$, *for the system* $\left\{\upsilon_k(x)\,\|\upsilon_k\|_q^{-1}\right\}_{k=1}^{\infty}$, $2 \le q < \infty$.

Lemma 2.1 is a special case of the result obtained in [7] for a second order differential operator.

Lemma 2.2. *(see* [5, 6]*) Let the system* $\{\upsilon_k(x)\}_{k=1}^{\infty}$ *consist of root functions of the formally conjugate operator* $L^* = -\frac{d^2}{dx^2} + \overline{q(x)}$, *the numbers* μ_k *satisfy the condition (1.1) and* $\rho_k = Re\,\mu_k \ge 2$. *Then the estimate*

$$\left|\overset{\wedge}{f_k}\right| \le const\left\{\omega_1\left(f,\,\rho_k^{-1}\right) + \rho_k^{-1}\,\|f\|_p\right\} \tag{2.1}$$

holds for every function $f(x) \in L_p(G)$, $p \ge 1$, *where const does not depend on* $f(x)$ *and* μ_k.

Let the system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy the conditions A_p without the requirement that the biorthogonal system $\{\upsilon_k(x)\}_{k=1}^{\infty}$ is a system of root functions of the formally conjugate operator $L^* = -\frac{d^2}{dx^2} + \overline{q(x)}$. Under these conditions, the estimate

$$\Delta_\nu(f,\,K) \le C_1(K)\left\{\|q\|_r\,\psi\left(f,\,\frac{\nu}{2},\,r\right) + \varphi_1(f,\,\nu)\right\} \tag{2.2}$$

has been obtained for the difference $\Delta_\nu(f,\,K)$ in [5],[6], where $C_1(K)$ is a positive constant independent of ν and the function $f(x)$, $f(x) \in L_p(G)$, $p \ge 1$.

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let us estimate the expressions $\varphi_1(f,\,\nu)$ and $\psi\left(f,\,\frac{\nu}{2},\,r\right)$. By the estimate (2.1) of Lemma 2.2, we have

$$\varphi_1(f,\,\nu) = \nu^{-1}\,\|f\|_p + \omega_1\left(f,\nu^{-1}\right) + \max_{\rho_k \ge \nu/2|}\left|\overset{\wedge}{f_k}\right| \le$$

$$\le \nu^{-1}\,\|f\|_p + \omega_1\left(f,\nu^{-1}\right) + const\,\max_{\rho_k \ge \nu/2}\left\{\omega_1\left(f,\rho_k^{-1}\right) + \rho_k^{-1}\,\|f\|_p\right\}.$$

Hence, taking into account the monotonicity of the function $\omega_1(f,t)$, we obtain

$$\varphi_1(f,\,\nu) \le const\,\left\{\nu^{-1}\,\|f\|_p + \omega_1\left(f,\,\nu^{-1}\right)\right\},\ p \ge 1. \tag{3.1}$$

Let $p > 1$ and $r = \infty$. Then, by the definition of $\psi\left(f,\,\frac{\nu}{2},\,r\right)$, for $r = \infty$ we have

$$\psi\left(f,\,\frac{\nu}{2},\,\infty\right) = \Omega_1\left(f,\frac{\nu}{2}\right) = \nu^{-1}\sum_{1 \le \rho_k \le \nu/2}\rho_k^{-1}\ln\rho_k\left|\overset{\wedge}{f_k}\right|.$$

Let us first apply the Hölder inequality, and then the Riesz inequality for $1 < p \le 2$, which holds by virtue of Lemma 2.1:

$$\psi\left(f,\,\frac{\nu}{2},\,\infty\right) \le \nu^{-1}\left(\sum_{1 \le \rho_k \le \nu/2}\rho_k^{-p}\ln^p\,\rho_k\right)^{\frac{1}{p}}\left(\sum_{1 \le \rho_k \le \nu/2}\left|\overset{\wedge}{f_k}\right|^q\right)^{\frac{1}{q}} \le$$

$$\leq \nu^{-1}\left(\sum_{1\leq\rho_k\leq\nu/2}\rho_k^{-p}\ln^p\,\rho_k\right)^{\frac{1}{p}}\left(\sum_{k=1}^{\infty}\left|\hat{f}_k\right|^q\right)^{\frac{1}{q}}\leq$$

$$\leq M\nu^{-1}\left\|f\right\|_p\left(\sum_{1\leq\rho_k\leq\nu/2}\rho_k^{-p}\ln^p\,\rho_k\right)^{\frac{1}{p}}.$$

Since the sum on the right-hand side is estimated from above by the convergent (by condition (1.2)) numerical series

$$\sum_{k=1}^{\infty}\rho_k^{-p}\ln^p\rho_k, p>1\,,$$

we have

$$\psi\left(f,\,\frac{\nu}{2},\,\infty\right)\leq const\,\nu^{-1}\,\left\|f\right\|_p. \tag{3.2}$$

For $p>2$ the estimate (3.2) remains true, because $L_p\,(G)$, $p>2$, is embedded into $L_2\,(G)$. Consequently, for $p>1$, $r=\infty$, from (2.2), (3,1), (3.2) we get the validity of the first part of (1.4).

Now let $p>1$, $1\leq r<\infty$. In this case, by definition, we have $\psi\left(f,\,\frac{\nu}{2},\,r\right)=\;\Omega\left(f,\,\frac{\nu}{2},\,1-\frac{1}{r}\right)$, i.e. $\psi\left(f,\,\frac{\nu}{2},\,r\right)=\;\nu^{-1}\sum\limits_{1\leq\rho_k\leq\frac{\nu}{2}}\rho_k^{1-\frac{1}{r}}\left|\hat{f}_k\right|$.

To estimate $\psi\left(f,\,\frac{\nu}{2},\;r\right)$ for $1<p\leq 2$, we first apply the Hölder inequality, and then the Riesz inequality (Lemma 2.2):

$$\psi\left(f,\,\frac{\nu}{2},\;r\right)=\;\nu^{-1}\sum_{1\leq\rho_k\leq\nu/2}\rho_k^{\frac{1}{r}-1}\left|\hat{f}_k\right|\leq$$

$$\nu^{-1}\left(\sum_{1\leq\rho_k\leq\nu/2}\rho_k^{\left(\frac{1}{r}-1\right)p}\right)^{\frac{1}{p}}\left(\sum_{1\leq\rho_k\leq\nu/2}\left|\hat{f}_k\right|^q\right)^{\frac{1}{q}}\leq$$

$$\leq M\left\|f\right\|_p\nu^{-1}\left(\sum_{1\leq\rho_k\leq\nu/2}\rho_k^{\left(\frac{1}{r}-1\right)p}\right)^{\frac{1}{p}}.$$

And, due to condition (1.2),

$$\left(\sum_{1\leq\rho_k\leq\nu/2}\rho_k^{\left(1-\frac{1}{r}\right)p}\right)^{\frac{1}{p}}\leq\;const\;\left(\sum_{i=1}^{\left[\frac{\nu}{2}\right]}i^{\left(\frac{1}{r}-1\right)p}\right)^{\frac{1}{p}}\leq$$

$$\leq const\begin{cases}\nu^{\frac{1}{r}-\frac{1}{q}} & for\;\;\left(1-\frac{1}{r}\right)\,p\neq 1\,,\\ \ln^{\frac{1}{q}}\,\nu & for\;\;\left(1-\frac{1}{r}\right)\,p=1\,.\end{cases}$$

Consequently, for $1 < p \leq 2$, $1 \leq r < \infty$, the estimate

$$\psi\left(f,\, \frac{\nu}{2},\, r\right) \leq const\, \|f\|_p \begin{cases} \nu^{\frac{1}{r}-\frac{1}{q}-1} & for \quad \left(1-\frac{1}{r}\right) p \neq 1\,, \\ \nu^{-1} \ln^{\frac{1}{q}} \nu & for \quad \left(1-\frac{1}{r}\right) p = 1 \end{cases} \tag{3.3}$$

holds.

Now let us estimate the expression $\psi\left(f,\, \frac{\nu}{2},\, r\right)$ for $p > 2$. By the estimate (see [8])

$$\|\upsilon_k\|_q \leq \|\upsilon_k\|_2 \leq const\, \|\upsilon_k\|_q\,, \;\; 1 \leq q < 2,$$

for $\left|\overset{\wedge}{f_k}\right|$ we have

$$\left|\overset{\wedge}{f_k}\right| = |f_k|\, \|\upsilon_k\|_q^{-1} \leq const\, |f_k|\, \|\upsilon_k\|_2^{-1}\,.$$

As the space $L_p(G)$ is embedded into $L_2(G)$ when $p > 2$, applying Bessel inequality we obtain

$$\left(\sum_{k=1}^{\infty} \left|\overset{\wedge}{f_k}\right|^2\right)^{\frac{1}{2}} \leq const \left(\sum_{k=1}^{\infty} |f_k|^2\, \|\upsilon_k\|_2^{-1}\right)^{\frac{1}{2}} \leq M\, \|f\|_2 \leq M\, \|f\|_p$$

for $f(x) \in L_p(G)$, $p > 2$.

Consequently, for $f(x) \in L_p(G)$, $p > 2$, the estimate

$$\psi\left(f,\, \frac{\nu}{2},\, r\right) \leq const\, \|f\|_p \begin{cases} \nu^{\frac{1}{r}-\frac{1}{2}-1} & for \quad 2\left(1-\frac{1}{r}\right) p \neq 1\,, \\ \nu^{-1} \ln^{\frac{1}{2}} \nu & for \quad 2\left(1-\frac{1}{r}\right) p = 1 \end{cases} \tag{3.4}$$

holds.

Combining the estimates (3.3) and (3.4), we obtain

$$\psi\left(f,\, \frac{\nu}{2},\, r\right) \leq const\, \|f\|_p \begin{cases} \nu^{\frac{1}{r}-\frac{1}{\alpha}-1} & for \quad \alpha'\left(\frac{1}{r}-1\right) \neq 1, \\ \nu^{-1} \ln^{\frac{1}{\alpha'}} \nu & for \quad \alpha'\left(\frac{1}{r}-1\right) = 1, \end{cases} \tag{3.5}$$

where $\alpha = \max\{2,\, q\}$, $\alpha' = \min\{2,\, p\}$, $p^{-1} + q^{-1} = 1$.

Considering the estimates (3.1) and (3.5) in (2.2), we obtain the second and the third parts of (1.4).

Let $p = 1$, $r = \infty$. In this case,

$$\psi\left(f,\, \frac{\nu}{2}\,,\, \infty\right) = \Omega_1\left(f,\, \frac{\nu}{2}\right) = \nu^{-1} \sum_{1 \leq \rho_k \leq \nu/2} \rho_k^{-1} \ln \rho_k \left|\overset{\wedge}{f_k}\right|\,.$$

Since $\left|\overset{\wedge}{f_k}\right| = |(f,\, \upsilon_k)|\, \|\upsilon_k\|_\infty^{-1} \leq \|f\|_1$, by the condition (1.2) we have

$$\psi\left(f,\, \frac{\nu}{2}\,,\, \infty\right) = \nu^{-1}\, \|f\|_1 \sum_{1 \leq \rho_k \leq \nu/2} \rho_k^{-1} \ln \rho_k \leq const\, \nu^{-1} \ln^2 \nu\, \|f\|_1\,.$$

Hence, from (3.1), (3.2) it follows that

$$\Delta_\nu(f,\, K) \leq C(K) \left\{ \nu^{-1} \ln^2 \nu\, \|f\|_1 + \omega_1\left(f,\, \nu^{-1}\right) \right\}.$$

In case $p=1$, $1\le r<\infty$, we have

$$\psi\left(f,\,\frac{\nu}{2}\,,\,r\right)\;=\;\nu^{-1}\sum_{1\le\rho_k\le\nu/2}\rho_k^{-1+\frac{1}{r}}\left|\overset{\wedge}{f_k}\right|\;\le\;const\;\nu^{\frac{1}{r}-1}\,\|f\|_1\,,$$

and, consequently, due to (3.1), (2.2), the estimate

$$\Delta_\nu\,(f,\,K)\;\le\;C\,(K)\,\left\{\nu^{\frac{1}{r}-1}\;\|f\|_1+\omega_1\,\left(f,\,\nu^{-1}\right)\right\}$$

holds. The estimate (1.5) is proved.

The estimate (1.6) is a corollary of (2.2) and (3.1), because $q\,(x)\equiv 0$. Theorem 1.1 is proved.

Proof of Theorem 1.2. Let us prove the estimate (1.7). By definition of $\varphi_1\,(f,\nu)$

$$\varphi_1\,(f,\nu)=\nu^{-1}\,\|f\|_p+\omega_1\left(f,\,\nu^{-1}\right)+$$

$$+\max_{\rho_k\ge\nu/2}\left|\overset{\wedge}{f_k}\right|=\nu^{-1}\,\|f\|_p+\varphi\,(f,\nu)\,,\;1\le p\le\infty. \tag{3.6}$$

Let $1<p\le 2$.

In case $r=\infty$, from the equality $\psi\left(f,\,\frac{\nu}{2},\,\infty\right)\;=\;\nu^{-1}\sum\limits_{1\le\rho_k\le\nu/2}\rho_k^{-1}\ln\rho_k\left|\overset{\wedge}{f_k}\right|$, due to the Riesz inequality, we obtain

$$\psi\left(f,\,\frac{\nu}{2}\,,\,\infty\right)=\nu^{-1}\;\|f\|_p\left(\sum_{1\le\rho_k\le\nu/2}\rho_k^{-p}\,\ln^p\,\rho_k\right)^{\frac{1}{p}}\;\le\;const\,\nu^{-1}\;\|f\|_p\,.$$

Consequently, from (3.6) and (2.2), taking into account the last inequality, we obtain the first part of (1.7).

In case $1\le r<\infty$, as in the proof of Theorem 1.1, we first apply the Hölder inequality, and then the Riesz inequality to prove the estimate

$$\psi\left(f,\,\frac{\nu}{2},\,r\right)\le const\;\|f\|_p\begin{cases}\nu^{\frac{1}{r}-\frac{1}{q}-1} & for\quad\left(1-\frac{1}{r}\right)\,p\ne 1\,,\\[2ex] \nu^{-1}\ln^{\frac{1}{q}}\,\nu & for\quad\left(1-\frac{1}{r}\right)\,p=1\,.\end{cases}$$

Hence, from (3.6), (2.2) we get the second and the third parts of (1.7). The estimate (1.7) is proved.

Let $p=1$ or $p>2$. In case $r=\infty$, due to the inequality $\left|\overset{\wedge}{f_k}\right|\le\|f\|_p$ and the condition (1.2), we have

$$\psi\left(f,\,\frac{\nu}{2},\,\infty\right)=\Omega_1\left(f,\,\frac{\nu}{2}\right)=\nu^{-1}\sum_{1\le\rho_k\le\nu/2}\rho_k^{-1}\ln\rho_k\left|\overset{\wedge}{f_k}\right|\le$$

$$\le\nu^{-1}\,\|f\|_p\sum_{1\le\rho_k\le\nu/2}\rho_k^{-1}\,\ln\,\rho_k\le\;const\,\nu^{-1}\ln^2\nu\;\|f\|_p\,.$$

Hence, from (3.6), (2.2) we get the first part of (1.8).

And, in case $1 \leq r < \infty$, due to the inequality $\left|\overset{\wedge}{f_k}\right| \leq \|f\|_p$ and the condition (1.2), we have

$$\psi\left(f, \frac{\nu}{2}, r\right) = \nu^{-1} \sum_{1 \leq \rho_k \leq \nu/2} \rho_k^{\frac{1}{r}-1} \left|\overset{\wedge}{f_k}\right| \leq$$

$$\leq \nu^{-1} \|f\|_p \sum_{1 \leq \rho_k \leq \nu/2} \rho_k^{\frac{1}{r}-1} \leq const\, \nu^{\frac{1}{r}-1} \|f\|_p .$$

Consequently, due to the last inequality, (3.6) and (2.2), the second part of the estimate (1.8) also holds. And the estimate (1.9) is a corollary of (3.6) and (2.2), because $q(x) \equiv 0$. Theorem 1.2 is proved.

Example 3.1 (see [3]). Consider the spectral problem

$$-u'' = \lambda u, \; x \in (0,1)$$

$$u(1) = 0$$

$$(a - \lambda)\, u'(0) + \lambda\, b\, u(0) = 0, \; a, b > 0.$$

As is known, this problem has only the following eigenfunctions:

$$u_k(x) = \sqrt{2}\, \sin\left(\sqrt{\lambda_k}\,(1-x)\right), \; k = 0, 1, 2, \ldots .$$

The eigenvalues $\lambda_k > 0$ are the roots of the equation $tg\, \sqrt{\lambda} = \frac{(a-\lambda)}{(b)\sqrt{\lambda}}$. We consider any one of these eigenfunctions as an eigenfunction with zero index, and we enumerate the rest of them in ascending order of corresponding eigenvalues.

For the eigenvalues, starting from some number N_0, the following estimate holds:

$$\frac{\pi}{2} + \pi(k-1) < \sqrt{\lambda_k} < \frac{\pi}{2} + \pi(k-1) + \frac{b}{\left(\frac{\pi}{4} + \pi(k-1)\right)} \tag{3.7}$$

Biorthogonal (biorthonormal) system $\{\upsilon_k(x)\}_{k=1}^{\infty}$ of $\{u_k(x)\}_{k=1}^{\infty}$ is:

$$\upsilon_k(x) = \sqrt{2}\left(\sin\left(\sqrt{\lambda_k}\,(1-x)\right)\right) -$$

$$- \left.\frac{\sqrt{\lambda_0}\, \cos\sqrt{\lambda_k}\, \sin\left(\sqrt{\lambda_0}\,(1-x)\right)}{\sqrt{\lambda_k}\, \cos\sqrt{\lambda_0}}\right) \Big/ \left(1 + \frac{\cos^2\sqrt{\lambda_k}}{b} + \frac{a\, \cos^2\sqrt{\lambda_k}}{b\, \lambda_k}\right). \tag{3.8}$$

As the system $\{u_k(x)\}_{k=1}^{\infty}$ forms a basis for $L_p(0, 1)$, $1 < p < \infty$ (a Riesz basis in case $p = 2$), this system is complete and minimal in $L_p(0, 1)$, $1 < p < \infty$. Consequently, it is complete in $L_p(0, 1)$, $1 \leq p < \infty$. We have $\{\upsilon_k(x)\}_{k=1}^{\infty} \subset L_\infty(0,1)$. Therefore, the system $\{u_k(x)\}_{k=1}^{\infty}$ is complete and minimal in $L_p(0,1)$, $1 \leq p < \infty$. By (3.7), $\mu_k = \sqrt{\lambda_k}$, $k = 1, 2, \ldots$, satisfy the condition 2). Due to the uniform boundedness of the systems $\{u_k(x)\}_{k=1}^{\infty}$ and $\{\upsilon_k(x)\}_{k=1}^{\infty}$, the conditions (1.3) hold. Therefore, by Theorem 1.2, the estimate (1.9) ($q(x) \equiv 0$) holds for an arbitrary function $f(x) \in L_p(0,1)$, $p \geq 1$.

Note that the system $\{\upsilon_k(x)\}_{k=1}^{\infty}$ is not a system of eigenfunctions of the formally conjugate operator $L^* = -\frac{d^2}{dx^2}$.

Let us estimate the coefficients $\overset{\wedge}{f_k} = (f, \upsilon_k)\, \|\upsilon_k\|_\infty^{-1}$, $f(x) \in L_1(0,1)$. As $\{\upsilon_k(x)\}_{k=1}^{\infty}$ also forms a Riesz basis for $L_2(0,1)$ there exist constants α_1, $\alpha_2 > 0$

such that $\alpha_1 \le \|\upsilon_k\|_2 \le \alpha_2$, $k = 1, 2, \ldots$. Therefore, the following two-sided estimate holds:

$$\left|\hat{f}_k\right| \le |(f, \upsilon_k)| \; \|\upsilon_k\|_2^{-1} \le \frac{1}{\alpha_1} \left|(f, \upsilon_k)\right|.$$

Due to (3.8),

$$|(f,\, \upsilon_k)| \le \; const \left[\left|\left(f,\, \sin\sqrt{\lambda_k}\,(1-x)\right)\right| + \|f\|_1 \, \mu_k^{-1}\right] \le$$

$$\le const \left[\omega_1 \left(f,\, \mu_k^{-1}\right) + \mu_k^{-1} \|f\|_1 + \mu_k^{-1} \|f\|_1\right] \le const \left[\omega_1 \left(f,\, \mu_k^{-1}\right) + \mu_k^{-1} \|f\|_1\right].$$

Consequently, the estimate $\max\limits_{\mu_k \ge \nu/2} \left|\hat{f}_k\right| \le const \; \left\{\omega_1 \left(f, \nu^{-1}\right) + \nu^{-1} \|f\|_1\right\}$ holds.

Taking this into account in (1.9), we obtain

$$\Delta_\nu (f, K) \le C(K) \left\{\omega_1 \left(f, \nu^{-1}\right) + \nu^{-1} \|f\|_1\right\}, \tag{3.9}$$

where $f(x) \in L_1(0, 1)$.

From (3.9) we obtain, in particular, the following estimates:

$$\Delta_\nu (f, K) \le C(K)\, \nu^{-\alpha} \|f\|_{H_1^\alpha(G)}, \; \alpha \in (0, 1),$$

if $f(x) \in H_1^\alpha(G)$, where $H_1^\alpha(G)$ is a Nikolski class, and

$$\Delta_\nu (f,\, K) \le C(K) \; \nu^{-\alpha} \|f\|_{B^\alpha_{1,\,\theta}(G)}, \; \alpha \in (0, 1), \; \theta \in [1, \infty],$$

if $f(x) \in B^\alpha_{1,\theta}(G)$, where $B^\alpha_{1,\theta}(G)$ is a Besov class.

Example 3.2 [4]. Consider the spectral problem

$$u''(x) + \lambda\, u(x) = 0,$$

$$u(0) = \beta u(1), \; u'(0) = u'(1), \; \beta \ne 1.$$

All the eigenvalues λ_k, except for λ_0, have multiplicity 2: every multiple eigenvalue λ_k corresponds to one eigenfunction and one associated function. $\mu_0 = \sqrt{\lambda_0} = 0$, $\mu_{2k-1} = \mu_{2k} = 2\pi k$, $k = 1, 2, \ldots$,

$$u_0(x) = \frac{2\left((1-\beta)\, x + \beta\right)}{(1+\beta)},$$

$$u_{2k-1}(x) = \sin 2\pi kx\,, \quad u_{2k}(x) = \frac{4\left((1-\beta)\, x + \beta\right)\, \cos 2\pi kx}{(1+\beta)}.$$

Biorthogonal (biorthonormal) system

$$\upsilon_0(x) = 1, \; \upsilon_{2k-1}(x) = \frac{4\left((\beta-1)\, x + 1\right)\, \sin 2\pi kx}{(\beta+1)},$$

$$\upsilon_{2k}(x) = \cos 2\pi kx, \; k = 1, 2, \ldots,$$

consists of root functions of the formally conjugate operator $L^* = -\frac{d^2}{dx^2}$. The system $\{u_k(x)\}_{k=1}^\infty$ forms a basis in $L_p(0,1)$, $1 < p < \infty$, which is a Riesz basis when $p = 2$. Due to the uniform boundedness of the systems $\{u_k(x)\}_{k=1}^\infty$ and $\{\upsilon_k(x)\}_{k=1}^\infty$, the conditions 3) hold. For every fixed p, $p \in [1, \infty)$, all conditions of Theorem 1.1 are satisfied.

Applying this theorem with $p = 1$ $(q(x) = 0)$, we have

$$\Delta_\nu (f, K) \le C(K) \; \left\{\nu^{-1} \; \|f\|_1 + \omega_1 \left(f, \nu^{-1}\right)\right\}.$$

Example 3.3. Let us consider the problem

$$u'' + \mu^2 u = 0, \quad G = (0,1)$$
$$u(0) = u(1) = 0.$$

It is obvious that the orthonormal system of eigenfunctions of this problem is

$$u_n(x) = \sqrt{2}\,\sin \pi n x, \; n = 1, 2, \ldots,$$

and $\mu_n^2 = n^2\pi^2, \; n = 1, 2, \ldots$ are eigenvalues.

Let $f(x)$ be an arbitrary function from the class $H_1^\alpha(G), \; 0 < \alpha < 1$. By virtue of Lemma 2.2 we have

$$\left|\overset{\wedge}{f_n}\right| = |f_n| \le const\,(\pi n)^{-\alpha}\,\|f\|_{H_1^\alpha(G)}\,.$$

And by virtue of Theorem 1.1 (see estimate (1.6))

$$\Delta_\nu(f,K) \le C(K)\,\nu^{-\alpha}\,\|f\|_{H_1^\alpha(G)}\,. \tag{3.10}$$

Let us prove that estimate (3.10) cannot be improved. Let there be a function $\beta(\nu), \; \beta(\nu) \to 0, \; \nu \to \infty$, such that for arbitrary $f \in H_1^\alpha(0,1)$

$$\Delta_\nu(f,K) \le C(K)\,\beta(\nu)\,\nu^{-\alpha}\,\|f\|_{H_1^\alpha(G)}\,. \tag{3.11}$$

Let us consider the problem

$$\tilde{u}'' + \left(\mu^2 - 1\right)\tilde{u} = 0, \; \tilde{u}(1) = \tilde{u}(0) = 0.$$

For this problem

$$\tilde{u}_n(x) = \sqrt{2}\,\sin \pi n x, \quad n = 1, 2, \ldots; \; \tilde{\mu}_n^2 = (\pi n)^2 + 1.$$

Therefore, the difference $\tilde{\Delta}_\nu(f,K)$ is estimated

$$\tilde{\Delta}_\nu(f,K) \le C(K)\,\beta(\nu)\,\nu^{-\alpha}\,\|f\|_{H_1^\alpha(G)}\,.$$

From here and from (3.11), by virtue of the triangle inequality, we obtain

$$\|\sigma_\nu(f,\cdot) - \tilde{\sigma}_\nu(f,\cdot)\|_{C(K)} \le C_1(K)\,\beta(\nu)\,\nu^{-\alpha}\,\|f\|_{H_1^\alpha(G)}\,.$$

Assuming here $\nu = \pi n$ we get

$$|f_n|\,\left\|\sqrt{2}\,\sin \pi n x\right\|_{C(K)} = \|\,\sigma_{\pi n}(f,\cdot) - \tilde{\sigma}_{\pi n}(f,\,\cdot)\,\|_{C(K)} \le$$
$$\le C_2(K)\;\beta(\pi n)\;n^{-\alpha}\,\|f\|_{H_1^\alpha(G)}\,.$$

Since $\left\|\sqrt{2}\,\sin \pi n x\right\|_{C(K)} \ge C_3(K) > 0$, the inequality

$$C_3(K)\;|f_n| \le C_2(K)\;\beta(\pi n)\;n^{-\alpha}\,\|f\|_{H_1^\alpha(G)} \tag{3.12}$$

is satisfied.

Let us select the functions $f(x) = x^{\alpha-1}, \; 0 < \alpha < 1 \; \left(x^{\alpha-1} \in H_1^\alpha(G)\right)$ and estimate $|f_n|$ from below:

$$|f_n| = \sqrt{2}\left|\int_0^1 x^{\alpha-1}\,\sin \pi n x\;dx\right| = \frac{\sqrt{2}}{(\pi n)^\alpha}\left|\int_0^{\pi n} \frac{\sin t}{t^{1-\alpha}}\,dt\right| =$$
$$= \frac{\sqrt{2}}{(\pi n)^\alpha}\left|\int_0^\infty \frac{\sin t}{t^{1-\alpha}}\,dt - \int_{\pi n}^\infty \frac{\sin t}{t^{1-\alpha}}\,dt\right| \ge$$

$$\geq \frac{\sqrt{2}}{(\pi n)^{\alpha}} \left(\Gamma(\alpha)\sin\frac{\pi\alpha}{2} + o(1)\right).$$

Taking this estimate into account in (3.12), we arrive inequality $\beta(\pi n) \geq C(\alpha) > 0$ for sufficiently large n. And this contradicts $\beta(\pi n) \to 0,\ n \to \infty$. Thus, the estimate (3.10) cannot be improved in the class $H_1^{\alpha}(G)$, $0 < \alpha < 1$.

By virtue of Theorem 1.1 (see estimate (1.6)) for any function $f(x) \in B_{1,\theta}^{\alpha}(G)$, $0 < \alpha < 1,\ \theta \in [1, \infty)$,

$$\Delta_{\nu}(f, K) \leq C(K)\ \ \nu^{-\alpha}\ \|f\|_{B_{1,\theta}^{\alpha}(G)}. \tag{3.13}$$

Let us prove that this estimate is accurate in the class $B_{1,\theta}^{\alpha}(G)$. Proceeding in the same way as in the case $f \in H_1^{\alpha}(G)$, we obtain instead of (3.10) the inequality

$$C_0\,|f_n| \leq C_4(K)\,\gamma(\pi n)\, n^{-\alpha}\,\|f\|_{B_{1,\theta}^{\alpha}(G)},\ ,f \in B_{1,\theta}^{\alpha}(G), \tag{3.14}$$

where $\gamma(\pi n) \to 0,\ n \to \infty$.

Let us select the function $f(x) = \exp(-i\pi n x)$ and estimate $\omega_1(f, t)$:

$$\omega_1(f, t) = \sup_{0<h\leq t} \int_0^{1-h} |\exp(-i\pi n(x+h)) - \exp(-i\pi n x)|\,dx \leq$$

$$\leq \sup_{0<h\leq t}\left|\sin\frac{\pi n h}{2}\right| \leq \min\{1, \pi n t\}$$

Then when $\pi n > h_0^{-1}$ $(h_0 > 0)$ we get

$$\|f\|_{B_{1,\theta}^{\alpha}} = \|f\|_1 + \left(\int_0^{h_0} \frac{\omega_1^{\theta}(f,t)}{t^{1+\theta\alpha}}dt\right)^{1/\theta} \leq$$

$$\leq 1 + \left(\int_0^{1/\pi n} \frac{(\pi n t)^{\theta}}{t^{1+\theta\alpha}}dt + \int_{1/\pi n}^{h_0} \frac{dt}{t^{1+\theta\alpha}}\right)^{1/\theta} \leq$$

$$\leq 1 + \left((\pi n)^{\theta}(\pi n)^{\alpha\theta-\theta} + (\pi n)^{\theta\alpha}\right)^{1/\theta} \leq 3(\pi n)^{\alpha};$$

$$|f_n| = \left|\sqrt{2}\int_0^1 \exp(-i\pi n x)\,\sin\pi n x\,dx\right| =$$

$$= \frac{\sqrt{2}}{2}\left|1 + \frac{\exp(-2\pi n i) - 1}{2\pi n i}\right| = \frac{\sqrt{2}}{2} + o\left(n^{-1}\right).$$

Taking into account these relations in inequality (3.14), we obtain $\gamma(\pi n) \geq$ *const* > 0. The obtained contradiction proves the accuracy of estimate (3.13) in the class $B_{1,\theta}^{\alpha}(G)$, $0 <\ \alpha < 1$, $\theta \in [1, \infty)$.

References

[1] V.A. Il'in, Necessary and sufficient conditions for the basis property and equiconvergence with the trigonometric series of spectral decompositions. I, *Diff.uravn.* **16** (1980), no. 5, 771-794.

[2] V.A. Il'in, Necessary and sufficient conditions for the basis property and equiconvergence with the trigonometric series of spectral decompositions. II, *Diff.uravn.* **16** (1980), no.6, 980-1009.

[3] N. Yu. Kapustin, E.I. Moiseev, The basis property in L_p of the systems of eigenfunctions corresponding to two problems with a spectral parameter in the boundary condition, *Diff.uravn.* **36** (2000),no. 10, 1357-1360.

[4] T.B. Kasumov, V.S. Mirzoyev, On one generalization of Ionkin examples, Theses of the scientific conference dedicated to the 100^{th} Anniversary of Academician Ashraf Huseynov, Baku, October 25, 2007, 87-88. (in Russian)

[5] V.M. Kurbanov, Equiconvergence of biorthogonal expansions in root functions of differential operators I, *Diff.uravn.* **35** (1999), no.12, 1619-1633.

[6] V.M. Kurbanov, Equiconvergence of biorthogonal expansions in root functions of differential operators II, *Diff.uravn.* **36** (2000), no.3, 319-335.

[7] V.M. Kurbanov, Kh.R. Gojayeva, Riesz property criterion for the system of eigen and associated functions of second order ordinary differential operator, *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.- Tech. Math. Sci. Mathematics*, **45** (2025), no.4, 123-131.

[8] N.B. Kerimov, Some properties of eigen and associated functions of ordinary differential operators, *Dokl. Akad. Nauk SSSR,* **291** (1986), no. 5, 1054-1056.

[9] I.S. Lomov, The local convergence of biortothogonal series related to differential operators with nonsmooth coefficients I, *Differ.Equations,* **37** (2001), no. 3, 351-366.

[10] I.S. Lomov, The local convergence of biortothogonal series related to differential operators with nonsmooth coefficients II, *Differ.Equations,* **37** (2001), no. 5, 680-694.

[11] V.S. Rykhlov, Rate of equiconvergence for differential operators with nonzero coefficient of the $(n-1)$-th derivative, *Differ.Equations,* **26** (1990), no. 6, 704-715.

[12] V.S. Rykhlov, Equiconvergence rate in terms of general moduli of continuity for differential operators, *Res. Math.,* **29** (1996), 153-168.

[13] V.S. Rykhlov, Equiconvergence of expansions in root functions of a differential operator and in a trigonometric Fourier series, *Sovrem. Mat. Fundam. Napravl.*, **71** (2025), no.3, 452-477.

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