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# HYPERBOLIC EQUATION WITH DELTA-INTERACTION

AGIL KH. KHANMAMEDOV AND KHATIRA E. ABBASOVA

**Abstract**. A Goursat-type problem for an inhomogeneous string vibration equation with discontinuous conditions is considered. Its unique solvability is proven. An explicit solution is found.

## 1. Introduction and main result

Hyperbolic equations with the Dirac delta function ($\delta$-function) model physical phenomena (sound, electromagnetic waves, string vibrations) with pulsed or instantaneous effects (see [5]). Such equations make it possible to study the propagation of waves in media where there are sharp inhomogeneities, and how, for example, a shock wave arises, where the solution may have discontinuities. For example, an equation of the form

$$u_{xx}(x,t)-u_{tt}(x,t)=f(t)\,\delta(x-x_0)$$

describes the transverse vibrations of a string, with a concentrated force $f(t)$ applied at point $x_0$(see [9]). This means that at the point of application of the concentrated force, the first derivatives undergo a discontinuity and two conjugation conditions are satisfied:

$$u(x_0+0,t)-u(x_0-0,t)=0,$$

$$u_x(x_0+0,t)-u_x(x_0-0,t)=f(t).$$

The first of which expresses the continuity of the string, the second determines the magnitude of the bend of the string at point $x_0$.

Consider the inhomogeneous wave equation

$$u_{xx}(x,t)-u_{tt}(x,t)=F(x,t),\ x\neq 0,\ t>x,\ t\neq 0, \tag{1.1}$$

(1) in which the function $F(x,t)$ differentiable with respect to $t$ satisfies the following conditions:
(2) $F(x,t)=0$ for $t<x$,
(3) $\int\limits_a^{+\infty} d\tau \int\limits_\tau^{+\infty} |F(\tau,u)|\,du<\infty$, $\int\limits_a^{+\infty} |F_t(u,u+v)|\,du<\infty$, for any $a\in(-\infty,+\infty)$, $v\geq 0$.

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For equation (1.1) we pose the following problem:

$$u\left(x,x\right)=\varphi\left(x\right),x\in\left(-\infty,+\infty\right), \tag{1.2}$$

$$u\left(+0,t\right)-u\left(-0,t\right)=0,u_x\left(+0,t\right)-u_x\left(-0,t\right)=\psi_1\left(t\right),t\geq 0, \tag{1.3}$$

$$u\left(x,+0\right)-u\left(x,-0\right)=0,u_t\left(x,+0\right)-u_t\left(x,-0\right)=\psi_2\left(x\right),\ x\leq 0, \tag{1.4}$$

$$\lim_{x+t\to+\infty}u\left(x,t\right)=0, \tag{1.5}$$

where

$$\varphi\left(x\right)\in C^{(2)}\left(-\infty,+\infty\right),\ \psi_1\left(t\right)\in C^{(1)}\left[0,+\infty\right),\psi_2\left(x\right)\in C^{(1)}\left(-\infty,0\right],$$

$$\lim_{x\to+\infty}\varphi\left(x\right)=\lim_{x\to+\infty}\psi_1\left(x\right)=0. \tag{1.6}$$

Such problems arise in particular when constructing transformation operators for one-dimensional Schrödinger equations with delta-shaped potentials. Typically, the kernels of such transformation operators satisfy certain Goursat-type problems for second-order hyperbolic equations (see [1-8]). Note that the Schrödinger equation with various discontinuous conditions was studied in works [2], [3]. In these works, a different approach was used to find representations of Jost-type solutions that are not transformation operators. Therefore, the study of problem (1.1)-(1.5), on the one hand, is of independent interest, and, on the other hand, it is interesting from the point of view for further research of transformation operators. In the absence of a discontinuous condition a similar problem was studied in [4]. The presence of discontinuity conditions requires constructing solutions in separate areas and gluing them together. This circumstance, in turn, creates certain mathematical difficulties.

In this paper, the solvability of problem (1.1)-(1.5) is proved and an explicit solution is found.

The main result of the present paper is as follows.

**Theorem 1.1.** *Let conditions* (1.6) *be satisfied and* $\psi_1\left(0\right)=-\psi_2\left(0\right)$, $\psi_1'\left(0\right)=-\psi_2'\left(0\right)$. *Then problem* (1.1)-(1.5) *has a unique solution* $u\left(x,t\right)$, *representable in the form*

$$u\left(x,t\right)=\varphi\left(\frac{x+t}{2}\right)+\frac{1}{2}\int\limits_{\max(0,t+x)}^{t-x}\psi_1\left(u\right)du-\frac{1}{2}\int\limits_{\max(x-t,x+t)}^{0}\psi_2\left(u\right)du+$$

$$+\frac{1}{2}\int\limits_{x}^{+\infty}d\tau\int\limits_{t-\tau+x}^{t+\tau-x}F\left(\tau,\xi\right)d\xi\,,\ x<0,\ t>x, \tag{1.7}$$

$$u\left(x,t\right)=\varphi\left(\frac{x+t}{2}\right)+\frac{1}{2}\int\limits_{x}^{+\infty}d\tau\int\limits_{t-\tau+x}^{t+\tau-x}F\left(\tau,\xi\right)d\xi,\ 0<x<t. \tag{1.8}$$

*In this case, we will consider the value of the integral to be equal to zero if the upper limit of integration is less than the lower limit.*

## 2. Proof of the theorem

Let us note first of all that the difference between two solutions to problem (1.1)-(1.5) satisfies the corresponding homogeneous equation and in this case the discontinuity conditions disappear. Therefore, the uniqueness of the solution follows from the results for the case without discontinuous conditions (see [4], Chapter I, paragraph 3). Let us prove the existence of a solution. First of all, as in [4, Chapter I, paragraph 3), we find that for $t > x > 0$ the solution $u(x,t)$ is represented in the form

$$u(x,t) = \varphi\left(\frac{x+t}{2}\right) + \frac{1}{2}\int_{x}^{+\infty} d\tau \int_{t-\tau+x}^{t+\tau-x} F(\tau,\xi)\, d\xi, \; t > x > 0 \tag{2.1}$$

Let us construct a solution in the region $x < 0, \; t > -x$. It is not difficult to see that

$$u_0(x,t) = \varphi\left(\frac{x+t}{2}\right) + \frac{1}{2}\int_{x}^{+\infty} d\tau \int_{t-\tau+x}^{t+\tau-x} F(\tau,\xi)\, d\xi$$

is a particular solution of equation (1.1). To obtain the general solution for equation (1.1), the general solution of the homogeneous equation should be added to the particular solution $u_0(x,t)$. This gives

$$u(x,t) = \varphi\left(\frac{x+t}{2}\right) + \frac{1}{2}\int_{x}^{+\infty} d\tau \int_{t-\tau+x}^{t+\tau-x} F(\tau,\xi)\, d\xi + \theta_1(t-x) + \theta_2(t+x), \tag{2.2}$$

where $\theta_1$ and $\theta_2$ are arbitrary twice differentiable functions. Since $u(+0,t) = u(-0,t)$, then from (2.1), (2.2) we get

$$\theta_1(t) + \theta_2(t) = 0,$$

i.e.

$$\theta_1(t) = -\theta_2(t), \; t > 0. \tag{2.3}$$

On the other hand, using the second equality from (1.3), we obtain

$$\theta_1'(t) - \theta_2'(t) = \psi_1(t).$$

Hence,

$$\theta_1'(t) = \frac{1}{2}\psi_1(t).$$

From here and from (2.3), we obtain

$$\theta_1(t) = \frac{1}{2}\int_0^t \psi_1(u)\, du + \theta_1(0),$$

$$\theta_2(t) = \frac{1}{2}\int_t^0 \psi_1(u)\, du + \theta_2(0).$$

Then from (2.2) it follows that

$$u(x,t)=\varphi\left(\frac{x+t}{2}\right)+\frac{1}{2}\int\limits_{x}^{+\infty}d\tau\int\limits_{t-\tau+x}^{t+\tau-x}F(\tau,\xi)\,d\xi+$$

$$+\frac{1}{2}\int\limits_{t+x}^{t-x}\psi_1(u)\,du,\ x<0,\ t>-x. \tag{2.4}$$

Let us construct a solution to equation (1.1) in the region $x<0$, $0<t<-x$. We will seek a solution $u(x,t)$ in the form

$$u(x,t)=\varphi\left(\frac{x+t}{2}\right)+\frac{1}{2}\int\limits_{x}^{+\infty}d\tau\int\limits_{t-\tau+x}^{t+\tau-x}F(\tau,\xi)\,d\xi\ +\alpha_1(t-x)+\alpha_2(t+x).$$

Using the equality $u(x,-x+0)=u(x,-x-0)$, we obtain

$$\alpha_1(-2x)+\alpha_2(0)=\frac{1}{2}\int\limits_{0}^{-2x}\psi_1(u)\,du,$$

$$\alpha_1(0)+\alpha_2(0)=0.$$

From the last two equalities it follows that

$$\alpha_1(x)=\frac{1}{2}\int\limits_{0}^{x}\psi_1(u)\,du+\alpha_1(0).$$

Hence,

$$u(x,t)=\varphi\left(\frac{x+t}{2}\right)+\frac{1}{2}\int\limits_{x}^{+\infty}d\tau\int\limits_{t-\tau+x}^{t+\tau-x}F(\tau,\xi)\,d\xi\ +\frac{1}{2}\int\limits_{0}^{t-x}\psi_1(u)\,du+$$

$$+\alpha_1(0)+\alpha_2(t+x),\ x<0,\ \ 0<t<-x. \tag{2.5}$$

Further, in the region $x<t<0$ we look for a general solution $u(x,t)$ in the form

$$u(x,t)=\frac{1}{2}\int\limits_{x}^{+\infty}d\tau\int\limits_{t-\tau+x}^{t+\tau-x}F(\tau,\xi)\,d\xi+\beta_1(t-x)+\beta_2(t+x).$$

By virtue of (1.2) we have:

$$\beta_1(0)+\beta_2(2x)=\varphi(x).$$

Therefore, in the region $x<t<0$ the solution $u(x,t)$ is represented as

$$u(x,t)=\varphi\left(\frac{x+t}{2}\right)+\frac{1}{2}\int\limits_{x}^{+\infty}d\tau\int\limits_{t-\tau+x}^{t+\tau-x}F(\tau,\xi)\,d\xi+$$

$$+\frac{1}{2}\int\limits_{0}^{t-x}\psi_1(u)\,du+\beta(x-t), \tag{2.6}$$

where $\beta(x) = \beta_1(-x) - \frac{1}{2}\int\limits_0^{-x} \psi_1(u)\,du - \beta_1(0)$. Then using (1.4), (2.5), (2.6) we have

$$\alpha_1(0) + \alpha_2(x) = \beta(x),$$
$$\alpha_2'(x) + \beta'(x) = \psi_2(x).$$

It follows that

$$\alpha_2(x) = -\frac{1}{2}\int\limits_x^0 \psi_2(s)\,ds + \alpha_1(0),$$

$$\beta(x) = -\frac{1}{2}\int\limits_x^0 \psi_2(s)\,ds + \alpha_1(0) + \alpha_2(0) = -\frac{1}{2}\int\limits_x^0 \psi_2(s)\,ds.$$

Substituting the last equalities into (2.5), (2.6) we obtain

$$u(x,t) = \varphi\left(\frac{x+t}{2}\right) + \frac{1}{2}\int\limits_x^{+\infty} d\tau \int\limits_{t-\tau+x}^{t+\tau-x} F(\tau,\xi)\,d\xi + \frac{1}{2}\int\limits_0^{t-x} \psi_1(u)\,du -$$
$$-\frac{1}{2}\int\limits_{x+t}^0 \psi_2(s)\,ds,\ x<0,\ 0<t<-x. \tag{2.7}$$

$$u(x,t) = \varphi\left(\frac{x+t}{2}\right) + \frac{1}{2}\int\limits_x^{+\infty} d\tau \int\limits_{t-\tau+x}^{t+\tau-x} F(\tau,\xi)\,d\xi +$$
$$+\frac{1}{2}\int\limits_0^{t-x} \psi_1(u)\,du - \frac{1}{2}\int\limits_{x-t}^0 \psi_2(s)\,ds,\ x<t<0. \tag{2.8}$$

Finally, due to the equality $u_x(x,-x+0) = u_x(x,-x-0)$, from (2.2), (2.7), (2.8) we find

$$-\frac{1}{2}\psi_1(-2x) - \frac{1}{2}\psi_1(0) = -\frac{1}{2}\psi_1(-2x) + \frac{1}{2}\psi_2(0).$$

Hence, $\psi_1(0) = -\psi_2(0)$. Similarly, the condition $\psi_1'(0) = -\psi_2'(0)$ ensures the equality $u_{xx}(x,-x+0) = u_{xx}(x,-x-0)$. This completes the proof of the theorem.

*Remark* 2.1. In the case $\psi_1(0) \neq -\psi_2(0)$, there exists a twice differentiable solution $u(x,t)$ for $x\neq 0,\ t\neq 0,\ t\neq -x(x<0)$ to problem (1.1)-(1.5). Moreover, the partial derivatives $u_x(x,t),\ u_t(x,t)$ at the point $t=-x(x<0)$ suffer discontinuities of the first kind.

*Remark* 2.2. Consider the equation

$$-y'' + \alpha\delta(x)\,y + q(x)\,y = \lambda y,\ -\infty<x<\infty,\ \lambda\in C, \tag{2.9}$$

where $\alpha$ is a real constant, $\delta(x)$ is the Dirac delta function, and the real potential $q(x)$ satisfies the conditions

$$q(x) \in C^{(1)}(-\infty, \infty), \int_{-\infty}^{\infty} (1+|x|)\,|q(x)|\,dx < \infty.$$

If the transformation operator that transforms solutions of the equation $-y'' + \alpha\delta(x)\,y = \lambda y$, $-\infty < x < \infty$, $\lambda \in C$, into solutions of equation (2.9), is sought in the form

$$(I + \mathrm{K})\,f(x) = f(x) + \int_{x}^{\infty} K(x,t)\,f(t)\,dt,$$

then it can be shown that the kernel $K(x,t)$ satisfies the problem

$$\frac{\partial K(x,t)}{\partial x^2} - \frac{\partial K(x,t)}{\partial t^2} - [\delta(x) - \delta(t) + q(x)]\;K(x,t) = 0,\; t > x,$$

$$K(x,x) = \frac{1}{2}\int_{x}^{\infty} q(t)\;dt,$$

$$\lim_{x+t\to\infty} K(x,t) = 0.$$

The last problem can be reduced to a problem of the form (1.1)-(1.5), where $F(x,t) = q(x)\,K(x,t)$, $\psi_1(t) = \alpha K(0,t)$, $\psi_2(x) = \alpha K(x,0)$. As a result, a Volterra-type integral equation is obtained for the kernel $K(x,t)$, which can be solved by the method of successive approximations.

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Agil Kh. Khanmamedov
*Baku State University, Baku, Azerbaijan*
E-mail address: agil_khanmamedov@yahoo.com

Khatira E. Abbasova
*Azerbaijan State University of Economics (UNEC), Baku, Azerbaijan*
E-mail address: abbasova_xatira@unec.edu.az

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