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SUFFICIENT CONDITIONS FOR A MINIMUM IN NONSMOOTH PROBLEMS OF THE CALCULUS OF VARIATIONS

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Abstract. In this paper, nonsmooth problems of the calculus of variations are investigated. The concepts of strong and weak extremals in a nonsmooth variational problem are introduced. For the investigation of these extremals for a minimum, a method is presented. With the help of this method, various sufficient conditions for a minimum are obtained. The effectiveness of the obtained results is demonstrated by examples.

1. Introduction

Consider the classical vector problem of the calculus of variations

$$S\big(x(\cdot)\big) = \int_{t_0}^{t_1} L\big(t, x(t), \dot{x}(t)\big)\, dt \to \min, \tag{1.1}$$

$$x(t_0) = x_0, \quad x(t_1) = x_1, \quad x_0, x_1 \in \mathbb{R}^n, \tag{1.2}$$

where $\mathbb{R}^n$ is an n-dimensional Euclidean space, and x_0, x_1, t_0, t_1 are given points. Here, with respect to the given function

$$L(\cdot) : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} := (-\infty, +\infty)$$

called the integrand, it is assumed that it is continuous with respect to the collection of variables; $x(\cdot) : [t_0, t_1] =: I \to \mathbb{R}^n$ is a piecewise smooth vector function, i.e. it is continuous, and its derivative is continuous everywhere on I except for a finite number of points $\tau_i \in (t_0, t_1)$; moreover, at the points τ_i the derivative $\dot{x}(\cdot)$ has discontinuities of the first kind (at the points t_0 and t_1, the value of the derivative $\dot{x}(\cdot)$ is understood as the finite right-hand and left-hand derivative, respectively). The set of such functions will be denoted by $KC^1(I, \mathbb{R}^n)$.

Functions $x(\cdot) \in KC^1(I, \mathbb{R}^n)$ satisfying the conditions (1.2) will be called admissible.

Let us recall (see the example on [5, p. 107]) several concepts that are more characteristic of the classical calculus of variations. An admissible function $\bar{x}(\cdot)$

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is called a strong (weak) local minimum in the problem (1.1), (1.2) if there exists a number $\delta > 0$ such that the inequality

$$S\big(x(\cdot)\big) \geq S\big(\bar{x}(\cdot)\big)$$

holds for all admissible functions $x(\cdot)$ for which

$$\|x(\cdot) - \bar{x}(\cdot)\|_{C(I,\mathbb{R}^n)} \leq \delta \left(\max\left\{\|x(\cdot) - \bar{x}(\cdot)\|_{C(I,\mathbb{R}^n)}, \|\dot{x}(\cdot) - \dot{\bar{x}}(\cdot)\|_{L_\infty(I,\mathbb{R}^n)}\right\} \leq \delta\right).$$

In such cases, we shall say that the admissible function $\bar{x}(\cdot)$ provides a strong (weak) local minimum in the problem (1.1), (1.2) with a δ-neighborhood. Further, if for all admissible functions $x(\cdot)$ the inequality $S(x(\cdot)) \geq S(\bar{x}(\cdot))$ holds, then we shall say that the admissible function $\bar{x}(\cdot)$ is an absolute minimum in the problem (1.1), (1.2).

It is clear that every strong local minimum is also weak, but the converse is not always true (see the example [11]).

Let us also recall that the concept of a strong local extremum was introduced into the calculus of variations by Weierstrass. He obtained a necessary condition, as well as a sufficient condition, for a strong local extremum in smooth problems of the calculus of variations (i.e. in problems in which, as a rule, it is at least assumed that the integrand $L(\cdot)$ is continuously differentiable with respect to the collection of variables). These conditions are formulated by means of a special function called the Weierstrass function

$$E(t, x, y, z) = L(t, x, z) - L(t, x, y) - L_y^T(t, x, y)(z - y), \tag{1.3}$$

where $L_y(\cdot)$ is the derivative of the integrand $L(t, x, y)$ with respect to the variable y, and the symbol T denotes the transposition operation.

We shall call a variational problem nonsmooth if the integrand of this problem is nondifferentiable with respect to at least one of its arguments.

It is important to note that problems of nonsmooth calculus of variations arise in various problems of nonlinear mechanics, the theory of economic planning, computer science, quantum mechanics, and other fields.

Analyzing a number of works [1–17, 19–21] devoted to smooth and nonsmooth variational problems, one can note that obtaining new sufficient conditions for an extremum in the problem (1.1), (1.2) remains relevant even today. Naturally, it becomes more important to obtain such results for the problem (1.1), (1.2) that do not follow as consequences of the general theory of optimal control.

This idea is also implemented in the present work. In work [10], for the nonsmooth problem (1.1), (1.2), a necessary condition for a minimum was obtained, which is formulated as follows: let the integrand $L(t, x, \dot{x})$ be continuous with respect to the collection of variables. Then: (i) if the admissible function $\bar{x}(\cdot)$ is a strong local minimum in the problem (1.1), (1.2), then the inequality

$$Q\big(t, \lambda, \xi; \bar{x}(\cdot)\big) \geq 0, \quad \forall (t, \lambda, \xi) \in I_1 \times [0, 1] \times \mathbb{R}^n; \tag{1.4}$$

holds; (ii) if the admissible function $\bar{x}(\cdot)$ is a weak local minimum in the problem (1.1), (1.2), then there exists a number $\delta > 0$ for which the inequality

$$Q\big(t, \lambda, \xi; \bar{x}(\cdot)\big) \geq 0, \quad \forall (t, \lambda, \xi) \in I_1 \times \left[0, \frac{1}{2}\right] \times B_\delta(0), \tag{1.5}$$

holds, where the set $B_\delta(0)$ is the closed ball of radius δ centered at the point $0 \in \mathbb{R}^n$, and $I_1 = [t_0, t_1] \setminus \{t\}$, where $\{t\}$ is a finite set. Next, the function $Q(\cdot)$ is defined as follows

$$\begin{aligned} Q\big(t, \lambda, \xi; \bar{x}(\cdot)\big) &= \lambda \left[L\big(t, \bar{x}(t), \dot{\bar{x}}(t) + \xi\big) - \bar{L}(t)\right] \\ &\quad + (1-\lambda)\left[L\left(t, \bar{x}(t), \dot{\bar{x}}(t) + \frac{\lambda}{\lambda - 1}\xi\right) - \bar{L}(t)\right], \qquad (1.6) \\ &(t, \lambda, \xi) \in I_1 \times [0, 1) \times \mathbb{R}^n. \end{aligned}$$

where

$$\bar{L}(t) := L\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big).$$

Definition 1.1. An admissible control $\bar{x}(\cdot)$ will be called a strong (weak) local extremal in the nonsmooth problem (1.1), (1.2) if $\bar{x}(\cdot)$ is a solution of the inequality (1.4) ((1.5)).

In the present work, a new approach is proposed for investigating extremals, in the sense of Definition 1.1, for a minimum. As a result, sufficient conditions are obtained for an absolute minimum, as well as for a strong and weak local minimum.

The contents of this article are presented according to the following scheme. In the second section, an increment formula for the functional (1.1) is obtained. In the third section, various sufficient conditions for a minimum are obtained for extremals in the sense of Definition 1.1. In the last, fourth, section, some analyses of the sufficient conditions obtained in the third section are presented and are confirmed by illustrative examples.

2. Increment formula for the functional (1.1)

Let the admissible function $\bar{x}(\cdot)$ be an extremal in the problem (1.1), (1.2), i.e. the inequality (1.4) ((1.5)) holds. We calculate the increment of the functional (1.1) along $\bar{x}(\cdot)$.

We have

$$\begin{aligned} \Delta S\big(h(\cdot); \bar{x}(\cdot)\big) &= \int_{t_0}^{t_1} \Big[L\big(t, \bar{x}(t), \dot{\bar{x}}(t) + \dot{h}(t)\big) - \bar{L}(t)\Big]\, dt \\ &\quad + \int_{t_0}^{t_1} \Big[L\big(t, \bar{x}(t) + h(t), \dot{\bar{x}}(t) + \dot{h}(t)\big) - L\big(t, \bar{x}(t), \dot{\bar{x}}(t) + \dot{h}(t)\big)\Big]\, dt, \end{aligned} \qquad (2.1)$$

$$\bar{L}(t) = L\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big),$$

where $h(\cdot) \in KC^1(I, \mathbb{R}^n)$ and $h(t_0) = h(t_1) = 0$.

Consider the identity of the form

$$\int_{t_0}^{t_1} \frac{d}{dt}\left[a^T(t)h(t) + h^T(t)A(t)h(t)\right] dt = 0, \quad \forall h(\cdot) \in KC^1(I, \mathbb{R}^n), \qquad (2.2)$$

$$h(t_0) = h(t_1) = 0,$$

where the vector function $a(t)$, $t \in I$, and the matrix function $A(t)$, $t \in I$, are piecewise smooth; moreover, $A^T(\cdot) = A(\cdot)$, i.e. it is a symmetric matrix. Using the method developed in [12], we take the identity (2.2) into account in (2.1). Then the increment $\Delta S\big(h(\cdot);\bar{x}(\cdot)\big)$ takes the form:

$$\begin{aligned}\Delta S\big(h(\cdot);\bar{x}(\cdot)\big) = \int_{t_0}^{t_1} \Big\{ &\big[L\big(t,\bar{x}(t),\dot{\bar{x}}(t)+\dot{h}(t)\big) - \bar{L}(t) + a^T(t)\dot{h}(t)\big] \\ &+ \big[L\big(t,\bar{x}(t)+h(t),\dot{\bar{x}}(t)+\dot{h}(t)\big) - L\big(t,\bar{x}(t),\dot{\bar{x}}(t)+\dot{h}(t)\big) \\ &+ \dot{a}^T(t)h(t) + 2h^T(t)A(t)\dot{h}(t) + h^T(t)\dot{A}(t)h(t)\big]\Big\}dt. \qquad (2.3)\end{aligned}$$

It should be noted that the increment formula (2.3) is important for obtaining sufficient conditions in the nonsmooth problem (1.1), (1.2).

3. Sufficient conditions for a minimum

Based on the increment formula (2.3), we prove the following theorem.

Theorem 3.1. *Let the integrand $L(t,x,\dot{x})$ be continuous with respect to the collection of variables on the set $I\times\mathbb{R}^n\times\mathbb{R}^n$, and let the admissible function $\bar{x}(\cdot)$ be an extremal of the problem* (1.1), (1.2) *in the sense of Definition 1.1. In addition, suppose that there exist a smooth vector function $a(t)$ and a smooth $n\times n$-dimensional function $A(t)$ for which the inequality*

$$\begin{aligned}&L\big(t,\bar{x}(t),\dot{\bar{x}}(t)+\xi\big) - \bar{L}(t) + a^T(t)\xi \\ &\quad + \big[L\big(t,\bar{x}(t)+\eta,\dot{\bar{x}}(t)+\xi\big) - L\big(t,\bar{x}(t),\dot{\bar{x}}(t)+\xi\big) + \dot{a}^T(t)\eta \\ &\quad + 2\eta^T A(t)\xi\big] + \eta^T\dot{A}(t)\eta \geq 0, \forall t\in I_1, \qquad (3.1)\end{aligned}$$

$$\forall(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^n\,(\forall(\xi,\eta)\in\mathbb{R}^n\times B_\delta(0)\,(\forall(\xi,\eta)\in B_\delta(0)\times B_\delta(0)))\,,$$

holds. Then: (i) if the inequality (3.1) *holds for all $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^n$, then the strong local extremal in the sense of Definition 1.1 is an absolute minimum in the problem* (1.1), (1.2)*; (ii) if the inequality* (3.1) *holds for all $(\xi,\eta)\in\mathbb{R}^n\times B_\delta(0)$ ($(\xi,\eta)\in B_\delta(0)\times B_\delta(0)$), then the strong (weak) local extremal in the sense of Definition 1.1 is a strong (weak) local minimum in the problem* (1.1), (1.2).

Proof. By the assumptions of Theorem 3.1, the increment formula (2.3) is valid. Therefore, taking into account inequality (3.1), we obtain the following inequality:

$$\Delta S\big(h(\cdot);\bar{x}(\cdot)\big) \geq 0, \quad \forall h(\cdot)\in KC^1(I,\mathbb{R}^n),$$

$$\forall\big(h(t),\dot{h}(t)\big)\in B_\delta(0)\times\mathbb{R}^n, \quad t\in I_1,$$

$$\forall\big(h(t),\dot{h}(t)\big)\in B_\delta(0)\times B_\delta(0), \quad t\in I,$$

where

$$h(t_0)=h(t_1)=0.$$

The theorem is proved. □

Corollary 3.1. *Let the functions $L(\cdot)$ and $L_x(\cdot)$ be continuous with respect to the collection of variables on the set $I\times\mathbb{R}^n\times\mathbb{R}^n$, and let the admissible function $\bar{x}(\cdot)$ be an extremal of the problem* (1.1), (1.2) *in the sense of Definition 1.1. In addition,* *suppose that there exists a piecewise smooth vector function* $a(t) \in KC^1(I, \mathbb{R}^n)$ *for which the inequality*

$$\begin{aligned} &L\big(t, \bar{x}(t), \dot{\bar{x}}(t) + \xi\big) - \bar{L}(t) + a^T(t)\xi \\ &\quad + \left[L_x^T\big(t, \bar{x}(t) + \theta\eta, \dot{\bar{x}}(t) + \xi\big) + \dot{a}^T(t)\right]\eta \geq 0, \forall\theta \in (0,1), \end{aligned} \tag{3.2}$$

$\forall(t, \xi, \eta) \in I_1 \times \mathbb{R}^n \times \mathbb{R}^n \left((t, \xi, \eta) \in I_1 \times \mathbb{R}^n \times B_\delta(0) \left((t, \xi, \eta) \in I_1 \times \mathbb{R}^n \times B_\delta(0)\right)\right).$

holds. Then:

(i) if the inequality (3.2) *holds for all* $(t, \xi, \eta) \in I_1 \times \mathbb{R}^n \times \mathbb{R}^n$*, then the strong local extremal* $\bar{x}(\cdot)$*, in the sense of Definition 1.1, is an absolute minimum in the problem* (1.1)*,* (1.2)*;*

(ii) if the inequality (3.2) *holds for all* $(t, \xi, \eta) \in I_1 \times \mathbb{R}^n \times B_\delta(0)$*, then the strong (weak) local extremal* $\bar{x}(\cdot)$ *in the sense of Definition 1.1 is a strong (weak) local minimum in the problem* (1.1)*,* (1.2)*.*

Proof. Let us take into account the assumptions of Corollary 3.1 and set

$$A(\cdot) = 0$$

in the increment formula (2.3). Then we obtain, by virtue of inequality (3.2) and Taylor's formula with the Lagrange remainder term, the inequality holds:

$$\Delta S\big(h(\cdot); \bar{x}(\cdot)\big) \geq 0, \quad \forall h(\cdot) \in KC^1(I, \mathbb{R}^n),$$

$$\forall\big(h(t), \dot{h}(t)\big) \in B_\delta(0) \times \mathbb{R}^n, \quad t \in I_1,$$

$$\forall\big(h(t), \dot{h}(t)\big) \in B_\delta(0) \times B_\delta(0), \quad t \in I.$$

Thus, the proof of Corollary 3.1 follows. □

Corollary 3.2. *Suppose that the functions* $L(\cdot)$*,* $L_x(\cdot)$*, and* $L_{\dot{x}}(\cdot)$ *are continuous with respect to the aggregate of variables on the set* $I \times \mathbb{R}^n \times \mathbb{R}^n$*, and that the admissible function* $\bar{x}(\cdot)$ *is an extremal of problem* (1.1)*,* (1.2)*. in the sense of Definition 1.1. Moreover, suppose that there exists an* $n \times n$ *piecewise smooth symmetric matrix* $A(t)$*,* $t \in I$*, for which the inequality*

$$\begin{aligned} &E(L)\big(t, \bar{x}(t), \dot{\bar{x}}(t) + \xi\big) + \Big[2\xi^T A(t) + L_x^T\big(t, \bar{x}(t) + \theta\eta, \dot{\bar{x}}(t) + \xi\big) \\ &\quad - L_x^T\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big)\Big]\eta + \eta^T A(t)\eta \geq 0, \end{aligned} \tag{3.3}$$

holds for $\forall(t, \theta) \in I_1 \times (0,1)$*, and*

$$\forall(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \left(\forall(\xi, \eta) \in \mathbb{R}^n \times B_\delta(0) \left(\forall(\xi, \eta) \in B_\delta(0) \times B_\delta(0)\right)\right),$$

and, furthermore, along $\bar{x}(\cdot)$ *the first variation of functional* (1.1)*, for all* $h(\cdot) \in KC^1(I, \mathbb{R}^n)$ *with* $h(t_0) = h(t_1) = 0$*, is equal to zero; that is, the equality*

$$\int_{t_0}^{t_1} \Big[L_x^T\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big)h(t) + L_{\dot{x}}^T\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big)\dot{h}(t)\Big]\, dt = 0. \tag{3.4}$$

holds.

Then:

(i) if inequality (3.3) *holds for all* $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$*, then the strong local extremal* $\bar{x}(\cdot)$ *is an absolute minimum in problem* (1.1)*,* (1.2)*;*

(ii) if inequality (3.3) *holds for all* $(\xi,\eta)\in\mathbb{R}^n\times B_\delta(0)$ $\big((\xi,\eta)\in B_\delta(0)\times B_\delta(0)\big)$, *then the strong (weak) local extremal* $\bar{x}(\cdot)$ *is a strong (weak) local minimum in problem* (1.1), (1.2).

Here the function $E(L)(\cdot)$ *is defined by formula* (1.3), *namely in the form*

$$\begin{aligned} E(L)(t,\bar{x}(t),\dot{\bar{x}}(t)+\xi) &= L(t,\bar{x}(t),\dot{\bar{x}}(t)+\xi)-L(t,\bar{x}(t),\dot{\bar{x}}(t)) \\ &\quad - L_{\dot{x}}^T(t,\bar{x}(t),\dot{\bar{x}}(t))\,\xi. \end{aligned} \tag{3.5}$$

Proof. In view of assumption (3.4), instead of identity (2.2), we may consider the identity

$$\int_{t_0}^{t_1}\left\{-\overline{L}_x^T(t)h(t)-\overline{L}_{\dot{x}}^T(t)\dot{h}(t)+\frac{d}{dt}\left[h^T(t)A(t)h(t)\right]\right\}dt=0, \tag{3.6}$$

$$\forall h(\cdot)\in KC^1(I,\mathbb{R}^n),\qquad h(t_0)=h(t_1)=0,$$

where

$$\overline{L}_y(t)=L_y\big(t,\overline{x}(t),\dot{\overline{x}}(t)\big),\qquad y\in\{x,\dot{x}\},$$

Next, by Taylor's formula, we obtain

$$\begin{aligned} &L\big(t,\overline{x}(t)+h(t),\dot{\overline{x}}(t)+\dot{h}(t)\big)-L\big(t,\overline{x}(t),\dot{\overline{x}}(t)+\dot{h}(t)\big) \\ &\quad = L_x^T\big(t,\overline{x}(t)+\theta h(t),\dot{\overline{x}}(t)+\dot{h}(t)\big)h(t),\quad \theta\in(0,1). \end{aligned} \tag{3.7}$$

Let us take (3.5)–(3.7) into account in the increment formula (2.3). Then we obtain a new form of the increment $\Delta S\big(h(\cdot);\overline{x}(\cdot)\big)$. Therefore, by inequality (3.3), the following inequality is valid:

$$\Delta S\big(h(\cdot);\overline{x}(\cdot)\big)\geq 0,\qquad \forall h(\cdot)\in KC^1(I,\mathbb{R}^n),$$

$$\Big(\forall\big(h(t),\dot{h}(t)\big)\in B_\delta(0)\times\mathbb{R}^n,\quad t\in I_1$$

$$\big(\forall\big(h(t),\dot{h}(t)\big)\in B_\delta(0)\times B_\delta(0),\quad t\in I_1\big)\Big).$$

Corollary 3.2 is proved. □

4. Conclusion

First, let us discuss the obtained results. Thus:

1. The assumption used, $x(\cdot)\in KC^1(I,\mathbb{R}^n)$, is excessive. It can be weakened.
2. A similar assertion, Corollary 3.2, was obtained in [12].
3. The results obtained in this work can be directly formulated for a variational problem depending on derivatives of higher orders, since, by means of a change of variables, it is reduced to problem (1.1), (1.2) (see, for example, [10]).

Now we also demonstrate the effectiveness of the obtained results by means of examples.

Example 1.

Consider the problem

$$\int_0^1\left[\dot{x}^2-x|\sin\dot{x}|+|x|\right]dt\to\min,\qquad x(0)=x(1)=0, \tag{4.1}$$

where

$$L(t,x,\dot{x}) = \dot{x}^2 - x|\sin\dot{x}| + |x|, \qquad x(t) \in KC^1([0,1],\mathbb{R}).$$

We investigate the admissible function $\bar{x}(\cdot) = 0$ for a minimum. First, we show that it satisfies inequalities (1.4) and (1.5). It is clear that if $\bar{x}(\cdot) = 0$ does not satisfy inequality (1.5), that is, if it does not satisfy the necessary condition for a minimum, then $\bar{x}(\cdot) = 0$ is not a minimum in the nonsmooth problem (4.1).

We present the following calculations. Namely,

$$L(t,\bar{x}(t),\dot{\bar{x}}(t)+\xi) - \bar{L}(t) = \xi^2,$$

$$L\left(t,\bar{x}(t),\dot{\bar{x}}(t)+\frac{\lambda}{1-\lambda}\xi\right) - \bar{L}(t) = \frac{\lambda^2}{(1-\lambda)^2}\xi^2.$$

According to (1.6), we have

$$Q(t,\lambda,\xi;\bar{x}(\cdot)) = \lambda\xi^2 + \frac{\lambda^2}{1-\lambda}\xi^2.$$

Hence, we obtain that inequality (1.4) is valid. Therefore, by Definition 1.1, the admissible function

$$\bar{x}(\cdot) = 0$$

is a strong local extremal.

Next, for all

$$(t,\lambda,\xi) \in [0,1]\times[0,1]\times\mathbb{R}$$

along

$$\bar{x}(\cdot) = 0,$$

inequality (3.1) is satisfied, that is, for the case

$$a(t) = A(t) = 0, \qquad t \in [0,1],$$

the following inequality holds:

$$\lambda\xi^2 - \eta\left|\sin\left(\frac{\lambda}{\lambda-1}\xi\right)\right| + |\eta| \geq 0, \qquad \forall(t,\lambda,\xi,\eta) \in [0,1]\times[0,1)\times\mathbb{R}\times\mathbb{R}.$$

Thus, by Theorem 3.1, we obtain that the strong local extremal $\bar{x}(\cdot) = 0$ is an absolute minimum in problem (4.1).

Example 2. Consider the problem

$$\int_0^1 \left[(1-\dot{x})^2 - x^2 + |x|\right] dt \to \min, \qquad x(0) = x(1) = 0, \tag{4.2}$$

where

$$L(t,x,\dot{x}) = (1-\dot{x})^2 - x^2 + |x|, \qquad x(t) \in KC^1([0,1],\mathbb{R}).$$

We investigate the admissible function $\bar{x}(t) = 0$ for a minimum. First, we show that it satisfies inequality (1.4) or (1.5). It is clear that if $\bar{x}(\cdot) = 0$ does not satisfy inequality (1.5), that is, if it does not satisfy the necessary condition for a minimum, then $\bar{x}(\cdot) = 0$ is not even a weak local minimum in the nonsmooth problem (4.2).

We present the following calculations. Along $\bar{x}(\cdot)=0$, we have

$$L\big(t,\bar{x}(t),\dot{\bar{x}}(t)+\xi\big)-\bar{L}(t)=(1-\xi)^2-1=\xi^2-2\xi,$$

$$L\left(t,\bar{x}(t),\dot{\bar{x}}(t)+\frac{\lambda}{\lambda-1}\xi\right)-\bar{L}(t)=\left(1-\frac{\lambda}{\lambda-1}\xi\right)^2-1$$
$$=\left(\frac{\lambda}{\lambda-1}\right)^2\xi^2+2\frac{\lambda}{1-\lambda}\xi.$$

According to (1.6), we obtain

$$Q\big(t,\lambda,\xi;\bar{x}(\cdot)\big)=\lambda\xi^2-2\lambda\xi+\frac{\lambda^2}{1-\lambda}\xi^2+2\lambda\xi\geq 0,\qquad \forall(\lambda,\xi)\in[0,1]\times\mathbb{R}.$$

Consequently, inequality (1.4) is satisfied. Therefore, by Definition 1.1, the function $\bar{x}(\cdot)=0$ is a strong local extremal.

Now we verify the fulfillment of inequality (3.1). Let $a(t)=2$, $A(t)=0$, $t\in[0,1]$. Then

$$\xi^2-2\xi+a(t)\xi+|\eta|-\eta^2+\dot{a}(t)\eta+A(t)\xi\eta+\dot{A}(t)\eta^2$$
$$=\xi^2+|\eta|-\eta^2\geq 0,\qquad \forall(\xi,\eta)\in\mathbb{R}\times[-1,1].$$

Hence, by Theorem 3.1, we have that the strong local extremal $\bar{x}(\cdot)=0$ is a strong local minimum in the nonsmooth problem (4.2).

Example 3.

Consider the problem

$$\int_0^1\big[\dot{x}^3+|\dot{x}|-\dot{x}^2+|x|\big]\,dt\to\min,\qquad x(0)=x(1)=0,\tag{4.3}$$

where

$$L(t,x,\dot{x})=\dot{x}^3+|\dot{x}|-\dot{x}^2+|x|,\qquad x(t)\in KC^1([0,1],\mathbb{R}).$$

We investigate the admissible function

$$\bar{x}(t)=0,\qquad t\in[0,1],$$

for a minimum. In this case, inequality (1.5) has the form

$$Q(t,\lambda,\xi;\bar{x}(\cdot))=\lambda\xi^3-\frac{\lambda^3}{(1-\lambda)^2}\xi^3+\lambda|\xi|+\lambda|\xi|$$
$$=\lambda\xi^3\frac{1-2\lambda}{(1-\lambda)^2}+2\lambda|\xi|\geq 0,\qquad \forall\lambda\in\left[0,\frac{1}{2}\right],\quad \forall\xi\in[-1,1].$$

Therefore, by Definition 1.1, the admissible function

$$\bar{x}(\cdot)=0$$

is a weak local extremal.

Next, let

$$a(t)=A(t)=0.$$

Then it is not difficult to obtain that for all

$$(t,\xi,\eta)\in[0,1]\times[-1,1]\times[-1,1],$$

inequality (3.1) is satisfied; namely, the following inequality holds:

$$\xi^3+|\xi|-\eta^2+|\eta|\geq 0,\qquad \forall(\xi,\eta)\in[-1,1]\times[-1,1].$$

Thus, by Theorem 3.1, we obtain that the weak local extremal

$$\bar{x}(\cdot) = 0$$

is a weak local minimum in the nonsmooth problem (4.3).

It should be noted that in problems (4.1)–(4.3), the integrands are not differentiable with respect to x. Therefore, Pontryagin's maximum principle [18] and a number of well-known necessary as well as sufficient conditions are not directly applicable.

References

[1] G. A. Bliss, *Lectures on the Calculus of Variations*, The University of Chicago Press, Chicago, 1945.

[2] J. F. Bonnans, X. Dupuis, L. Pfeiffer, Second-order sufficient conditions for strong solutions to optimal control problems, *ESAIM Control Optim. Calc. Var.* **20** (2014), no. 3, 704–724.

[3] P. Bousquet, The Euler equation in the multiple integrals calculus of variations, *SIAM J. Control Optim.* **51** (2015), no. 2, 1047–1062.

[4] M. R. Hestenes, *Calculus of Variations and Optimal Control Theory*, Wiley, New York, 1966.

[5] A. D. Ioffe, V. M. Tikhomirov, *Theory of Extremum Problems*, Nauka, Moscow, 1974.

[6] V. F. Krotov, V. I. Gurman, *Methods and Problems of Optimal Control*, Science, Moscow, 1973.

[7] M. A. Lavrentev, L. A. Lusternik, *The Course of Variational Calculus*, 1950.

[8] E. S. Levitin, A. A. Milyutin, N. P. Osmolovsky, On necessary and sufficient conditions for a minimum, *Dokl. Akad. Nauk SSSR* **210** (1973), no. 5, 1173–1176.

[9] M. J. Mardanov, T. K. Melikov, H. F. Guliyev, On comparing Pontryagin's maximum principle with the basic necessary conditions of variational calculus, *Adv. Math. Models Appl.* **10** (2025), no. 3, 450–455.

[10] M. J. Mardanov, T. K. Melikov, S. T. Malik, Necessary conditions for the extremum in non-smooth problems of variational calculus, *J. Comput. Appl. Math.* **399** (2022), 113723.

[11] M. J. Mardanov, T. K. Melikov, S. T. Malik, Necessary conditions for a minimum in classical calculus of variations in the presence of various types of degenerations, *J. Comput. Appl. Math. 418* (2023), 114668.

[12] M. J. Mardanov, T. K. Melikov, S. T. Malik, On sufficient conditions in the classical problem of the calculus of variations, arXiv:2512.21918, 2025.

[13] M. J. Mardanov, S. T. Malik, Necessary optimality conditions for a new type of singular controls in optimal control problems, *Int. J. Control* **99** (2026), no. 2, 519–530.

[14] P. Mauro, Nuovi criteri sufficienti en un classico problema di calcolo dell variazioni, *Ann. Mat. Pura Appl.* **53** (1961), 119–137.

[15] P. Mauro, Criteri sufficienti per il minimo assoluto di un integrale pluridimensionale del primo ordine net vettore minimaute, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I 6 (1962), no. 11.

[16] C. B. Morrey Jr., *Multiple Integrals in the Calculus of Variations*, Springer, Berlin, Heidelberg, 1966.

[17] N. P. Osmolovskii, H. Maurer, *Applications to Regular and Bang-Bang Control: Second-Order Necessary and Sufficient Optimality Conditions in Calculus of Variations and Optimal Control*, SIAM, Philadelphia, 2012.

[18] L. S. Pontryagin, V. G. Boltyanskiy, R. V. Gamkrelidze, E. F. Mishchenko, *Mathematical Theory of Optimal Processes*, John Wiley & Sons, New York, 1962.
[19] H. W. William, Second-order sufficient optimality conditions in the calculus of variations, *J. Optim. Theory Appl.* **205** (2025), no. 1, 1–9.
[20] L. C. Young, *Lectures on Variational Calculus and Optimal Control Theory*, Mir, Moscow, 1974.
[21] V. Zeidan, Sufficient conditions for variational problems with variable endpoints: coupled points, *Appl. Math. Optim.* **27** (1993), no. 2, 191–209.

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